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On the number of periodic trajectories for a Hamiltonian flow on a convex energy surface

By Ivar Ekeland and Jean-Michel Lasry

Abstract

In this paper, we look for periodic solutions, with prescribed energy $h \in \mathbf{R}$, of Hamilton's equations:

$$\dot{x} = \frac{\partial H}{\partial p}(x, p), \qquad \dot{p} = -\frac{\partial H}{\partial x}(x, p). \tag{H}$$

It is assumed that the Hamiltonian H is convex on $\mathbf{R}^n \times \mathbf{R}^n$, and that the origin (0, 0) is an isolated equilibrium. It is also assumed that some ball B around the origin can be found such that the energy surface $H^{-1}(h)$ lies outside B but inside $\sqrt{2}\,B$. Under these assumptions, we prove that there are at least n distinct periodic orbits of the Hamiltonian flow (H) with energy level h.

I. Introduction

The evolution of many conservative systems in mechanics or physics can be described by Hamilton's equations:

$$\begin{cases} \dot{x}_i = \dfrac{\partial H}{\partial p_i}(x, p), & 1 \leqq i \leqq n, \\ \dot{p}_i = -\dfrac{\partial H}{\partial x_i}(x, p), & 1 \leqq i \leqq n. \end{cases} \tag{H}$$

Here (x, p) belongs to $\mathbf{R}^n \times \mathbf{R}^n$, the so called phase space, and n is the number of degrees of freedom. The first n components $x = (x_1, \cdots, x_n)$ represent position variables, and the n last ones $p = (p_1, \cdots, p_n)$ momentum variables. The function $H: \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{R}$, the Hamiltonian, represents the energy of the system. Indeed, it is an immediate consequence of equation (H) that H is an integral of motion, i.e., $H(x(t), p(t))$ is constant along any solution of (H).

Equations (H) can be written in a more concise way. Introduce the

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phase variable $u = (x, p)$ in $\mathbf{R}^{2n}$ and the symplectic linear map σ of $\mathbf{R}^{2n}$ into itself:

$$\sigma u = \sigma(x, p) = (p, -x) . \tag{1}$$

We denote by H' the euclidian gradient of H. Equations (1) now become:

$$\dot{u} = \sigma H'(u) . \tag{H}$$

In this paper, we are interested in periodic solutions of equation (H). We will assume that the origin is an equilibrium, i.e., $H'(0) = 0$, so that equation (H) has the constant (and hence periodic) solution $u(t) = 0$ for all t. The problem then is to find non-constant periodic solutions. It has been tackled from two directions. One can either prescribe the period T or the energy level h.

For the fixed-period problem, we refer the reader to the recent papers of Rabinowitz ([19], [20]), Clarke-Ekeland ([6]) and Ekeland ([7]). We simply mention that these papers prove the existence, for each $T > 0$, of some non-constant T-periodic solution of (H), under various assumptions on the shape and growth of H.

We will be dealing with the fixed-energy problem. It goes back to Liapunov, who encountered it in his study of equilibrium stability. From now on, we scale the Hamiltonian by setting $H(0) = 0$. We shall consider trajectories of equation (H) rather than solutions: if $u(t)$ is a solution of (H), a translation in time, $v(t) = u(t + \theta)$, yields a different solution but the same trajectory.

Liapunov ([14]) and Horn ([12]) have shown that, if H is analytic, and the matrix of second derivatives $H''(0)$ is non-resonant,(*) then there is some $\varepsilon > 0$ such that, for all $h \neq 0$ within the range $(-\varepsilon, \varepsilon)$, there are at least n distinct periodic trajectories with energy level h. In a celebrated paper, Weinstein ([24]; see also Moser, [16]) has proved the same result, assuming only that H is C^2 and $H''(0)$ is positive definite.

