

ON THE NUMBER OF PAIRS OF PARTITIONS OF  $n$   
WITHOUT COMMON SUBSUMS

BY

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**1. Introduction.** To every (unrestricted or unequal) partition  $\tau$  of  $n$ :

$$n = n_1 + \dots + n_t$$

we assign the multiset  $\mathcal{N} = \{n_1, \dots, n_t\}$ .

Two partitions  $\tau_1, \tau_2$  of the same number  $n$  are said to be *additively independent* if for  $\mathcal{N}'_1 \subset \mathcal{N}_1, \mathcal{N}'_2 \subset \mathcal{N}_2$  the sum of the elements of  $\mathcal{N}'_1$  can be equal to the sum of the elements of  $\mathcal{N}'_2$  only in the two cases  $\mathcal{N}'_1 = \mathcal{N}'_2 = \emptyset$  (so that both sums are 0) and  $\mathcal{N}'_1 = \mathcal{N}_1, \mathcal{N}'_2 = \mathcal{N}_2$  (so that both sums are  $n$ ).

We denote by  $\pi(n)$  the set of unrestricted partitions of  $n$ , and by  $\pi'(n)$  the set of unequal partitions of  $n$ , and as usual we set

$$p(n) = \text{card } \pi(n), \quad q(n) = \text{card } \pi'(n).$$

Let  $G(n)$  and  $H(n)$  denote the number of pairs of additively independent unrestricted or unequal partitions of  $n$ , respectively. We shall prove

**THEOREM 1.** *For all integers  $k$  there are coefficients  $\alpha_1, \dots, \alpha_k$  such that*

$$(1.1) \quad G(n) = 2p(n) \left( 1 + \frac{\alpha_1}{\sqrt{n}} + \frac{\alpha_2}{n} + \dots + \frac{\alpha_k}{n^{k/2}} + O\left(\frac{1}{n^{(k+1)/2}}\right) \right)$$

with

$$\alpha_1 = \frac{\pi}{\sqrt{6}} = 1.28\dots, \quad \alpha_2 = \frac{17}{12}\pi^2 - 1 = 12.98\dots$$

*The coefficients  $\alpha_3, \dots, \alpha_{17}$  have been computed by the computer algebra system MAPLE:*

$$\alpha_3 = \frac{1}{\sqrt{6}} \left( \frac{337}{36}\pi^3 - \frac{1019}{48}\pi + \frac{3}{2\pi} \right) = 91.46\dots,$$

$$\alpha_4 = \frac{7889}{864}\pi^4 - \frac{12115}{288}\pi^2 + \frac{509}{24} + \frac{3}{4\pi^2} = 495.53\dots,$$

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$$\begin{aligned}
\alpha_5 &= 10450.82, & \alpha_6 &= 43427.98, & \alpha_7 &= -848498.0, \\
\alpha_8 &= 7.67 \cdot 10^7, & \alpha_9 &= -1.897 \cdot 10^9, & \alpha_{10} &= 4.42 \cdot 10^{10}, \\
\alpha_{11} &= -7.28 \cdot 10^{11}, & \alpha_{12} &= 1.23 \cdot 10^{13}, & \alpha_{13} &= -4.04 \cdot 10^{14}, \\
\alpha_{14} &= 2.53 \cdot 10^{16}, & \alpha_{15} &= -1.42 \cdot 10^{18}, & \alpha_{16} &= 6.51 \cdot 10^{19}, \\
\alpha_{17} &= -2.53 \cdot 10^{21}.
\end{aligned}$$

THEOREM 2. *There exists a real number  $c$  such that*

$$(1.2) \quad H(n) = cq(n)(1 + O(\log^2 n/\sqrt{n})).$$

*The value of  $c$  satisfies  $13.83 \leq c \leq 14.29$ .*

To any set or multiset  $\mathcal{A}$  of integers, we associate

$$S(\mathcal{A}) = \sum_{a \in \mathcal{A}} a \quad \text{and} \quad |\mathcal{A}| = \text{card } \mathcal{A} = \sum_{a \in \mathcal{A}} 1.$$

If  $\tau$  is an unrestricted or unequal partition of  $n$ , and  $\mathcal{N}$  is the set of its parts, we associate to them  $\mathcal{P}(\tau) = \mathcal{P}(\mathcal{N})$ , the set of non-zero subsums:

$$\mathcal{P}(\tau) = \mathcal{P}(\mathcal{N}) = \{S(\mathcal{N}') : \mathcal{N}' \neq \emptyset, \mathcal{N}' \subset \mathcal{N}\}.$$

We shall say that the partition  $\tau$  *represents*  $a$  if  $a \in \mathcal{P}(\tau)$ . We observe the obvious symmetry:

$$0 < x < n \text{ and } x \in \mathcal{P}(\tau) \Rightarrow n - x \in \mathcal{P}(\tau),$$

and it is convenient to introduce

$$\mathcal{P}^*(\tau) = \mathcal{P}^*(\mathcal{N}) = \{x \in \mathcal{P}(\tau) : x \leq n/2\}.$$

It is easy to see that two partitions  $\tau_1$  and  $\tau_2$  are additively independent if and only if

$$\mathcal{P}^*(\tau_1) \cap \mathcal{P}^*(\tau_2) = \emptyset.$$

Let us denote the set of positive integers by  $\mathbb{N} = \{1, 2, \dots\}$  and for  $N \in \mathbb{N}$  the set of positive integers up to  $N$  by  $\mathbb{N}_N = \{1, \dots, N\}$ .

We say that a partition  $\tau$  of  $n$  is *practical* if  $\mathcal{P}(\tau) = \mathbb{N}_n$ . It has been proved that almost all unrestricted partitions of  $n$  are practical (cf. [10], or [5]), that is, the number  $\tilde{p}(n)$  of practical unrestricted partitions of  $n$  satisfies

$$\tilde{p}(n) \sim p(n).$$

This is no longer true for unequal partitions, because  $1 \notin \mathcal{P}(\tau)$  for about half of the partitions.

Let us calculate  $G(7)$ . We have  $p(7) = 15$ ,  $\tilde{p}(7) = 8$ .

$\sigma_i$	$\mathcal{P}^*(\sigma_i)$	$\{j : \mathcal{P}^*(\sigma_i) \cap \mathcal{P}^*(\sigma_j) = \emptyset\}$	number of $j$ 's
$\sigma_1 = 7$	$\emptyset$	1, ..., 15	15
$\sigma_2 = 6 + 1$	1	1, 3, 5, 9	4
$\sigma_3 = 5 + 2$	2	1, 2, 5, 8	4
$\sigma_4 = 5 + 1 + 1$	1, 2	1, 5	2
$\sigma_5 = 4 + 3$	3	1, 2, 3, 4	4
$\sigma_6 = 4 + 2 + 1$	1, 2, 3	1	1
$\sigma_7 = 4 + 1 + 1 + 1$	1, 2, 3	1	1
$\sigma_8 = 3 + 3 + 1$	1, 3	1, 3	2
$\sigma_9 = 3 + 2 + 2$	2, 3	1, 2	2
$\sigma_{10} = 3 + 2 + 1 + 1$	1, 2, 3	1	1
$\sigma_{11} = 3 + 1 + 1 + 1 + 1$	1, 2, 3	1	1
$\sigma_{12} = 2 + 2 + 2 + 1$	1, 2, 3	1	1
$\sigma_{13} = 2 + 2 + 1 + 1 + 1$	1, 2, 3	1	1
$\sigma_{14} = 2 + 1 + 1 + 1 + 1 + 1$	1, 2, 3	1	1
$\sigma_{15} = 1 + 1 + 1 + 1 + 1 + 1 + 1$	1, 2, 3	1	1
Total:			41

We see that  $G(7) = 41$ . It is clear that  $G(n)$  is always an odd number.

From the above table it can be guessed that the main contribution to  $G(n)$  is given by the pairs  $(\tau', \tau'')$  such that either  $\tau'$  or  $\tau''$  is the partition with only one part. The number of such couples is  $2p(n) - 1$ , and this explains the first term of (1.1).

