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On the number of irreducible factors of a polynomial II

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Dedicated to the memory of Jacek Szarski

Abstract. For a polynomial f with integral coefficients and $f(0) \neq 0$ the number of its irreducible factors counted with or without multiplicities and the maximal multiplicity in question are estimated in terms of its degree $|f|$ and of the sum of squares of its coefficients $||f||$ roughly by $\sqrt{|f|\log||f||}$. The estimates in their exact form are nearly best possible.

This paper is a sequel to [9] and the same notation is used. For a polynomial f with integral coefficients $|f|$ is its degree, $||f||$ the sum of squares of its coefficients; if $|f| \geqslant 1$ then $\Omega(f)$ and $\omega(f)$ are the number of irreducible factors of f counted with or without multiplicities, respectively, $\Omega_1(f)$ and $\omega_1(f)$ the relevant numbers of irreducible non-cyclotomic factors. Finally, $m(f)$ is the maximal multiplicity of a factor of f. Clearly,

$$\max\{\omega(f), m(f)\} \leqslant \Omega(f) \leqslant |f|$$

and it has been proved in [9] that if $f(0) \neq 0$, then

$$\Omega_1(f) \leqslant \sqrt{26\,|f|\log 7\,|f|\log||f||},$$
$$\omega(f)-\omega_1(f) \ll |f|^{1/2}, \tag{1}$$
$$\Omega(f)-\Omega_1(f) \ll |f|^{3/4}(\log\log|f|)^{1/2}(\log||f||)^{1/2} \qquad (|f| \geqslant 3).$$

Here as everywhere in the sequel the constants implicit in the Vinogradov symbol $\ll$ are absolute.

It has been conjectured in [9] that for every ε between 0 and 1

$$\omega(f) = o(|f|^{\varepsilon})(\log||f||)^{1-\varepsilon}, \tag{2}$$

$$\Omega(f) = O(|f|^{\varepsilon})(\log||f||)^{1-\varepsilon}. \tag{3}$$

Dobrowolski's result on the Lehmer problem about algebraic integers implies (see [1], Theorem 2)

$$\Omega_1(f) = O(|f|^{\varepsilon})(\log||f||)^{1-\varepsilon} \tag{4}$$

for every ε between 0 and 1; on the other hand, he has disproved conjectures (2) and (3) for every $\varepsilon < 1/2$ or $\varepsilon \leqslant 1/2$, respectively. For $m(f)$ there is an estimate implicit in a theorem of Erdös and Turán [3], namely

$$(5)\qquad m(f) < 16\sqrt{n\log|a_0 a_n|^{-1/2}\sum_{i=0}^{n}|a_i|}, \qquad \text{where } f(x) = \sum_{i=0}^{n} a_i x^{n-i}$$

(this has been pointed out to the author by M. Mignotte).

An easy modification of the proof of (4) gives

THEOREM 1. *For every ε between 0 and 1 and every polynomial f with integral coefficients and $f(0) \neq 0$*

$$\omega_1(f) = o(|f|^{\varepsilon})(\log\|f\|)^{1-\varepsilon}.$$

Our main result is

THEOREM 2. *For every polynomial f with integral coefficients and $f(0) \neq 0$*

$$(6)\qquad \omega(f) = o(\sqrt{|f|\log\|f\|}) \quad (|f| \to \infty),$$

$$(7)\qquad m(f) \ll \sqrt{|f|\log\|f\|},$$

$$(8)\qquad \Omega(f) \ll \sqrt{|f|\log\|f\|\log r}\log\log r,$$

$$r = \max\left(3, \frac{|f|}{\log\|f\|}\right).$$

COROLLARY. *For every ε between 0 and 1/2*

$$(9)\qquad \Omega(f) = O(|f|^{1/2+\varepsilon})(\log\|f\|)^{1/2-\varepsilon}.$$

The given estimates are quite precise, as it is shown by the examples

$$f_1(x) = \prod_{i=1}^{n}(x-i), \quad f_2(x) = (x-1)^n, \quad f_3(x) = \prod_{i=1}^{n}(x^i-1)^{n-i+1}.$$