All these results are local, holding only in an unspecified neighborhood of the origin, and so the question arises whether there is a global version, valid for any $h > 0$. Successively Seifert ([23]) for split Hamiltonians, $H(x, p) = \sum a_{ij}(x)p_i p_j + V(x)$ with $[a_{ij}(x)]$ positive definite, Weinstein ([25]) for C^2 strictly convex Hamiltonians, Clarke ([5]) for general convex Hamiltonians, have proved the existence of at least *one* periodic trajectory for each energy level $h > 0$. Rabinowitz, however, has a result which goes beyond convex Hamiltonians ([19], [20]). He has also shown how to deduce the

(*) the eigenvalues $\pm i w_1, \cdots, \pm i w_n$ of the matrix $\sigma H''(0)$ are purely imaginary, and no quotient w_i/w_j, $i \neq j$, is an integer.

fixed-energy case from the fixed-period case.

The basic result of this paper (Theorem IV. 1) is that, if H is convex and satisfies the uniform estimate

$$au^2 < H(u) < 2au^2 \qquad \text{for all} \quad u \neq 0$$

for some $a > 0$, then there are at least n distinct periodic orbits on each energy level $h > 0$. This theorem, and its corollaries, constitute a first step towards a global version of Weinstein's theorem.

The proof uses the Legendre transform G of H with respect to the variable $u = (x, p)$. Performing such a transformation with respect to the variable p alone is classical: this is how one reverts to the Lagrangian from the Hamiltonian formulation. However, performing this transformation with respect to both variables together is a new idea. It was introduced by Aubin-Ekeland ([1]) in another context, and by Clarke ([5]) who used it to find periodic solutions. His method was developed in the papers [6] and [7] where it is shown that the classical principle of least action:

$$\text{extremize} \int_0^T [p\dot{x} - H(x, p)]dt \tag{8}$$

has a dual formulation:

$$\text{extremize} \int_0^T [G(-\dot{q}, \dot{y}) - \dot{y}q]dt\ . \tag{9}$$

Note at this point that the Legendre transformation $H \mapsto G$ loses a derivative, e.g., if H is C^1, then G is merely continuous. This creates technical difficulties which are dealt with by standard methods and tools of convex analysis.

Prescribing suitable boundary conditions, one gets two problems in the calculus of variations which are equivalent: there are explicit formulas which transform an extremal of one problem into an extremal of the other. For instance, in the case when one looks for T-periodic solutions of equation (H), the appropriate boundary conditions are $\int_0^T \dot{u}(t)dt = 0$ for the first problem, and $\int_0^T \dot{v}(t)dt = 0$ for the second, with $v = (y, q)$.

From a technical point of view, it is better to change the time scale, so as to deal always with the interval (0, 1). We will work in the Hilbert space E of all functions $v = (y, q)$ such that:

$$\dot{v} \in L^2(0, 1; \mathbf{R}^{2n})\ , \qquad \int_0^1 \dot{v}(t)dt = 0 = \int_0^1 v(t)dt\ , \tag{10}$$

on which we define the functionals J and K by:

$$J(v) = \int_0^1 -\dot{y}(t)q(t)dt \,, \tag{11}$$

$$K(v) = \int_0^1 G(-\dot{q}(t),\, \dot{y}(t))dt \,. \tag{12}$$

The functional I_T used in the papers [6] and [7] to find T-periodic solutions is $K + TJ$. Critical points of I_T yield, after rescaling time, T-periodic solutions of equations (H). In this paper, where the period T is no longer prescribed but the energy level h is, we begin by studying the particular case where the Hamiltonian H, and therefore the function G, is positively homogeneous of degree two. This analysis is carried out in Section II. The peculiarity of the homogeneous case is that, if u is a solution of (H), then so is λu for any $\lambda > 0$, so that any particular energy level can be reached simply by rescaling. We therefore use a functional where neither the prescribed energy level nor the period appear explicitly, namely:

$$I = J + (K + a)K \,. \tag{13}$$

Here $a > 0$ is an ad hoc constant. The choice of this functional is partly arbitrary, and partly supported by heuristic reasons which we now develop.

Let us figure out the behavior of the functional I. Note first that it is the sum of the quadratic form J, which is neither positive nor negative, and the convex functional $(a + K)K$, which may well be non-differentiable. So I itself is neither convex nor differentiable in general.