A table of  $G(n)$  for  $n \leq 70$  is given at the end of the paper. As the other tables appearing in this paper, it has been calculated by M. Deléglise, and we are pleased to thank him very much. The method used to compute  $G(n)$  is a double back-tracking. It is not clear from this table that  $G(n)/p(n)$  tends to 2, but this phenomenon can be explained by the large size of the coefficients  $\alpha_i$  in (1.1).

One of the main tools in the proof of both Theorems 1 and 2 is Lemma 3 below. It is a result in additive number theory which has already been used in [8] and [4] (cf. also [12] and [13]).

Let  $P$  be a non-empty subset of  $\mathbb{N}$ , and  $a = \max P$ . We define  $\mathcal{R}(n, P)$  as the set of unrestricted partitions  $\tau$  of  $n$  such that  $\mathcal{P}(\tau) \cap P = \emptyset$  and we set  $R(n, P) = |\mathcal{R}(n, P)|$ . This quantity has been extensively studied by J. Dixmier, and we shall use the following results:

There exist three integers  $d(P)$ ,  $\varphi(P)$  and  $u(P)$ , and a polynomial

$$(1.3) \quad f(P; X) = \sum_{i=0}^{d(P)} a_i X^i \in \mathbb{Z}[X]$$

such that ([1], 4.2)

$$(1.4) \quad R(n, P) = \sum_{i=0}^{d(P)} a_i p(n-i).$$

For  $P$  fixed and  $n \rightarrow \infty$  we have (cf. [1], 4.8)

$$(1.5) \quad R(n, P) \sim p(n)(\pi/\sqrt{6n})^{\varphi(P)} u(P),$$

$$(1.6) \quad [a/2] + 1 \leq \varphi(P) \leq a$$

where  $[x]$  denotes the integral part of  $x$ .

In [1], 4.10, an algorithm to calculate  $f(P; X)$  is given, and at the end of [1] a table of  $f(P; X)$  is given for all  $P \subset \mathbb{N}_5$ . Some more information about  $f(P; X)$  can be found in [2].

We denote  $R(n, \{a\})$  by  $R(n, a)$ . For  $a$  fixed and  $n \rightarrow \infty$  we have (cf. [1], 4.22)

$$(1.7) \quad R(n, a) \sim p(n)(\pi/\sqrt{6n})^{\varphi(a)} u(a),$$

$$(1.8) \quad \varphi(a) = [a/2] + 1.$$

Various estimates for  $u(a)$  can be found in [1] and [3], but we shall not use them here.

Clearly, as  $a = \max P \in P$ , we have

$$(1.9) \quad R(n, P) \leq R(n, a).$$

The number of unequal partitions of  $n$  which do not represent any element of  $P$  will be denoted by  $Q(n, P)$ , and we shall write  $Q(n, a)$  instead of  $Q(n, \{a\})$ . Obviously, as in (1.9) we have

$$(1.10) \quad Q(n, P) \leq Q(n, a) \quad \text{for } a = \max P.$$

We shall need the following results, valid if  $a$  goes to infinity with  $n$ : there exists  $\varepsilon_0 > 0$  such that uniformly for  $1 \leq a \leq \varepsilon_0 \sqrt{n}$  we have (cf. [7], Theorem 1)

$$(1.11) \quad \log \frac{R(n, a)}{p(n)} \leq \varphi(a) \log \frac{\pi a}{\sqrt{6n}} + O(1/\sqrt{n}),$$

$$(1.12) \quad Q(n, a) \leq q(n) \exp \left( -a \log \frac{2}{\sqrt{3}} + \pi \frac{a^2}{8\sqrt{3}n} + O(1) \right).$$

For  $a = a(n)$  such that  $\sqrt{n} \log n \leq a \leq n - \sqrt{n} \log n$  we have uniformly (this is a consequence of Theorems 1 and 2 of [8])

$$(1.13) \quad R(n, a) = p([n/2])^{1+o(1)},$$

$$(1.14) \quad Q(n, a) = q([n/2])^{1+o(1)}.$$

For all  $\varepsilon > 0$  there exists  $\delta < 1$  such that for all  $n \geq 1$  and  $a$  with  $\varepsilon\sqrt{n} \leq a \leq n - \varepsilon\sqrt{n}$  we have (cf. [4], 2.1)

$$(1.15) \quad R(n, a) \leq p(n)^\delta,$$

$$(1.16) \quad Q(n, a) \leq q(n)^\delta.$$

In fact, (1.16) is not proved in [4]; but, starting with (1.12), (1.16) can be proved in the same way as (1.15).

From the famous result of Hardy and Ramanujan (cf. [11]) we know that

$$(1.17) \quad p(n) \sim \frac{1}{4\sqrt{3}n} \exp(C\sqrt{n})$$

where  $C = \pi\sqrt{2/3} = 2.56\dots$  (throughout the paper). As explained in [5], it is also possible to deduce from the result of Hardy and Ramanujan an asymptotic expansion for  $p(n - \mu)/p(n)$ , where  $\mu$  is a fixed integer:

$$(1.18) \quad \frac{p(n - \mu)}{p(n)} = 1 + \sum_{i=1}^k \beta_i n^{-i/2} + O(n^{-(k+1)/2}).$$

The first values of  $\beta_i$  are

$$(1.19) \quad \begin{aligned} \beta_1 &= -\frac{C\mu}{2}, & \beta_2 &= \mu + \frac{C^2\mu^2}{8}, \\ \beta_3 &= -\frac{C^3\mu^3}{48} - \frac{5}{8}C\mu^2 - \frac{1}{96} \frac{(48 + C^2)\mu}{C}. \end{aligned}$$

From Hardy and Ramanujan we also know that

$$(1.20) \quad q(n) \sim \frac{1}{4(3n^3)^{1/4}} \exp(\pi\sqrt{n/3}).$$

It is easy to deduce from (1.20) that for  $h = h(n) = o(n^{3/4})$  we have (cf. [9], Lemma 3)

$$(1.21) \quad q(n + h) \sim q(n) \exp\left(\frac{\pi h}{2\sqrt{3n}}\right).$$

We finally introduce  $\rho(n, m)$ , which is the number of partitions of  $n$  into unequal parts  $\geq m$ . From ([9], Theorem 1) we know that for all  $n \geq 1$  and  $1 \leq m \leq n$ ,

$$(1.22) \quad \frac{1}{2^{m-1}} q(n) \leq \rho(n, m) \leq \frac{1}{2^{m-1}} q\left(n + \frac{m(m-1)}{2}\right).$$

Perhaps, in Theorem 2 it is possible to get an asymptotic expansion of  $H(n)$  in the powers of  $1/\sqrt{n}$ . In a letter to J.-L. Nicolas, M. Szalay explains how to obtain such an asymptotic expansion for  $\rho(n, 2)$  by an analytic method. If this result can be extended to  $\rho(n, m)$  for  $m = O(\log^2 n)$ , then our proof yields an asymptotic expansion for  $H(n)$ .

As in [8], we have not tried to choose optimally the parameters in the proofs of Theorems 1 and 2.

## 2. The lower bound

PROPOSITION 1. For  $h_0 < n/2$  we have

$$(2.1) \quad G(n) \geq 2 \sum_{h=0}^{h_0} \sum_{\sigma \in \pi(h)} R(n, \mathcal{P}(\sigma)) - E(h_0)$$

with

$$(2.2) \quad E(h_0) \leq ((1 + h_0)p(h_0))^2,$$

$$(2.3) \quad E(h_0) \leq G(2h_0).$$

Similarly, for unequal partitions we have

$$(2.4) \quad H(n) \geq 2 \sum_{h=0}^{h_0} \sum_{\sigma \in \pi'(h)} Q(n, \mathcal{P}(\sigma)) - E'(h_0)$$

with

$$(2.5) \quad E'(h_0) \leq ((1 + h_0)q(h_0))^2,$$

$$(2.6) \quad E'(h_0) \leq H(2h_0 + 1).$$

Proof. We shall prove (2.1)–(2.3). The proof of (2.4)–(2.6) is the same, just considering unequal partitions instead of unrestricted ones.