Here $|f_1| = |f_2| = n$, $|f_3| \sim n^3$, $\log\|f_1\| \ll n\log n$, $\log\|f_2\| \ll n$, $\log\|f_3\| \ll n\log n$ (see [1], p. 401), $\omega(f_1) = n$, $m(f_2) = n$, $\Omega(f_3) \gg n^2\log n$. Hence for $f = f_1$ the left-hand side of (6) is n, the right-hand side is $o\big(n(\log n)^{1/2}\big)$, for $f = f_2$ the left-hand side of (7) and (9) is n, the right-hand side is $\ll n$, for $f = f_3$ the left-hand side of (8) is $\gg n^2\log n$, the right-hand side is $\ll n^2\log n\cdot\log\log n$. Note that (7) is weaker than (5), but the proof will be entirely different. There is still room to conjecture that

$$\Omega(f) \ll \sqrt{|f|\log\|f\|\log r}.$$

Proof of Theorem 1. Put

$$M(f) = |a_f|\prod_{i=1}^{n}\max\{1, |\alpha_i|\}$$

where $a_1, a_2, \ldots, a_n$ are the zeros of the polynomial f listed with proper multiplicity and a_f is its leading coefficient.

Let

$$f(x) = f_0(x)\prod_{i=1}^{\omega_1} f_i(x)^{\beta_i}, \tag{10}$$

where f_i are for $i > 0$ distinct non-cyclotomic polynomials and f_0 is a product of cyclotomic factors. Then

$$M(f) = \prod_{i=1}^{\omega_1} M(f_i)^{\beta_i}.$$

Landau [4] showed that $M(f) \leqslant \|f\|^{1/2}$, thus

$$\log\|f\| \geqslant \sum_{i=1}^{\omega_1} \log M(f_i).$$

On the other hand, by comparison of degrees in (10),

$$|f| \geqslant \sum_{i=1}^{\omega_1} |f_i|.$$

By Hölder's inequality

$$\sum_{i=1}^{\omega_1} |f_i|^{\varepsilon}\big(\log M(f_i)\big)^{1-\varepsilon} \leqslant \Big(\sum_{i=1}^{\omega_1} |f_i|\Big)^{\varepsilon}\Big(\sum_{i=1}^{\omega_1} \log M(f_i)\Big)^{1-\varepsilon} \leqslant |f|^{\varepsilon}(\log\|f\|)^{1-\varepsilon}. \tag{11}$$

Take an arbitrary $A > 0$ and consider separately $i \leqslant \omega_1$ such that

$$|f_i|^{\varepsilon}\big(\log M(f_i)\big)^{1-\varepsilon} \geqslant A \tag{12}$$

and the remaining ones. The number of i's satisfying (12) does not exceed, in virtue of (11), $|f|^{\varepsilon}(\log\|f\|)^{1-\varepsilon}/A$.

On the other hand, by Dobrowolski's theorem ([1], Theorem 1) either $|f_i| \leqslant 2$ or

$$\log M(f_i) \gg \left(\frac{\log\log|f_i|}{\log|f_i|}\right)^3,$$

whence for a certain $c(\varepsilon) > 1$

$$\log M(f_i) \geqslant \frac{1}{c(\varepsilon)\,|f_i|^{\varepsilon}}.$$

Thus the negation of (12) gives $|f_i|^{\varepsilon^2} < c(\varepsilon)A$ and also $\log M(f_i) < A^{1/(1-\varepsilon)}$.

The number of polynomials g with integral coefficients with bounded $|g|$ and $M(g)$ is finite; hence the number of i's for which (12) fails does not exceed $B(\varepsilon, A)$ independent of f. Eventually we obtain

$$\omega_1 = \omega_1(f) \leqslant \frac{|f|^{\varepsilon}(\log\|f\|)^{1-\varepsilon}}{A} + B(\varepsilon, A)$$

and since A is arbitrary, the theorem follows.

For the proof of Theorem 2 we need several lemmata.

LEMMA 1. *Let $\Phi_n(x)$ be the n-th cyclotomic polynomial, $\Phi_n(x)^{\varepsilon_n}\|f(x)$. Then for every prime p*

$$\varepsilon_n \leqslant \frac{\varphi(np)}{\varphi(n)\log p}\left(\log\binom{|f|}{\varepsilon_{np}} + \log\|f\|\right),$$

where φ is Euler's function.