The behavior near zero or infinity is easily investigated. At infinity, the fourth-order term K^2 takes precedence over the second-order terms J and aK, so that $I(v)/\|v\| \to +\infty$ when $\|v\| \to \infty$. Near the origin, on the other hand, the second-order terms take precedence, so that the functional I behaves like $J + aK$: this is a linear combination of J and K, similar to the one we used in the fixed-period problem. Finally, the critical points of I (assuming for the moment that K is differentiable) are given by:

$$I' = J' + (a + 2K)K' = 0 \,. \tag{14}$$

Setting $(a + 2K) = T^{-1}$, we rewrite this equation as $K' + TJ' = 0$. This is just $I'_T = 0$, which has been seen to give T-periodic solutions of equation (H). Summing up, the critical points of I, i.e., equation (14), give rise to periodic solutions of the Hamiltonian system, and what we will show is that this procedure does indeed yield at least n distinct periodic trajectories with a given energy level.

This is where we want to stress the difference between periodic trajectories, which are closed curves in $\mathbf{R}^{2n}$, and periodic solutions, which are mappings from $S^1 = \mathbf{R}/\mathbf{Z}$ into $\mathbf{R}^{2n}$. What a critical point of I gives us is some number $T > 0$ and some 1-periodic function $u(t)$ such that:

$$\frac{1}{T}\dot{u} = \sigma H'(u) , \tag{15}$$

so that $t \to u(t/T)$ is a T-periodic solution of equation (H). But two different solutions (T_1, u_1) and (T_2, u_2) can in fact describe the same periodic trajectory:

— either because

$$T_1 = k_1 T , \qquad T_2 = k_2 T , \qquad u_1(t) = u(k_1 t) , \qquad u_2(t) = u(k_2 t) ,$$

with k_1 and k_2 integers, not both equal to 1. In this case, the points $u_1(t)$, $u_2(t)$ and $u_2(t)$ run along the same closed curve in $\mathbf{R}^{2n}$, with $u_i(t)$ doing k_i revolutions ($i = 1, 2$) while $u(t)$ does only one.

— or because $T_1 = T$ and $u_1(t) = u_2(t + \theta)$, with $\theta \neq 0$ modulo 1. In this case, the points $u_1(t)$ and $u_2(t)$ run after each other on the same curve with a fixed time lag.

— or a combination of the two preceding cases.

In other words, we first have to eliminate from consideration the critical points of I which will lead to pairs (T, u) where T is not the true minimal period of u. This is done in Section II by restricting ourselves to the subset Ω of E where $I < 0$. This is where the particular choice of the constant $a > 0$ in formula (13) comes in. If it is small enough, the subset Ω will not be empty. If it is large enough, every $u \in \Omega$ will have 1 as its minimal period. That a can be chosen just right will follow from the estimate (2).

We next have to take into account the action of the group S^1; this means we have to identify functions $u \in E$ which differ only by a translation in time. This is done in Section III, where we show how this group action can actually be turned to our advantage. Indeed, the very existence of this group action, and the fact that it leaves the functional I invariant, put strong constraints on the topological situation, which enables us to show that I has at least n distinct families of critical points, defining distinct trajectories of (H) on the prescribed energy level.

The result is stated in Section III as an abstract theorem on critical points of functionals on Hilbert space which are invariant by an S^1-action. The result is first proved in the case where the functional is C^1, along classical lines: it is essentially the proof Clark ([4]) gave for even functions ($\mathbf{Z}_2$-action), where we replace the notion of genus by the index theory of Fadell-Rabinowitz ([11]). The case, where the functional splits into the sum of two terms, a smooth one and a convex one, follows by a regularization procedure. We construct a new functional, which has the same critical points as the original one, but which is C^1 and falls within the range of the earlier result. This abstract theorem will apply to the functional I defined by (13), without any smoothness assumption on G.

The proof then is complete for the case when the Hamiltonian H, and its Legendre transform G, are positively homogeneous of degree two. In the last section, IV, we show how to infer the general case from this particular one. This is done by using a trick, which we learned from Rabinowitz, although it seems to be well-known among specialists. Finally, we compare our result to Weinstein's theorem, to grasp the meaning of condition (2) and give food for future thought.