To each  $h$  with  $0 \leq h \leq h_0$  and each partition  $\sigma \in \pi(h)$  we associate a partition of  $n$ , say  $\tau \in \pi(n)$ , by adding to  $\sigma$  a large element  $n - h > n/2$ . So,  $\mathcal{P}^*(\tau) = \mathcal{P}(\sigma)$ . Clearly, there are  $R(n, \mathcal{P}(\sigma))$  partitions of  $n$  which are additively independent of  $\tau$ .

Now, we may consider all the pairs  $(\tau', \tau'')$  in  $\pi(n)$  with either  $\tau' = \tau$  and  $\tau'' \in \mathcal{R}(n, \mathcal{P}(\sigma))$ , or  $\tau'' = \tau$  and  $\tau' \in \mathcal{R}(n, \mathcal{P}(\sigma))$ . The number of such pairs is the first term on the right hand side of (2.1).

But some pairs are counted twice: for instance, the pair  $\tau' = n$ ,  $\tau'' = 1 + (n - 1)$ , if  $h_0 \geq 1$ . Denote the number of such pairs by  $E(h_0)$ . To be counted twice, the pair  $(\tau', \tau'')$  must be of the following form: there exist  $h'$  and  $h''$  with  $0 \leq h' \leq h_0$ ,  $0 \leq h'' \leq h_0$  and  $\sigma' \in \pi(h')$ ,  $\sigma'' \in \pi(h'')$  such that  $\tau' = \sigma' + (n - h')$ ,  $\tau'' = \sigma'' + (n - h'')$  and

$$(2.7) \quad \mathcal{P}(\sigma') \cap \mathcal{P}(\sigma'') = \emptyset.$$

If we neglect (2.7), the number of such exceptions is

$$\left( \sum_{h=0}^{h_0} p(h) \right)^2 \leq ((h_0 + 1)p(h_0))^2,$$

which proves (2.2).

To prove (2.3), we associate with  $\sigma'$  and  $\sigma''$  two partitions  $\xi'$  and  $\xi''$  of  $2h_0$ , by adding a large element  $2h_0 - h'$  or  $2h_0 - h''$ . We have

$$(2.8) \quad \mathcal{P}^*(\xi') = \mathcal{P}(\sigma')$$

and similarly,  $\mathcal{P}^*(\xi'') = \mathcal{P}(\sigma'')$ . (2.8) can be easily seen when  $h' < h_0$ , and it also holds when  $h' = h_0$ , because in this case  $h_0 = h' \in \mathcal{P}(\sigma')$ .

From (2.7) and (2.8) we see that  $\xi'$  and  $\xi''$  are two additively independent partitions of  $2h_0$ . Observing that to distinct  $\sigma'$  correspond distinct  $\xi'$  completes the proof of (2.3).

To prove (2.6), it suffices to observe that  $\xi'$  and  $\xi''$  belong to  $\pi'(2h_0 + 1)$ , because if  $h' = h_0$  and  $\sigma' = h_0$ , then  $\xi'$  must have unequal parts.

### 3. The upper bound

LEMMA 1. *Let  $s(m)$  denote the smallest integer which does not divide  $m$ . If  $m$  is not a divisor of 12, we have*

$$(3.1) \quad s(m) \leq \frac{7}{\log 60} \log m \leq 1.71 \log m.$$

Proof. This is an improvement of Lemma 1 of [8]. After Chebyshev, we define

$$\psi(x) = \sum_{p^m \leq x} \log p,$$

and it follows from Chebyshev's results that  $\psi(x)/x \geq (\log 60)/7$  for  $x \geq 5$ . Thus for  $s(m) \geq 6$  we have

$$\log m \geq \psi(s(m) - 1) = \psi(s(m) - c) \geq \frac{\log 60}{7} (s(m) - c)$$

for any  $c$  such that  $0 < c < 1$ . Letting  $c$  tend to 0 we obtain

$$(3.2) \quad s(m) \geq 6 \Rightarrow s(m) \leq \frac{7}{\log 60} \log m.$$

It remains to prove the lemma for  $s(m) \leq 5$ . But in this case (3.1) holds for  $\log m \geq \frac{5}{7} \log 60$ , i.e., for  $m \geq 19$ . Calculating  $s(m)$  for  $2 \leq m \leq 18$  completes the proof.

LEMMA 2. *Let  $\tau$  be a partition of  $n$ . Consider the following property:*

$$(3.3) \quad \tau \text{ has at least } \frac{\sqrt{n}}{100} \text{ distinct parts not exceeding } 100\sqrt{n}.$$

(i) *The number of unrestricted partitions of  $n$  which do not satisfy (3.3) is  $O(p(n)^{1/4})$ .*

(ii) *The number of unequal partitions of  $n$  which do not satisfy (3.3) is  $O(q(n)^{1/4})$ .*

*Proof.* We shall use Lemma 4 of [8], which claims that if  $Z(n, t, m)$  denotes the number of unrestricted partitions of  $n$  such that at most  $t$  distinct parts not exceeding  $m$  may occur, and  $t < m/2$ , then

$$Z(n, t, m) \leq 6tn^2 \binom{m}{t} p(n, t) p(n, n/m).$$

Now, if  $m = [100\sqrt{n}]$  and  $t = [\sqrt{n}/100]$ , then

$$\binom{m}{t} \leq \frac{m^t}{t!} \leq \frac{m^t}{t^t e^{-t}} = \left(\frac{me}{t}\right)^t \leq \exp\left(\frac{\sqrt{n}}{100} \log(10000e)\right) \leq \exp(0.11\sqrt{n}).$$

By Lemma 2 of [8],  $p(n, \sqrt{n}/100) \leq \exp(0.12\sqrt{n})$ , and therefore, for  $n$  large enough,

$$Z(n, \sqrt{n}/100, 100\sqrt{n}) \leq \exp(0.4\sqrt{n}),$$

which together with (1.17) proves (i).

To prove (ii), we observe that a partition of  $n$  which does not satisfy (3.3) has at most  $\sqrt{n}/100$  parts smaller than  $100\sqrt{n}$ , and also at most  $\sqrt{n}/100$  parts bigger than  $100\sqrt{n}$ . Such a partition has at most  $2\sqrt{n}/100$  parts, and their number is bounded by  $p(n, \sqrt{n}/50) \leq \exp(0.21\sqrt{n})$  by Lemma 2 of [8], which, in view of (1.20), completes the proof of (ii).

LEMMA 3. *Let  $N \in \mathbb{N}$ ,  $N \geq 2500$ , and  $m \in \mathbb{N}$  satisfy*

$$(3.4) \quad 7Ns(m) \leq m \leq 10^3 N^2 s(m)^{-2},$$

*and suppose that*

$$(3.5) \quad \mathcal{A} \subset \mathbb{N}_N,$$

$$(3.6) \quad |\mathcal{A}| \geq 10^4 N/s(m).$$

*Then  $m \in \mathcal{P}(\mathcal{A})$ .*

*Proof.* This is Lemma 10 of [8].

LEMMA 4. *Let  $m$  and  $n$  be two positive integers,  $n \geq 2$ , such that*

$$(3.7) \quad 1500\sqrt{n} \log n \leq m \leq 2 \cdot 10^6 n (\log n)^{-2},$$

$$(3.8) \quad s(m) > 10^8,$$

*and suppose that*

$$(3.9) \quad \mathcal{A} \subset \mathbb{N}_{[100\sqrt{n}]},$$

$$(3.10) \quad |\mathcal{A}| \geq \sqrt{n}/100.$$

*Then  $m \in \mathcal{P}(\mathcal{A})$ .*

*Proof.* We set  $N = [100\sqrt{n}]$ , so we have  $99\sqrt{n} \leq N \leq 100\sqrt{n}$ . We apply Lemma 3: (3.5) comes from (3.9), and (3.6) follows from (3.8) and (3.10). Moreover, from (3.2) and (3.8) we have  $\log m > \frac{1}{2} \cdot 10^8$ , and (3.7) implies  $n \geq m$ .



Further, by Lemma 1,

$$\frac{10^3 N^2}{s(m)^2} \geq \frac{10^3(9801)n}{(2 \log m)^2} \geq 2 \cdot 10^6 \frac{n}{(\log n)^2} \geq m$$

and

$$7Ns(m) \leq 700\sqrt{n}(2 \log m) < 1500\sqrt{n} \log n \leq m,$$

so that (3.4) holds, and Lemma 3 yields  $m \in \mathcal{P}(\mathcal{A})$ .