Proof. Let

$$f(x) = \Phi_{np}(x)^{\varepsilon_{np}}\Phi_n(x)^{\varepsilon_n}g(x), \qquad g \in Z[x].$$

Differentiating ε_{np} times and substituting afterwards for x a primitive npth root of unity, ζ_{np}, we get

$$f^{(\varepsilon_{np})}(\zeta_{np}) = (\varepsilon_{np})!\,\Phi'_{np}(\zeta_{np})^{\varepsilon_{np}}\Phi_n(\zeta_{np})g(\zeta_{np}). \tag{13}$$

Taking norms from $Q(\zeta_{np})$ to the rational field Q, we obtain

$$\left|N\left(f^{\varepsilon_{np}}(\zeta_{np})\right)\right| \leqslant \left|\overline{f^{(\varepsilon_{np})}(\zeta_{np})}\right|^{\varphi(np)} \leqslant \left(\varepsilon_{np}!\binom{|f|}{\varepsilon_{np}}\|f\|\right)^{\varphi(np)}, \tag{14}$$

$$\left|N\left(\Phi'_{np}(\zeta_{np})^{\varepsilon_{np}}g(\zeta_{np})\right)\right| \geqslant 1. \tag{15}$$

On the other hand, since $\Phi_n(x) = \prod_{d|n}(x^d-1)^{\mu\left(\frac{n}{d}\right)}$, we have

$$N\Phi_n(\zeta_{np}) = \prod_{\substack{j=1\\(j,np)=1}}^{np}\Phi_n(\zeta_{np}^j) = \prod_{\substack{j=1\\(j,np)=1}}^{np}\prod_{d|n}(\zeta_{np}^{dj}-1)^{\mu\left(\frac{n}{d}\right)}$$

$$= \pm\prod_{d|n}\prod_{\substack{j=1\\(j,np)=1}}^{np}(1-\zeta_{\frac{n}{d}p}^{j})^{\mu\left(\frac{n}{d}\right)} = \pm\prod_{d|n}\Phi_{\frac{n}{d}p}(1)^{\mu\left(\frac{n}{d}\right)\frac{\varphi(np)}{\varphi\left(\frac{n}{d}p\right)}}.$$

However,

$$\Phi_r(1) = \begin{cases} q & \text{if } r = q^{\alpha},\ q \text{ prime},\\ 0 & \text{if } r = 1,\\ 1 & \text{otherwise;}\end{cases}$$

hence

$$N\Phi_n(\zeta_{np}) = \begin{cases} \pm\Phi_p(1)^{\varphi(np)/\varphi(p)} & \text{if } p\nmid n,\\ \pm\Phi_p(1)^{\varphi(np)/\varphi(p)}\Phi_{p^2}(1)^{-\varphi(np)/\varphi(p^2)} & \text{if } p\,|\,n.\end{cases}$$

and

$$|N\Phi_n(\zeta_{np})| = p^{\varphi(n)}. \tag{16}$$

The lemma follows from (13)–(16).

LEMMA 2. *There exists an absolute constant $c \geqslant e^2$ such that in the notation of Lemma 1 we have*

$$\sum{}^{*} \frac{\varepsilon_n^2 \varphi(n)}{\omega(n)+1} \log \frac{\varepsilon_n}{\log\|f\|} \ll |f| \log\|f\| \log \frac{|f|}{\log\|f\|},$$

where $\omega(n)$ is the number of distinct prime factors of n, $\sum^{*}$ is taken over all n with $\varepsilon_n > c\log\|f\|$.

Proof. Let us assume that $\varepsilon_n \geqslant e^2 \log\|f\|$ and define b_n by the equation

$$\frac{b_n}{\log b_n} = \frac{\varepsilon_n}{2\log\|f\|} \geqslant \frac{e^2}{2} > \frac{2}{\log 2}. \tag{17}$$

Since $x/\log x$ is increasing for $x \geqslant e$, we have

$$b_n \geqslant \frac{\varepsilon_n}{2\log\|f\|} \log \frac{\varepsilon_n}{\log\|f\|}. \tag{18}$$

Let us consider prime numbers $p \leqslant b_n$. For such primes p we have

$$\frac{p}{\log p} \leqslant \frac{\varepsilon_n}{2\log\|f\|};$$

hence by Lemma 1

$$\varepsilon_{np} \log \frac{e|f|}{\varepsilon_{np}} \geqslant \log \binom{|f|}{\varepsilon_{np}} \geqslant \frac{\varphi(n)\log p}{\varphi(np)} \varepsilon_n - \log\|f\| \geqslant \log\|f\|.$$