The reader who wants *to avoid convex analysis* may read this paper under the added assumption that the Hamiltonian H is C^2, with $H''(u)$ positive definite, uniformly for $u \in \mathbf{R}^{2n}$:

$$\exists c > 0 \colon \forall (u, v) \quad \big(H''(u)v, v\big) \geqq c\|v\|^2 . \tag{16}$$

This smooths everything down: G becomes C^1, and I is C^1. One can replace subgradients by ordinary gradients everywhere, and just use Theorem III. 1 instead of Theorem III. 2. The details are given in an earlier version of this paper ([8]).

II. The case of a homogeneous Hamiltonian

A. *Main statement*

In this section, we shall deal with a particular case, which already contains the whole difficulty. It is the case when the Hamilton H, still convex, is also positively homogeneous of degree two. If certain estimates on H are satisfied, the corresponding Hamiltonian equations will be shown to have at least m $(1 \leqq m \leqq n)$ distinct periodic trajectories on each energy surface. In Section IV, the same conclusion will be seen to hold, with the same estimates, but without the homogeneity assumption.

THEOREM 1. *Let $H\colon \mathbf{R}^{2n} \to \mathbf{R}_+$ be a C^1 convex function, positively homogeneous of degree two, with an isolated zero at the origin:*

$$\forall \lambda > 0 , \qquad \forall u \neq 0 , \qquad H(\lambda u) = \lambda^2 H(u) > 0 . \tag{1}$$

Assume there exist constants β, γ and an integer m with:

$$0 < \beta < 2\gamma \qquad \text{and} \qquad 1 \leqq m \leqq n , \tag{2}$$

$$\forall u \in \mathbf{R}^{2n} , \qquad \gamma \|\Pi_m u\|^2 \leqq H(u) \leqq \beta \|u\|^2 \tag{3}$$

where Π_m is the projection defined by:

$$\begin{aligned} &\Pi_m(x_1, \cdots, x_n, p_1, \cdots, p_n) \\ &\qquad = (x_1, \cdots, x_m, 0, \cdots, 0, p_1, \cdots, p_m, 0, \cdots, 0) . \end{aligned} \tag{4}$$

Then there are at least m distinct periodic trajectories $u^1, \cdots, u^m$ of

Hamilton's equations (H) *on the energy level* 1:

$$\dot{u}^i = \sigma H'(u^i) , \tag{5}$$

$$H(u^i(t)) = 1 , \quad \textit{for all} \quad t \in \mathbf{R} , \tag{6}$$

$$u^i(t + T_i) = u^i(t) , \quad \textit{for all} \quad t \in \mathbf{R} . \tag{7}$$

Moreover, T_i is the minimal period of u^i, and satisfies the following estimate:

$$\pi\beta^{-1} \leqq T_i \leqq 2\pi\beta^{-1} . \tag{8}$$

■

Let us first note that the assumption on H will imply that there is some constant $\alpha > 0$, namely the minimum value of H on the unit sphere, such that:

$$\forall u \in \mathbf{R}^{2n},\ H(u) \geqq \alpha \| u \|^2 . \tag{9}$$

We now explain what we mean by trajectories. We say that two maps u_1 and u_2: $\mathbf{R} \to \mathbf{R}^{2n}$ are equivalent if there exists some C^1 diffeomorphism ψ: $\mathbf{R} \to \mathbf{R}$ such that $u_1 = u_2 \circ \psi$. If u_1 and u_2 are equivalent solutions of the same autonomous differential equation with a right-hand side continuous and nonvanishing along the solutions (for instance equation (5), (6)—recall that $u \neq 0$ implies $H'(u) \neq 0$) then $\psi(t) = t + t_0$ for some constant t_0 and all t; if the right-hand side is locally Lipschitzian, then two solutions having one common point are equivalent.

A periodic trajectory of system (H) is the equivalence class of a periodic solution of (H).

B. *The function G*

The proof uses the Legendre transform G of the function H with respect to $u = (x, p)$. Since H is convex, G is defined by Fenchel's formula:

$$G(v) = \mathrm{Sup}_u \{(u, v) - H(u)\} \tag{10}$$

where from now Sup will stand briefly for $\mathrm{Sup}_{u \in \mathbf{R}^{2n}}$, and $(\cdot, \cdot)$ for the scalar product in $\mathbf{R}^{2n}$.