LEMMA 5. *If  $d \in \mathbb{N}$  and  $n_1, \dots, n_d$  are integers, then there is a sum of the form  $n_{i_1} + \dots + n_{i_t}$  ( $1 \leq i_1 < \dots < i_t \leq d$ ) such that  $d$  divides  $n_{i_1} + \dots + n_{i_t}$ .*

PROOF. Apply the pigeon-hole principle to the  $d+1$  sums  $\sum_{i=0}^k n_i$ , for  $0 \leq k \leq d$ , modulo  $d$ .

LEMMA 6. *Let  $D = \exp(\psi(10^8))$  denote the least common multiple of the numbers  $1, 2, \dots, 10^8$ . Assume that  $\tau_1$  and  $\tau_2$  are two additively independent partitions of  $n \geq 3$ , and  $\tau_1$  satisfies (3.3). To  $\tau_1$  and  $\tau_2$  assign the multisets  $\mathcal{N}_1$  and  $\mathcal{N}_2$  of their parts. Then  $\mathcal{N}_2$  can be written as*

$$(3.11) \quad \mathcal{N}_2 = \mathcal{N}' \cup \mathcal{N}''$$

where

$$(3.12) \quad S(\mathcal{N}') < 1500\sqrt{n} \log n$$

(or  $\mathcal{N}' = \emptyset$ ) and

$$(3.13) \quad |\mathcal{N}''| < D(\log n)^2.$$

PROOF. Let  $u$  denote the greatest integer such that

$$(3.14) \quad D|u,$$

$$(3.15) \quad u \leq 2 \cdot 10^6 n (\log n)^{-2},$$

and there is a subset  $\mathcal{N}^* \subset \mathcal{N}_2$  with

$$(3.16) \quad S(\mathcal{N}^*) = u;$$

if such an integer does not exist, then we write  $u = 0$ . We are going to show that  $\mathcal{N}' = \mathcal{N}^*$  (if  $u = 0$ , then  $\mathcal{N}' = \emptyset$ ) and  $\mathcal{N}'' = \mathcal{N}_2 \setminus \mathcal{N}'$  satisfy (3.11)–(3.13).

Here and everywhere the occurring subsets may also contain multiple elements; e.g., the multiplicity of  $m$  in  $\mathcal{N} \setminus \mathcal{N}'$  is the multiplicity in  $\mathcal{N}$  minus the multiplicity in  $\mathcal{N}'$ .

By (3.16),  $\mathcal{N}_2$  represents  $u$ . This implies that

$$(3.17) \quad S(\mathcal{N}') < 1500\sqrt{n} \log n$$

since otherwise, by Lemma 4 (with  $u$  in place of  $m$ , and  $\mathcal{A}$  being the subset of  $\mathcal{N}_1$  consisting of elements smaller than  $100\sqrt{n}$ ), it would follow from (3.14)

and (3.15) that also  $\tau_1$  represents  $u$ , and this is impossible since  $\tau_1$  and  $\tau_2$  are additively independent.

(3.11) holds trivially, while (3.12) holds by (3.17). To show that also (3.13) holds, assume indirectly that

$$(3.18) \quad |\mathcal{N}''| = |\mathcal{N}_2 \setminus \mathcal{N}'| \geq D(\log n)^2.$$

Write  $\mathcal{N}'' = \{n''_1, \dots, n''_v\}$  where  $n''_1 \leq \dots \leq n''_v$  and by (3.18),  $v = |\mathcal{N}''| \geq D(\log n)^2$ . Thus we have

$$(3.19) \quad (n''_1 + \dots + n''_D)[\log n]^2 \leq \sum_{j=0}^{[\log n]^2 - 1} \sum_{i=1}^D n''_{jD+i} \\ \leq n''_1 + \dots + n''_v = S(\mathcal{N}'') \leq S(\mathcal{N}_2) = n.$$

By Lemma 5, there is a (non-empty) subset  $\mathcal{N}_3 \subset \{n''_1, \dots, n''_D\}$  such that

$$(3.20) \quad D|S(\mathcal{N}_3).$$

Then writing  $u' = u + S(\mathcal{N}_3)$ , by (3.14) and (3.20) we have

$$(3.21) \quad D|(u + S(\mathcal{N}_3)) = u'.$$

Furthermore, it follows from (3.17) and (3.19) that for  $n \geq 3$ ,

$$(3.22) \quad u < u' = u + S(\mathcal{N}_3) = S(\mathcal{N}') + S(\mathcal{N}_3) \leq S(\mathcal{N}') + n''_1 + \dots + n''_D \\ \leq 1500\sqrt{n} \log n + n[\log n]^{-2} < 2 \cdot 10^6 n(\log n)^{-2}.$$

(3.21) and (3.22) contradict the maximum property of  $u$ , which proves (3.13) and completes the proof of the lemma.

LEMMA 7. For all  $\varepsilon > 0$  the number of pairs  $(\tau_1, \tau_2)$  of partitions of  $n$  such that  $\tau_1$  satisfies (3.3),  $\tau_1$  and  $\tau_2$  are additively independent and the greatest part of  $\tau_2$  is less than  $n - 3000\sqrt{n} \log n$  is

- (i)  $O_\varepsilon(p(n)^{1/\sqrt{2}+\varepsilon})$  for unrestricted partitions,
- (ii)  $O_\varepsilon(q(n)^{1/\sqrt{2}+\varepsilon})$  for unequal partitions.

PROOF. We shall prove (i); for unequal partitions, everything goes in the same way. To the couple  $(\tau_1, \tau_2)$  we assign the sets  $\mathcal{N}_1, \mathcal{N}_2, \mathcal{N}', \mathcal{N}''$  satisfying the conditions in Lemma 6. Write  $\mathcal{N}'' = \{n''_1, \dots, n''_v\}$  where  $n''_1 \leq \dots \leq n''_v$ , and define  $x$  by

$$(3.23) \quad S(\mathcal{N}') + (n''_1 + \dots + n''_{x-1}) < 1500\sqrt{n} \log n \\ \leq S(\mathcal{N}') + (n''_1 + \dots + n''_x)$$

(and  $x = 1$  for  $1500\sqrt{n} \log n \leq S(\mathcal{N}') + n''_1$ ). Let

$$(3.24) \quad a = S(\mathcal{N}') + n''_1 + \dots + n''_x.$$

By (3.23) and (3.24) we have

$$(3.25) \quad 1500\sqrt{n} \log n < a = S(\mathcal{N}') + (n_1'' + \dots + n_{x-1}'') + n_x'' \\ < 1500\sqrt{n} \log n + (n - 3000\sqrt{n} \log n) = n - 1500\sqrt{n} \log n.$$

By (3.24)  $\tau_2$  represents  $a$ . Since  $\tau_1$  and  $\tau_2$  are additively independent,  $\tau_1$  does not represent  $a$ , so that, for fixed  $a$ , it can be chosen in at most  $R(n, a)$  ways.

Writing  $k = S(\mathcal{N}')$ , by (3.12) we have

$$(3.26) \quad k < 1500\sqrt{n} \log n$$

and clearly  $\mathcal{N}'$  can be chosen in at most  $p(k)$  ways. Finally, all the elements of  $\mathcal{N}''$  can be chosen in at most  $n$  ways, and by (3.13) their number is at most  $D(\log n)^2$ , so that  $\mathcal{N}''$  can be chosen in at most

$$n^{D(\log n)^2} = \exp(D(\log n)^3)$$

ways.

Collecting the results above, and summing over the  $a$ 's in (3.25) and  $k$ 's in (3.26), we find that the number of pairs considered in this lemma is at most

$$\left( \sum_{1500\sqrt{n} \log n < a < n - 1500\sqrt{n} \log n} R(n, a) \right) \left( \sum_{k=1}^{1500\sqrt{n} \log n} p(k) \right) \exp(D(\log n)^3),$$

and by (1.13) and (1.17) this quantity is  $O(p(n)^{1/\sqrt{2}+\varepsilon})$  for all  $\varepsilon > 0$ .