It follows that

$$\frac{1}{\sqrt{e|f|/\varepsilon_{np}}} \geqslant \frac{\log e|f|/\varepsilon_{np}}{e|f|/\varepsilon_{np}} \geqslant \frac{\log\|f\|}{e|f|} \geqslant \left(\frac{\log\|f\|}{|f|}\right)^{3/2}$$

(note that $|f| \geqslant \varepsilon_n \geqslant e^2 \log\|f\|$); hence

$$\frac{e|f|}{\varepsilon_{np}} \leqslant \left(\frac{|f|}{\log\|f\|}\right)^3, \quad \log \frac{e|f|}{\varepsilon_{np}} \leqslant 3\log \frac{|f|}{\log\|f\|}$$

and by Lemma 1

$$3\varphi(np)\varepsilon_{np} \log \frac{|f|}{\log\|f\|} \geqslant \varphi(n)\varepsilon_n \log p - \varphi(n) p \log\|f\|.$$

Now

$$\sum_{p \leqslant b_n} \log p = b_n + O(b_n/\log b_n),$$

$$\sum_{p \leqslant b_n} p = \frac{b_n^2}{2\log b_n} + O\left(\frac{b_n^2}{(\log b_n)^2}\right).$$

Hence

$$3\log\frac{|f|}{\log\|f\|}\sum_{p\leqslant b_n}\varepsilon_{np}\varphi(np)$$
$$\geqslant b_n\varphi(n)\left(\varepsilon_n-\frac{b_n\log\|f\|}{2\log b_n}\right)+O\left(\frac{\varepsilon_n\varphi(n)b_n}{\log b_n}\right)+O\left(\frac{b_n^2}{(\log b_n)^2}\varphi(n)\log\|f\|\right)$$

and using (17) and (18)

$$(19)\qquad 3\log\frac{|f|}{\log\|f\|}\sum_{p\leqslant b_n}\varepsilon_{np}\varphi(np)\geqslant\frac{3}{4}\varepsilon_n\varphi(n)b_n+O\left(\frac{\varepsilon_n^2\varphi(n)}{\log\|f\|}\right)$$
$$\geqslant\frac{3}{8}\frac{\varepsilon_n^2\varphi(n)}{\log\|f\|}\log\frac{\varepsilon_n}{\log\|f\|}+O\left(\frac{\varepsilon_n^2\varphi(n)}{\log\|f\|}\right).$$

Let the constant implicit in the O symbol on the right-hand side of (19) be c_1. We now choose a constant $c\geqslant e^2$ such that

$$\log c\geqslant 4c_1.$$

Then if $\varepsilon_n>c\log\|f\|$, we get

$$c_1\frac{\varepsilon_n^2\varphi(n)}{\log f}\leqslant\frac{1}{4}\frac{\varepsilon_n^2\varphi(n)}{\log\|f\|}\log\frac{\varepsilon_n}{\log\|f\|}$$

and (19) gives

$$3\log\frac{|f|}{\log\|f\|}\sum_{p\leqslant b_n}\varepsilon_{np}\varphi(np)\geqslant\frac{1}{8}\frac{\varepsilon_n^2\varphi(n)}{\log\|f\|}\log\frac{\varepsilon_n}{\log\|f\|}.$$

Summing over n subject to the condition $\varepsilon_n>c\log\|f\|$, we obtain

$$(20)\qquad 3\log\frac{|f|}{\log\|f\|}\sum_n{}^{*}\sum_{p\leqslant b_n}\frac{\varepsilon_{np}\varphi(np)}{\omega(n)+1}$$
$$\geqslant\frac{1}{8}\sum{}^{*}\frac{\varepsilon_n^2\varphi(n)}{\log\|f\|(\omega(n+1))}\log\frac{\varepsilon_n}{\log\|f\|}.$$

On the other hand, for every m

$$1\geqslant\sum_{p|m}\frac{1}{\omega(m/p)+1};$$

hence

$$|f|\geqslant\sum\varepsilon_m\varphi(m)\geqslant\sum_n\sum_p\frac{\varepsilon_{np}\varphi(np)}{\omega(n)+1}.$$

This together with (20) gives the lemma.