Let us recall a few standard facts of convex analysis (see [15], [21], [10]). The function G: $\mathbf{R}^{2n} \to \mathbf{R} \cup \{+\infty\}$ is convex and lower semi-continuous. It need not be differentiable. Its subdifferential at a point $v \in \mathbf{R}^{2n}$ where $G(v) < +\infty$ is the set $\partial G(v)$ defined by:

$$\partial G(v) = \{u \in \mathbf{R}^{2n} \mid G(w) - G(v) \geqq (w - v, u) \quad \forall w\} . \tag{11}$$

It is a (possibly empty) closed convex set. It is compact and non-empty whenever G is continuous at v. The multi-valued map $v \to \partial G(v)$ has closed graph and is maximal monotone. Finally, we have Fenchel's reciprocity formulas:

$$H(u) = \mathrm{Sup}_v \{(u, v) - G(v)\} , \tag{12}$$

$$v = H'(u) \Longleftrightarrow H(u) + G(v) - (v, u) = 0 \Longleftrightarrow u \in \partial G(v) . \tag{13}$$

In our particular case, G inherits from H a few more properties:

LEMMA 1. *The function G is positively homogeneous of degree two and satisfies the estimates*:

$$\forall v \in \mathbf{R}^{2n},\ G(\Pi_m v) \leqq (4\gamma)^{-1} \| \Pi_m v \|^2 , \tag{14}$$

$$\forall v \in \mathbf{R}^{2n},\ (4\beta)^{-1} \| v \|^2 \leqq G(v) , \tag{15}$$

$$\forall v \in \mathbf{R}^{2n},\ G(v) \leqq (4\alpha)^{-1} \| v \|^2 . \tag{16}$$

Moreover, G is continuous and subdifferentiable everywhere. The equations

$$u \in \partial G(v) \qquad \text{and} \qquad v = H'(u) \tag{17}$$

are equivalent, and they imply that, for some uniform constants δ and δ',

$$\| u \| \leqq \delta \| v \| , \tag{18}$$

$$\| v \| \leqq \delta' \| u \| . \tag{19}$$

Proof. The first properties are obtained in a straightforward manner from Fenchel's formula (10). First we have homogeneity:

$$\begin{aligned} G(\lambda v) &= \mathrm{Sup}_u \{(u, \lambda v) - H(u)\} \\ &= \mathrm{Sup}_w \{(\lambda w, \lambda v) - H(\lambda w)\} \\ &= \lambda^2 \mathrm{Sup}_w \{(w, v) - H(w)\} \\ &= \lambda^2 G(v) . \end{aligned}$$

Then there is condition (14):

$$\begin{aligned} G(\Pi_m v) &= \mathrm{Sup}_u \{(u, \Pi_m v) - H(u)\} \\ &\leqq \mathrm{Sup}_u \{(\Pi_m u, v) - \gamma \| \Pi_m u \|^2\} \\ &= (4\gamma)^{-1} \| \Pi_m v \|^2 . \end{aligned}$$

Next we show condition (15):

$$\begin{aligned} G(v) &= \mathrm{Sup}_u \{(u, v) - H(u)\} \\ &\geqq \mathrm{Sup}_u \{(u, v) - \beta \| u \|^2\} \\ &= (4\beta)^{-1} \| v \|^2 . \end{aligned}$$

Finally condition (16) is obtained from condition (9). It implies immediately that $G(v) < +\infty$ for every $v \in E$. Since G is convex and finite-valued on $\mathbf{R}^{2n}$, it has to be continuous, and hence subdifferentiable everywhere. The equivalence of equations (17) is just Fenchel's reciprocity formula (13).

Since H is positively homogeneous of degree two, its derivative is positively homogeneous of degree one: $H'(\lambda v) = \lambda H'(v)$ for $\lambda \geqq 0$. Equation (19) follows, with:

$$\delta' = \operatorname{Max}\{\|H'(u)\| \mid \|u\| = 1\} .$$

One gets inequality (18) in the same way, from the equation $\partial G(\lambda v) = \lambda \partial G(v)$. It yields

$$\delta = \operatorname{Max}\{\|w\| \mid w \in \partial G(u),\ \|u\| = 1\} ,$$

a finite number, since the compact-valued mapping ∂G is upper semi-continuous. ■

C. *The functionals J and K*

We now consider in the Sobolev space $H^1_{\mathrm{loc}}(\mathbf{R}, \mathbf{R}^{2n})$, the subspace E of all functions u with period one and mean zero:

$$E = \left\{v \mid \dot{v} \in L^2(0, 1; \mathbf{R}^{2n}),\ \int_0^1 v(t)dt = 0 = \int_0^1 \dot{v}(t)dt\right\} . \tag{20}$$