To prove (ii) we just have to replace  $p(k)$  by  $q(k)$ ,  $R(n, a)$  by  $Q(n, a)$  and to use (1.14) and (1.20) instead of (1.13) and (1.17).

PROPOSITION 2. *For all  $\varepsilon > 0$  we have*

$$(i) \quad G(n) = 2 \sum_{h=0}^{3000\sqrt{n} \log n} \sum_{\sigma \in \pi(h)} R(n, \mathcal{P}(\sigma)) + O_\varepsilon(p(n)^{1/\sqrt{2}+\varepsilon}), \\ (ii) \quad H(n) = 2 \sum_{h=0}^{3000\sqrt{n} \log n} \sum_{\sigma \in \pi'(h)} Q(n, \mathcal{P}(\sigma)) + O_\varepsilon(q(n)^{1/\sqrt{2}+\varepsilon}).$$

Proof. First we observe that the lower bounds for  $G(n)$  and  $H(n)$  follow from Proposition 1, and more precisely from (2.1), (2.2) and (1.17), and respectively from (2.4), (2.5) and (1.20), with  $h_0 = \lceil 3000\sqrt{n} \log n \rceil$ .

For the upper bounds, we notice that if  $(\tau_1, \tau_2)$  is a pair of additively independent partitions of  $n$ , at least one of the following 5 cases must occur:

Case 1: Neither  $\tau_1$  nor  $\tau_2$  satisfies (3.3). From Lemma 2, the number of such pairs is  $O(p(n)^{1/2})$  for unrestricted partitions, and  $O(q(n)^{1/2})$  for unequal partitions.

Case 2:  $\tau_1$  satisfies (3.3), and the greatest element of  $\tau_2$  is less than  $n - 3000\sqrt{n} \log n$ . By Lemma 7, the number of such pairs is  $O(p(n)^{1/\sqrt{2}+\varepsilon})$  resp.  $O(q(n)^{1/\sqrt{2}+\varepsilon})$ .

Case 3:  $\tau_2$  satisfies (3.3) and the greatest element of  $\tau_1$  is less than  $n - 3000\sqrt{n} \log n$ . The number of such pairs is the same as for Case 2.

Case 4: The greatest element  $t$  of  $\tau_2$  is greater than or equal to  $n - 3000\sqrt{n} \log n$ , so that

$$h \stackrel{\text{def}}{=} n - t \leq 3000\sqrt{n} \log n.$$

For a fixed  $h$ ,  $\tau_2$  is the union of a large part  $t = n - h$  and of a partition  $\sigma \in \pi(h)$ . So  $\tau_1$  can be chosen in at most  $R(n, \mathcal{P}(\sigma))$  ways. The number of such pairs is at most

$$\sum_{h=0}^{3000\sqrt{n} \log n} \sum_{\sigma \in \pi(h)} R(n, \mathcal{P}(\sigma)).$$

Case 5: The greatest element of  $\tau_1$  is greater than or equal to  $n - 3000\sqrt{n} \log n$ . The number of pairs is the same as for Case 4.

For Cases 4 or 5 we have only considered unrestricted partitions. For unequal partitions, replace  $\pi(h)$  by  $\pi'(h)$  and  $R(n, \mathcal{P}(\sigma))$  by  $Q(n, \mathcal{P}(\sigma))$ .

**4. Proof of Theorem 1.** To obtain an asymptotic expansion of order  $k$ , we shall split the summation over  $h$  appearing in Proposition 2(i) into four subsums:

$S_1$  where  $h$  runs between 0 and  $h_0 \stackrel{\text{def}}{=} 2k - 1$ ,

$S_2$  where  $h$  runs between  $h_0 + 1 = 2k$  and  $h_0 + 4$ ,

$S_3$  where  $h$  runs between  $h_0 + 5$  and  $\varepsilon\sqrt{n}$  with  $\varepsilon$  smaller than  $\varepsilon_0$  considered in (1.11),

$S_4$  where  $h$  runs between  $\varepsilon\sqrt{n}$  and  $3000\sqrt{n} \log n$ .

We have to prove that  $S_2 + S_3 + S_4 = O(p(n)n^{-(k+1)/2})$ .

First, it follows from (1.9) that

$$\sum_{\sigma \in \pi(h)} R(n, \mathcal{P}(\sigma)) \leq p(h)R(n, h).$$

To estimate  $S_4$ , we use (1.15). So, for  $n$  large enough,

$$S_4 = \sum_{h=\lceil \varepsilon\sqrt{n} \rceil + 1}^{3000\sqrt{n} \log n} p(h)R(n, h) \leq n p([3000\sqrt{n} \log n]) p(n)^\delta$$

and  $S_4 = O(p(n)n^{-(k+1)/2})$  for all  $k$ , by (1.17). For  $S_3$ , we use (1.11):

$$R(n, h) = O\left(p(n) \exp\left(\frac{h}{2} \log \frac{\pi h}{\sqrt{6n}}\right)\right).$$

Now, from (1.17) we have  $p(h) = O(\exp(C\sqrt{h}))$  so that

$$p(h)R(n, h) = O(p(n) \exp g(h, n))$$

where

$$g(h, n) = C\sqrt{h} + \frac{h}{2} \log \frac{\pi h}{\sqrt{6n}}.$$

We notice that for  $\varepsilon$  small enough and  $n$  fixed,  $g(h, n)$  is decreasing in  $h$  for  $h \leq \varepsilon\sqrt{n}$ , therefore

$$S_3 \leq \sum_{h=h_0+5}^{\varepsilon\sqrt{n}} p(h)R(n, h) = O(\varepsilon\sqrt{n} p(n) \exp g(h_0 + 5, n))$$

and so  $S_3 = O(p(n)n^{-(k+1)/2})$ . The same estimate for  $S_2$  follows easily from (1.7).

It remains to obtain an asymptotic expansion of  $S_1$ , of order  $k$ . By using the polynomials introduced in (1.3) and setting

$$F(X) = \sum_{h=0}^{h_0} \sum_{\sigma \in \pi(h)} f(\mathcal{P}(\sigma); X) = \sum_{i=0}^{d'} b_i X^i,$$

we obtain by (1.4)

$$S_1 = \sum_{h=0}^{h_0} \sum_{\sigma \in \pi(h)} R(n, \mathcal{P}(\sigma)) = \sum_{i=0}^{d'} b_i p(n-i)$$

and by (1.18), each  $p(n-i)/p(n)$  can be expanded in the powers of  $1/\sqrt{n}$ .

For instance, we choose  $k = 2$ ,  $h_0 = 3$ . From Dixmier's table (cf. [1]) we have

$h$	$\sigma$	$\mathcal{P}(\sigma)$	$f(\mathcal{P}(\sigma); X)$
0		$\emptyset$	1
1	1	{1}	$1 - X$
2	2	{2}	$1 - 2X^2 + X^4$
2	1+1	{1, 2}	$1 - X - X^2 + X^3$
3	3	{3}	$1 - 3X^3 + X^5 + 2X^6 - X^8$
3	2+1	{1, 2, 3}	$1 - X - X^2 + X^4 + X^5 - X^6$
3	1+1+1	{1, 2, 3}	$1 - X - X^2 + X^4 + X^5 - X^6$

$$F(X) = 7 - 4X - 5X^2 - 2X^3 + 3X^4 + 3X^5 - X^8 = \sum_{\mu=0}^8 b_\mu X^\mu.$$

Then we have  $S_1 = \sum_{\mu=0}^8 b_\mu p(n - \mu)$ , and by (1.18) and (1.19)

$$S_1 = p(n) \left( s_0 - \frac{C}{2\sqrt{n}} s_1 + \left( s_1 + \frac{C^2}{8} s_2 \right) \frac{1}{n} + O\left(\frac{1}{n^{3/2}}\right) \right)$$

with

$$s_0 = \sum_{\mu=0}^8 b_\mu = 1, \quad s_1 = \sum_{\mu=0}^8 \mu b_\mu = -1, \quad s_2 = \sum_{\mu=0}^8 \mu^2 b_\mu = 17.$$

**5. Proof of Theorem 2.** Here we split the summation over  $h$  appearing in Proposition 2(ii) in the following way:

$S_1$  with  $0 \leq h \leq 5 \log n$ ,

$S_2$  with  $5 \log n < h \leq 10 \log n$ ,

$S_3$  with  $10 \log n < h \leq \varepsilon \sqrt{n}$  where  $\varepsilon$  is small enough, and

$S_4$  with  $\varepsilon \sqrt{n} < h \leq 3000 \sqrt{n} \log n$ .