LEMMA 3. *We have*

$$A(x) = \sum_{\varphi(n)\leqslant x} 1 \ll x$$

and for $x \geqslant 2$

$$B(x) = \sum_{\varphi(n)\leqslant x} \frac{1}{\varphi(n)} \ll \log x.$$

Proof. The estimate for $A(x)$ is due to Erdös [2]. The estimate for $B(x)$ is obtained by partial summation (see [8], p. 371) as follows:

$$B(x) = \frac{A(x)}{x} + \int_1^x \frac{A(\xi)}{\xi^2}\,d\xi \ll 1 + \int_1^x \frac{d\xi}{\xi} \ll \log x.$$

LEMMA 4. *For every* $a \geqslant 2$ *and* $x \geqslant 3$

$$\sum_{\varphi(n)\leqslant x} \frac{\omega(n)+1}{\varphi(n)\log\big(ax/\varphi(n)\big)} \ll (\log\log x)^2.$$

Proof. We shall first show that for $\xi \geqslant 3$

$$C(\xi) = \sum_{\varphi(n)\leqslant \xi} \omega(n) \ll \xi \log\log \xi. \tag{21}$$

By a theorem of Landau ([5], p. 216) for a suitable constant b and $\xi \geqslant 3$

$$\varphi(n) \leqslant \xi \quad \text{implies} \quad n \leqslant b\xi\log\log\xi = \eta > 1.$$

Now by Lemma 2 in [9]

$$\sum_{\substack{n\leqslant \eta \\ \omega(n)>10\log\log\eta}} \omega(n) \ll \frac{\eta}{\log \eta}.$$

Hence we get

$$\sum_{\substack{\varphi(n)\leqslant \xi \\ \omega(n)>10\log\log\eta}} \omega(n) \ll \frac{\eta}{\log\eta} \ll \frac{\xi\log\log\xi}{\log\xi}.$$

On the other hand, by Lemma 3

$$\sum_{\substack{\varphi(n)\leqslant \xi \\ \omega(n)\leqslant 10\log\log\eta}} \omega(n) \leqslant 10\log\log\eta\cdot A(\xi) \ll \xi\log\log\xi.$$

Thus (21) follows. Now using partial summation, we get

$$\sum_{\varphi(n)\leqslant x} \frac{\omega(n)}{\varphi(n)\log\big(ax/\varphi(n)\big)} \leqslant \frac{C(x)}{x\log a} + \int_1^x \frac{\log ax - \log\xi - 1}{\xi^2(\log ax/\xi)^2}\, C(\xi)\,d\xi$$

$$\ll \frac{x\log\log x}{x\log a} + \int_1^3 \frac{d\xi}{\xi^2 \log ax/\xi} + \int_3^x \frac{\xi \log\log \xi}{\xi^2 |\log ax/\xi|} d\xi$$

$$\ll \log\log x + \frac{1}{\log a} + \log\log x \int_3^x \frac{d\xi}{\xi \log ax - \log \xi}$$

$$\ll \log\log x - \log\log x \cdot \log(\log ax - \log \xi)|_3^x$$

$$= \log\log x - \log\log x \cdot \log\log a + \log\log x \cdot \log\log \frac{ax}{3}$$

$$\ll (\log\log x)^2.$$

Since

$$\sum_{\varphi(n)\leqslant x} \frac{\omega(n)+1}{\varphi(n)\log\big(ax/\varphi(n)\big)} \leqslant \frac{1}{\log ax} + 2 \sum_{\varphi(n)\leqslant x} \frac{\omega(n)}{\varphi(n)\log\big(ax/\varphi(n)\big)},$$

the lemma follows.

Proof of Theorem 2. By Theorem 1 with $\varepsilon = \frac{1}{2}$ we have

$$\omega_1(f) = o(\sqrt{|f|\log\|f\|}) \quad (|f| \to \infty),$$

thus in order to prove (6) it is enough to show that

$$\omega(f) - \omega_1(f) = o\big(\sqrt{|f|\log\|f\|}\big) \quad (|f| \to \infty).$$

Take a number A arbitrarily large. If $\log\|f\| \geqslant A^2$ we have by (1)

$$\omega(f) - \omega_1(f) \ll |f|^{1/2} \leqslant \frac{1}{A}\sqrt{|f|\log\|f\|}. \tag{22}$$