First, by (1.10), we notice that

$$\sum_{\sigma \in \pi'(h)} Q(n, \mathcal{P}(\sigma)) \leq q(h) Q(n, h).$$

Then we obtain an upper bound for  $S_4$  and  $S_3$  by using (1.14) and (1.12) in the same way as in Section 4:

$$(5.1) \quad S_3 + S_4 = O(q(n)/\sqrt{n}).$$

Now, for  $5 \log n < h \leq 10 \log n$ , (1.12) yields

$$Q(n, h) = O(q(n) \exp(-h \log(2/\sqrt{3})))$$

and (1.20) yields

$$(5.2) \quad q(h) = O(\exp(0.03h)),$$

so that  $q(h)Q(n, h) = O(q(n) \exp(-h/10))$  and

$$(5.3) \quad S_2 = O(q(n) \log n / \sqrt{n}).$$

To deal with  $S_1$ , we have to introduce a new definition.

For  $P \subset \mathbb{N}$  and  $h \geq 1$  let  $\mathcal{W}(h, P)$  denote the set of subsets  $\mathcal{A} \subset \{1, \dots, h\}$  such that  $\mathcal{P}(\mathcal{A}) \cap P = \emptyset$ . Then

$$(5.4) \quad Q(n, P) = \sum_{\mathcal{A} \subset \mathcal{W}(h, P), S(\mathcal{A}) \leq n} \rho(n - S(\mathcal{A}), h + 1)$$

provided that  $\max P \leq h$ . To prove this, we just have to distinguish for a partition of  $n$  the parts smaller than  $h$  and those larger than  $h + 1$ .

With (5.4) we have

$$(5.5) \quad S_1 = q(n) + \sum_{h=1}^{5 \log n} \sum_{\sigma \in \pi'(h)} \sum_{\mathcal{A} \subset \mathcal{W}(h, \mathcal{P}(\sigma))} \rho(n - S(\mathcal{A}), h + 1).$$

Further, remembering that  $\rho(n, m)$  is non-decreasing in  $n$ , observing that in (5.5) we have

$$(5.6) \quad S(\mathcal{A}) \leq h(h + 1)/2 = O(\log^2 n)$$

and using (1.22), we get

$$(5.7) \quad \rho(n - S(\mathcal{A}), h + 1) \leq \rho(n, h + 1) \leq \frac{1}{2^h} q \left( n + \frac{(h + 1)(h + 2)}{2} \right)$$

$$(5.8) \quad \rho(n - S(\mathcal{A}), h + 1) \geq \rho \left( n - \frac{h(h + 1)}{2}, h + 1 \right) \\ \geq \frac{1}{2^h} q \left( n - \frac{h(h + 1)}{2} \right).$$

Now, (5.6)–(5.8) and (1.21) give

$$(5.9) \quad \rho(n - S(\mathcal{A}), h + 1) = \frac{1}{2^h} q(n) (1 + O(\log^2 n / \sqrt{n})).$$

We set  $W(h, P) = |\mathcal{W}(h, P)|$  and we observe, as in the proof of Theorem 5 of [8], that if  $h = \max P$ , then

$$(5.10) \quad W(h, P) \leq 3^{h/2}.$$

Indeed, for all  $i$  with  $1 \leq i < h/2$  there are 3 possibilities:  $i \in \mathcal{A}$ ,  $h - i \notin \mathcal{A}$ ;  $i \notin \mathcal{A}$ ,  $h - i \in \mathcal{A}$ ; and  $i \notin \mathcal{A}$ ,  $(h - i) \notin \mathcal{A}$ .

Further, we define

$$(5.11) \quad z(h) = \sum_{\sigma \in \pi'(h)} W(h, \mathcal{P}(\sigma)) \leq q(h) 3^{h/2}$$

and  $z(0) = 1$ . We denote by  $c/2$  the sum of the convergent series

$$(5.12) \quad c/2 = \sum_{h=0}^{\infty} z(h) 2^{-h}.$$

From (5.2) and (5.11) we get

$$(5.13) \quad \sum_{h > 5 \log n} z(h) 2^{-h} = O \left( \sum_{h > 5 \log n} \exp(-h/10) \right) = O(1/\sqrt{n}).$$

Finally, (5.5) and (5.9) give

$$(5.14) \quad S_1 = q(n) \left( \sum_{h=0}^{5 \log n} z(h) 2^{-h} \right) (1 + O(\log^2 n / \sqrt{n})).$$

Combining this with (5.12) and (5.13) yields

$$(5.15) \quad S_1 = \frac{c}{2} q(n) (1 + O(\log^2 n / \sqrt{n})),$$

which together with Proposition 2(ii), (5.1) and (5.3) completes the proof of (1.2).

**6. Calculation of  $c$ .** The real number  $c$  is defined by (5.12). Unfortunately,  $z(h)$  is not easy to calculate, and M. Deléglise has calculated it for  $h \leq 40$  after a long running time of the computer.

The upper bound (5.11) is rather poor: for  $h = 40$  we have  $z(h)3^{-h/2} = 0.266$  and  $q(h) = 1113$ . So, we need an improved upper bound for  $z(h)$ .

LEMMA 8. *Let  $\sigma \in \pi'(h)$  and  $j = \text{card } \mathcal{P}(\sigma)$ . We have*

$$(6.1) \quad W(h, \mathcal{P}(\sigma)) \leq \frac{2}{3} \cdot 3^{(h-j+1)/2}.$$

PROOF. First recall that if  $x \in \mathcal{P}(\sigma)$ , then  $h - x \in \mathcal{P}(\sigma)$ . We shall consider 3 cases:

CASE 1:  $h$  is odd. By the above remark,  $j$  must be odd. Write  $\mathcal{P}(\sigma) = \{x_1, \dots, x_j\}$  with  $x_1 < \dots < x_{(j-1)/2} \leq (h-1)/2$  and  $x_i = h - x_{j-i}$  for  $i > (j-1)/2$ .

How to choose  $\mathcal{A} \in \mathcal{W}(h, \mathcal{P}(\sigma))$ ? For all  $i$  with  $1 \leq i \leq j$ ,  $x_i \in \mathcal{A}$  is impossible. For the  $\frac{1}{2}(h-1) - \frac{1}{2}(j-1) = \frac{1}{2}(h-j)$   $x$ 's up to  $\frac{1}{2}(h-1)$  and  $\neq x_i$  we have at most 3 possibilities:  $x \in \mathcal{A}$  and  $h-x \notin \mathcal{A}$ ;  $x \in \mathcal{A}$  and  $h-x \in \mathcal{A}$ ;  $x \notin \mathcal{A}$  and  $h-x \notin \mathcal{A}$ ; and thus,

$$(6.2) \quad W(h, \mathcal{P}(\sigma)) \leq 3^{(h-j)/2}.$$

CASE 2:  $h$  even,  $j$  odd. Necessarily we have  $h/2 \notin \mathcal{P}(\sigma)$ . Now, we have  $\frac{1}{2}(h-2) - \frac{1}{2}(j-1) = \frac{1}{2}(h-j-1)$   $x$ 's for which there are 3 possibilities, and for  $x = h/2$  we have two possibilities:  $x \in \mathcal{A}$  or  $x \notin \mathcal{A}$ , and thus

$$(6.3) \quad W(h, \mathcal{P}(\sigma)) \leq 2 \cdot 3^{(h-j-1)/2}.$$

CASE 3:  $h$  even,  $j$  even. We have  $h/2 \in \mathcal{P}(\sigma)$  and  $h/2 \notin \mathcal{A}$ . The number of free  $x$ 's is  $\frac{1}{2}(h-2) - \frac{1}{2}(j-2) = \frac{1}{2}(h-j)$ , and thus

$$(6.4) \quad W(h, \mathcal{P}(\sigma)) \leq 3^{(h-j)/2}.$$

It remains to observe that (6.2)–(6.4) imply (6.1).

Let us introduce now for  $1 \leq j \leq h$ ,

$$(6.5) \quad \rho(j) = \text{card}\{\sigma \in \pi'(h) : \text{card } \mathcal{P}(\sigma) = j\}.$$

It is easy to see that  $\rho(1) = 1$ ,  $\rho(2) = 0$ ,  $\rho(3) = [(h-1)/2]$ , the number of



partitions with 2 parts. We define

$$(6.6) \quad \widehat{s}(h) = \sum_{\sigma \in \pi'(h)} 3^{(1 - \text{card } \mathcal{P}(\sigma))/2} = \sum_{j=1}^h \rho(j) 3^{-(j-1)/2}.$$

It follows from (5.11), (6.6), and Lemma 8 that

$$(6.7) \quad \frac{z(h)}{2^h} \leq \frac{2}{3} \left( \frac{\sqrt{3}}{2} \right)^h \widehat{s}(h).$$

LEMMA 9. *For all real positive  $x$ , we have*

$$(6.8) \quad n \leq 3^{1/4} 3^{x/2} x^2 \Rightarrow \widehat{s}(n) \leq \frac{0.4}{\sqrt{n}} 3^{x^2/4} e^{2x},$$

and

$$(6.9) \quad \widehat{s}(n) \leq \frac{0.4}{\sqrt{n}} \exp\left(\frac{\log n (\log n + 4)}{\log 3}\right).$$

Proof. Consider a partition of  $n$  into  $m$  distinct parts. We clearly have  $m \leq \sqrt{2n}$ , and for such a partition  $\sigma$ ,  $\text{card } \mathcal{P}(\sigma) \geq m(m+1)/2$ . Indeed, if the parts are  $a_1 < \dots < a_m$ , the subsums  $a_1, \dots, a_m, a_m + a_1, \dots, a_m + a_{m-1}, a_m + a_{m-1} + a_1, \dots, a_m + a_{m-1} + a_{m-2}, \dots, a_m + a_{m-1} + \dots + a_1 = n$  are all distinct.

Now, it is known that the number of partitions of  $n$  into  $m$  distinct parts is at most

$$\frac{1}{m!} \binom{n-1}{m-1} = \frac{1}{m!} \frac{m}{n} \binom{n}{m}.$$

Therefore, from (6.6) we have

$$\widehat{s}(n) \leq \sum_{m=1}^{\sqrt{2n}} \frac{1}{m!} \frac{m}{n} \binom{n}{m} 3^{1/2 - m(m+1)/4}.$$

By Stirling's formula,  $m! \geq m^m e^{-m} \sqrt{2\pi m}$ , and thus,

$$(6.10) \quad \widehat{s}(n) \leq \frac{\sqrt{3}}{2\pi n} \sum_{m=1}^{\sqrt{2n}} \left( \frac{ne^2}{m^2 3^{(m+1)/4}} \right)^m.$$

Now, we set

$$y = x \left( \log(ne^2) - \frac{x+1}{4} \log 3 - 2 \log x \right),$$

$$y' = \log n - \frac{\log 3}{4} - \frac{x}{2} \log 3 - 2 \log x.$$

The derivative  $y'$  vanishes at  $x_0 = x_0(n)$ . It is easy to see that

$$(6.11) \quad x_0 < \frac{2 \log n}{\log 3},$$

$$(6.12) \quad x_0(n) \text{ is increasing in } n,$$

$$(6.13) \quad n = 3^{1/4} 3^{x_0/2} x_0^2.$$

Further, for  $x = x_0$  we have  $y = y_0 = x_0(2 + \frac{1}{4}x_0 \log 3)$ , so that (6.10) becomes

$$(6.14) \quad \widehat{s}(n) \leq \frac{\sqrt{6}}{2\pi\sqrt{n}} 3^{x_0^2/4} e^{2x_0} \leq \frac{0.4}{\sqrt{n}} 3^{x_0^2/4} e^{2x_0}.$$

Then (6.8) follows from (6.12)–(6.14), while (6.9) follows from (6.11) and (6.14).

Now, we write

$$\begin{aligned} c/2 &= \sum_{h=0}^{\infty} z(h)2^{-h} = \sum_0^{40} + \sum_{41}^{100} + \sum_{101}^{137} + \sum_{138}^{512} + \sum_{513}^{\infty} \\ &= T_1 + T_2 + T_3 + T_4 + T_5. \end{aligned}$$

From the table at the end of the paper we have  $T_1 = 6.9168\dots$ . The upper bound  $T_2 \leq 0.212$  has been obtained by using (6.7), and by computing the exact value of  $\widehat{s}(h)$  from (6.6) (cf. table at the end of the paper). For  $T_3$ , we use (6.8) with  $x = 3.7$ , which yields

$$n \leq 137 \Rightarrow \widehat{s}(n) \leq 28106/\sqrt{n},$$

so that by (6.7)

$$T_3 \leq \frac{2}{3} \frac{28106}{10} \sum_{h \geq 101} (\sqrt{3}/2)^h \leq 0.007.$$

Using (6.8) with  $x = 5$  yields

$$n \leq 512 \Rightarrow \widehat{s}(n) \leq 8.5 \cdot 10^6/\sqrt{n}$$

and allows us to get  $T_4 \leq 0.009$ .

For  $T_5$ , we use (6.9), and we observe that for  $n \geq 513$ ,

$$\frac{\log n(\log n + 4)}{\log 3} \leq 0.114n.$$

Therefore, by (6.7), we have

$$T_5 \leq \sum_{h \geq 513} \frac{2}{3} \frac{0.4}{\sqrt{513}} \exp\left(\left(0.114 - \log \frac{2}{\sqrt{3}}\right)h\right) \leq 4 \cdot 10^{-9}.$$

In conclusion,  $6.916 \leq c/2 \leq 7.145$ , which completes the proof of Theorem 2.

TABLE OF  $G(n)$ 

$n$	$p(n)$	$G(n)$	$G(n)/p(n)$	$n$	$p(n)$	$G(n)$	$G(n)/p(n)$
1	1	1	1.00	36	17977	151371	8.42
2	2	3	1.50	37	21637	171675	7.93
3	3	5	1.67	38	26015	214973	8.26
4	5	11	2.20	39	31185	251205	8.06
5	7	15	2.14	40	37338	313449	8.39
6	11	33	3.00	41	44583	356495	8.00
7	15	41	2.73	42	53174	447651	8.42
8	22	77	3.50	43	63261	506113	8.00
9	30	105	3.50	44	75175	625341	8.32
10	42	173	4.12	45	89134	721223	8.09
11	56	215	3.84	46	105558	868565	8.23
12	77	381	4.95	47	124754	989791	7.93
13	101	449	4.45	48	147273	1222899	8.30
14	135	699	5.18	49	173525	1372535	7.91
15	176	911	5.18	50	204226	1657831	8.12
16	231	1335	5.78	51	239943	1890863	7.88
17	297	1611	5.42	52	281589	2264913	8.04
18	385	2433	6.32	53	329931	2550905	7.73
19	490	2867	5.85	54	386155	3079125	7.97
20	627	4179	6.67	55	451276	3457885	7.66
21	792	5113	6.46	56	526823	4132983	7.85
22	1002	6903	6.89	57	614154	4662771	7.59
23	1255	8251	6.57	58	715220	5488969	7.67
24	1575	11769	7.47	59	831820	6172705	7.42
25	1958	13661	6.98	60	966467	7397379	7.65
26	2436	18177	7.46	61	1121505	8200197	7.31
27	3010	22011	7.31	62	1300156	9643057	7.42
28	3718	28997	7.80	63	1505499	10894619	7.24
29	4565	33711	7.38	64	1741630	12737677	7.31
30	5604	45251	8.07	65	2012558	14233625	7.07
31	6842	51891	7.58	66	2323520	16720939	7.20
32	8349	67697	8.11	67	2679689	18567877	6.93
33	10143	79499	7.84	68	3087735	21685005	7.02
34	12310	100123	8.13	69	3554345	24264927	6.83
35	14883	117307	7.88	70	4087968	28143245	6.88