If $\log\|f\| \leqslant A^2$ the number of distinct terms of $f(x)$ does not exceed $\|f\| \leqslant \exp A^2$. Let $f(x) = \sum_{i=0}^{k} a_i x^{a_i}$, $0 = a_0 < a_1 < \ldots < a_k$. Suppose that $\Phi_n(x) \mid f(x)$. Then $\sum_{i=0}^{k} a_i \zeta_n^{a_i} = 0$ and there exists a subset S of $\{1, \ldots, k\}$ such that

$$a_0 + \sum_{i\in S} a_i \zeta_n^{a_i} = 0,$$

but no subsum of the left-hand side vanishes. It follows from the result of Mann on sums of roots of unity ([6], Theorem 1, see also [7], Lemma 2) that $q = n/(n, \text{g.c.d.}_{i\in S}\, a_i)$ is square-free and is composed entirely of primes $\leqslant k+1$. Hence n is of the form qd, where $q \mid \prod_{p\leqslant k+1} p$ and $d \mid a_i$ for an $i \in S$. The number of pairs $\langle q, d\rangle$ satisfying these conditions is less than $2^{k+1} \sum_{i\in S} d(a_i)$, where $d(m)$ is the divisor function. Thus we get

$$\omega(f) - \omega_1(f) < 2^{k+1} \sum_{i=1}^{k} d(a_i).$$

However, $d(m) \ll m^{1/4}$, hence

$$\omega(f)-\omega_1(f)\leqslant 2^{k+1}\sum_{i=1}^{k} a_i^{1/4} \leqslant 2^{k+1}k|f|^{1/4} \leqslant |f|^{1/4}\exp\exp A^2.$$

If $|f|^{1/4} > 2A\exp\exp A^2$ we get

$$\omega(f)-\omega_1(f) < \frac{|f|^{1/2}}{2A} \leqslant \frac{\sqrt{|f|\log\|f\|}}{A}$$

which together with (22) proves (6).

In order to prove (7) and (8) let us observe that the multiplicity m of an irreducible non-cyclotomic factor of f does not exceed $\Omega_1(f)$. Hence by (4) with $\varepsilon = \frac{1}{2}$

$$m \leqslant \Omega_1(f) \ll \sqrt{|f|\log\|f\|}.$$

It remains to show in the notation of Lemma 1 that for every n

$$\varepsilon_n \ll \sqrt{|f|\log\|f\|} = B_1(f) \tag{23}$$

and

$$\sum \varepsilon_n \ll \sqrt{|f|\log\|f\|\log r}\log\log r = B_2(f), \qquad r = \max(3, |f|/\log\|f\|). \tag{24}$$

We set

$$l = \log r$$

and in order to prove (23) we distinguish three cases:

(i) $\varphi(n) > \sqrt{r}$,

(ii) $\varphi(n) \leqslant \sqrt{r}$ and $\varepsilon_n \leqslant c\log\|f\|$,

(iii) $\varphi(n) \leqslant \sqrt{r}$ and $\varepsilon_n > c\log\|f\|$

(c is the constant of Lemma 2). The obvious inequality

$$\sum \varepsilon_n\varphi(n) \leqslant |f| \tag{25}$$

gives in case (i) $\varepsilon_n \leqslant |f|/\sqrt{r} \leqslant B_1(f)$.

In case (ii) we get $\varepsilon_n \leqslant c\sqrt{r}\log\|f\| \leqslant B_1(f)$.

Finally, in case (iii) we have by Lemma 2 and (25)

$$\frac{\varepsilon_n^2\varphi(n)}{\omega(n)+1}\log\frac{\varepsilon_n}{\log\|f\|} \ll l|f|\log\|f\|$$

and, since $\dfrac{\varphi(n)}{\omega(n)+1} \geqslant \frac{1}{2}$,

$$\left(\frac{\varepsilon_n}{\log|f|}\right)^2\log\frac{\varepsilon_n}{\log\|f\|} \ll rl,$$

where the right-hand side is at least $c\log c > 1$, because by (iii)

$$|f| \geqslant \varepsilon_n \geqslant c\log\|f\|.$$

The function $x\log x$ is increasing for $x > 1$; hence $x^2\log x < y$ implies $x < 2\sqrt{y/\log y}$ for $y > 1$ and we get

$$\frac{\varepsilon_n}{\log\|f\|} \ll \sqrt{\frac{rl}{\log rl}} \ll \sqrt{r}, \qquad \varepsilon_n \ll \sqrt{r}\log\|f\| \leqslant B_1(f).$$

The proof of (23) and thus of (7) is complete.