TABLE OF  $H(n)$   
 $r_n := H(n)/q(n)$

$n$	$q(n)$	$H(n)$	$r_n$	$r_{n+1} - r_{n-1}$	$n$	$q(n)$	$H(n)$	$r_n$	$r_{n+1} - r_{n-1}$
1	1	1	1.00		41	1260	54541	43.29	3.71
2	1	1	1.00	0.50	42	1426	64497	45.23	3.81
3	2	3	1.50	0.50	43	1610	75823	47.10	3.82
4	2	3	1.50	0.83	44	1816	89067	49.05	4.01
5	3	7	2.33	0.75	45	2048	104665	51.11	4.13
6	4	9	2.25	0.67	46	2304	122527	53.18	3.98
7	5	15	3.00	0.92	47	2590	142677	55.09	4.03
8	6	19	3.17	1.12	48	2910	166471	57.21	4.09
9	8	33	4.12	1.13	49	3264	193149	59.18	4.36
10	10	43	4.30	1.13	50	3658	225237	61.57	4.30
11	12	63	5.25	1.10	51	4097	260071	63.48	4.17
12	15	81	5.40	1.25	52	4582	301201	65.74	4.23
13	18	117	6.50	1.46	53	5120	346697	67.71	4.44
14	22	151	6.86	1.54	54	5718	401399	70.20	4.40
15	27	217	8.04	1.67	55	6378	459917	72.11	4.23
16	32	273	8.53	1.72	56	7108	529029	74.43	4.31
17	38	371	9.76	1.93	57	7917	604999	76.42	4.49
18	46	481	10.46	1.85	58	8808	695093	78.92	4.39
19	54	627	11.61	1.90	59	9792	791261	80.81	4.48
20	64	791	12.36	1.82	60	10880	906317	83.30	4.40
21	76	1021	13.43	2.06	61	12076	1028939	85.21	4.45
22	89	1283	14.42	2.31	62	13394	1175301	87.75	4.41
23	104	1637	15.74	2.23	63	14848	1330657	89.62	4.40
24	122	2031	16.65	2.48	64	16444	1515269	92.15	4.42
25	142	2587	18.22	2.70	65	18200	1711531	94.04	4.47
26	165	3193	19.35	2.64	66	20132	1945243	96.62	4.45
27	192	4005	20.86	2.64	67	22250	2191343	98.49	4.40
28	222	4881	21.99	2.75	68	24576	2482699	101.02	4.43
29	256	6043	23.61	3.13	69	27130	2792127	102.92	4.40
30	296	7437	25.12	2.98	70	29927	3157955	105.52	4.44
31	340	9041	26.59	2.94	71	32992	3541887	107.36	4.41
32	390	10943	28.06	3.02	72	36352	3996105	109.93	4.43
33	448	13265	29.61	3.23	73	40026	4474421	111.79	4.46
34	512	16021	31.29	3.23	74	44046	5038449	114.39	4.49
35	585	19213	32.84	3.19	75	48446	5633187	116.28	4.38
36	668	23035	34.48	3.33	76	53250	6324539	118.77	4.29
37	760	27487	36.17	3.50	77	58499	7053295	120.57	4.47
38	864	32811	37.98	3.46	78	64234	7916409	123.24	4.42
39	982	28921	39.63	3.54	79	70488	8810353	124.99	4.30
40	1113	46213	41.52	3.66	80	77312	9860123	127.54	

TABLE OF  $z(h)$ 

$h$	$q(h)$	$z(h)$	$z(h)2^{-h}$	$\sum z(h)2^{-h}$
0	1	1	1.000000	1.000000
1	1	1	0.500000	1.500000
2	1	2	0.500000	2.000000
3	2	4	0.500000	2.500000
4	2	8	0.500000	3.000000
5	3	15	0.468750	3.468750
6	4	28	0.437500	3.906250
7	5	51	0.398438	4.304688
8	6	96	0.375000	4.679688
9	8	162	0.316406	4.996094
10	10	302	0.294922	5.291016
11	12	506	0.247070	5.538086
12	15	913	0.222900	5.760986
13	18	1509	0.184204	5.945190
14	22	2722	0.166138	6.111328
15	27	4339	0.132416	6.243744
16	32	7849	0.119766	6.363510
17	38	12459	0.095055	6.458565
18	46	21887	0.083492	6.542057
19	54	34721	0.066225	6.608282
20	64	61176	0.058342	6.666624
21	76	94817	0.045212	6.711836
22	89	166763	0.039759	6.751596
23	104	258428	0.030807	6.782403
24	122	448453	0.026730	6.809133
25	142	691043	0.020595	6.829727
26	165	1199147	0.017869	6.847596
27	192	1825810	0.013603	6.861199
28	222	3175164	0.011828	6.873028
29	256	4823668	0.008985	6.882013
30	296	8245366	0.007679	6.889692
31	340	12570653	0.005854	6.895545
32	390	21611259	0.005032	6.900577
33	448	32414428	0.003774	6.904351
34	512	55627306	0.003238	6.907589
35	585	83671722	0.002435	6.910024
36	668	142505471	0.002074	6.912097
37	760	214103771	0.001558	6.913655
38	864	364227805	0.001325	6.914980
39	982	542624438	0.000987	6.915967
40	1113	926297112	0.000842	6.916809

TABLE OF  $\widehat{s}(h)$  AND  $t(h) = \frac{2}{3}(\frac{3^{h/2}}{2^h})\widehat{s}(h)$ 

$h$	$t(h)$	$\sum_{i=41}^h t(i)$	$\widehat{s}(h)$	$h$	$t(h)$	$\sum_{i=41}^h t(i)$	$\widehat{s}(h)$
41	0.022594	0.022594	12.34	71	0.000688	0.205860	28.11
42	0.020468	0.043062	12.91	72	0.000617	0.206478	29.14
43	0.018116	0.061178	13.19	73	0.000539	0.207017	29.39
44	0.016377	0.077555	13.77	74	0.000484	0.207501	30.47
45	0.014496	0.092051	14.08	75	0.000423	0.207924	30.71
46	0.013106	0.105156	14.69	76	0.000379	0.208303	31.81
47	0.011567	0.116724	14.98	77	0.000331	0.208634	32.05
48	0.010450	0.127174	15.62	78	0.000297	0.208931	33.20
49	0.009216	0.136389	15.91	79	0.000259	0.209190	33.44
50	0.008327	0.144716	16.60	80	0.000232	0.209422	34.60
51	0.007333	0.152050	16.88	81	0.000202	0.209624	34.84
52	0.006614	0.158664	17.58	82	0.000181	0.209805	36.06
53	0.005821	0.164485	17.86	83	0.000158	0.209963	36.28
54	0.005254	0.169739	18.62	84	0.000141	0.210105	37.52
55	0.004616	0.174355	18.89	85	0.000123	0.210228	37.76
56	0.004159	0.178514	19.65	86	0.000110	0.210339	39.04
57	0.003655	0.182169	19.94	87	0.000096	0.210435	39.26
58	0.003293	0.185462	20.75	88	0.000086	0.210521	40.56
59	0.002889	0.188351	21.01	89	0.000075	0.210596	40.79
60	0.002602	0.190952	21.85	90	0.000067	0.210663	42.15
61	0.002281	0.193233	22.12	91	0.000058	0.210721	42.36
62	0.002053	0.195287	23.00	92	0.000052	0.210773	43.73
63	0.001799	0.197085	23.26	93	0.000045	0.210819	43.95
64	0.001618	0.198703	24.16	94	0.000041	0.210859	45.38
65	0.001417	0.200120	24.43	95	0.000035	0.210895	45.59
66	0.001274	0.201394	25.37	96	0.000032	0.210926	47.03
67	0.001115	0.202508	25.62	97	0.000027	0.210954	47.24
68	0.001001	0.203510	26.59	98	0.000025	0.210978	48.94
69	0.000876	0.204386	26.85	99	0.000021	0.211000	48.94
70	0.000787	0.205173	27.86	100	0.000019	0.211019	50.46

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