In order to prove (24) we observe that if $r/l \leqslant 9$, then

$$|f| \leqslant 3\sqrt{|f|\log\|f\|l}$$

and (24) follows from (25). Thus we assume that

$$r/l > 9, \tag{26}$$

and decompose the sum $\sum\varepsilon_n$ into three sums

$$\sum\varepsilon_n = \Sigma_1\varepsilon_n + \Sigma_2\varepsilon_n + \Sigma_3\varepsilon_n, \tag{27}$$

where Σ_1 is over n such that

$$\varphi(n) > \sqrt{\frac{r}{l}};$$

Σ_2 is over n such that

$$\varphi(n) \leqslant \sqrt{\frac{r}{l}} \quad \text{and} \quad \varepsilon_n \leqslant c\frac{\log\|f\|}{\varphi(n)}\sqrt{\frac{r}{l}},$$

Σ_3 is over n such that

$$\varphi(n) \leqslant \sqrt{\frac{r}{l}} \quad \text{and} \quad \varepsilon_n > c\frac{\log\|f\|}{\varphi(n)}\sqrt{\frac{r}{l}}.$$

By (25)

$$\Sigma_1\varepsilon_n < |f|/\sqrt{rl^{-1}} = B_2(f)(\log l)^{-1} \ll B_2(f). \tag{28}$$

For the sum $\Sigma_2\varepsilon_n$ we get the estimate

$$\Sigma_2\varepsilon_n \leqslant c\log\|f\|\sqrt{\frac{r}{l}}B\left(\sqrt{\frac{r}{l}}\right);$$

hence by Lemma 3

$$\Sigma_2\varepsilon_n \leqslant c\log\|f\|\sqrt{\frac{r}{l}}\log\frac{r}{l} \ll \log\|f\|\sqrt{rl} = B_2(f)(\log l)^{-1} \ll B_2(f). \tag{29}$$

To estimate $\Sigma_3\varepsilon_n$ we use Lemma 2. We have

$$\varepsilon_n \geqslant c\frac{\log\|f\|}{\varphi(n)}\sqrt{\frac{r}{l}} \geqslant c\log\|f\|;$$

hence

$$\Sigma_3 \frac{\varepsilon_n^2 \varphi(n)}{\omega(n)+1} \log \frac{\varepsilon_n}{\log \|f\|} < l|f| \log \|f\|,$$

and

$$\Sigma_3 \frac{\varepsilon_n^2 \varphi(n)}{\omega(n)+1} \log \frac{c}{\varphi(n)} \sqrt{r/l} \leqslant l|f| \log \|f\|.$$

On the other hand, by Lemma 4 with $x = \sqrt{r/l} \geqslant 3$

$$\Sigma_3 \frac{\omega(n)+1}{\varphi(n) \log\left(\frac{c}{\varphi(n)} \sqrt{r/l}\right)} \ll (\log\log\sqrt{r/l})^2 \ll (\log l)^2.$$

By the Schwarz inequality

$$\Sigma_3 \varepsilon_n \leqslant \sqrt{l|f|\log \|f\| \log l} = B_2(f),$$

which together with (27)–(29) proves (24), and hence (8).

Proof of the Corollary. If $|f|/\log\|f\| < 3$ we have $\Omega(f) \leqslant |f| \leqslant \sqrt{3|f|\log\|f\|}$. If $|f|/\log\|f\| \geqslant 3$, we have for every ε

$$\sqrt{\log \frac{|f|}{\log \|f\|} \log\log \frac{|f|}{\log \|f\|}} = O\left(\left(\frac{|f|}{\log \|f\|}\right)^{\varepsilon}\right);$$

hence

$$\Omega(f) = \sqrt{|f|\log\|f\|}\, O\left(\left(\frac{|f|}{\log\|f\|}\right)^{\varepsilon}\right) = O\left(|f|^{1/2+\varepsilon} (\log \|f\|)^{1/2-\varepsilon}\right).$$

Note added in proof. Taking in the proof of Theorem 1 $\varepsilon = \frac{1}{2}$, $A = \log\log|f|$ and in the proof of (6) $A = \frac{1}{2}\log\log|f|$ we get a quantitative version of (6)

$$\omega(f) \ll \sqrt{\frac{|f|\log\|f\|}{\log\log|f|}}.$$

It seems likely that $\log\log|f|$ can be replaced here by $\log|f|$.

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