It will be endowed with the inner product:

$$(u, v)_E = \int_0^1 \dot{u}(t)\dot{v}(t)dt \tag{21}$$

and the corresponding Hilbertian structure. We shall identify the functions $u \in E$ with their restrictions to $[0, 1]$. Note that the injection $E \to L^2(0, 1; \mathbf{R}^{2n})$ is compact.

Every function $v \in E$ has a Fourier expansion which converges in E:

$$\begin{cases} v(t) = \sum_{k \in \mathbf{Z}} v_k e^{2i\pi kt} \\ v_0 = 0 , \qquad v_{-k} = \bar{v}_k \in \mathbf{C} . \end{cases} \tag{22}$$

Similarly, the Fourier expansion for $\dot{v}$ in $L^2(0, 1; \mathbf{R}^{2n})$ is given by:

$$\dot{v}(t) = \sum_{k \in \mathbf{Z}} (2i\pi k)v_k e^{2i\pi kt} . \tag{23}$$

Using Plancherel's theorem, and denoting by $\|\cdot\|_2$ the L^2-norm, we get:

$$\|v\|_2^2 = \sum_{k \neq 0} |v_k|^2 , \tag{24}$$

$$\|\dot{v}\|_2^2 = \sum_{k \neq 0} 4\pi^2 k^2 |v_k|^2 . \tag{25}$$

Hence, using the fact that $v_0 = 0$, we get the estimate:

$$\forall v \in E,\ \|v\|_2 \leqq (2\pi)^{-1}\|\dot{v}\|_2 . \tag{26}$$

This estimate can be improved if there is no first-order term in the Fourier expansion. If $|v_1| = 0$, we have

$$\|v\|_2^2 = \sum_{k \geqq 2} |v_k|^2 , \tag{27}$$

$$\|v\|_2^2 = \sum_{k \geqq 2} 4\pi^2 k^2 |v_k|^2 , \tag{28}$$

and hence:

$$\forall v \in E\colon |v_1| = 0 \Longrightarrow \|v\|_2 \leqq (4\pi)^{-1}\|\dot{v}\|_2 . \tag{29}$$

Of course, the estimates (26) and (29) will also hold individually for each component of $v = (y, q)$. Note that the second estimate is sharper than the first by a factor 2. It is this fact which will enable us to distinguish from the point of view of functional analysis between the functions which have non-vanishing first Fourier coefficient (and hence have 1 as their minimal period), and the other ones. Loosely speaking, we will separate the normal modes from their harmonics.

We now define two functionals J and K on E:

$$J(v) = \int_0^1 -(\dot{y}(t), q(t))dt = \int_0^1 (y(t), \dot{q}(t))dt \tag{30}$$
$$= \frac{1}{2}\int_0^1 (v(t), \sigma\dot{v}(t))dt = \frac{1}{2}\int_0^1 (-\sigma v(t), \dot{v}(t))dt ;$$

$$K(v) = \int_0^1 G(-\dot{q}(t), \dot{y}(t))dt \tag{31}$$
$$= \int_0^1 G(-\sigma\dot{v}(t))dt .$$

The functional J is clearly a continuous quadratic form on E, and hence is C^∞. To compute its gradient in E, we write it as a scalar product:

$$J(v) = \frac{1}{2}(v, Av)_E , \tag{32}$$

where $Av = u$ is the unique solution in E of the equation $\dot{u} = -\sigma v$. The operator A: $E \to E$ is easily seen to be self-adjoint, so that:

$$J'(v) = Av . \tag{33}$$

Note that J takes positive values as well as negative ones. However, using inequality (26) we readily get an estimate for J:

$$|J(v)| \leqq \frac{1}{4\pi}\|\dot{v}\|_2^2 = \frac{1}{4\pi}\|v\|_E^2 . \tag{34}$$

The functional K: $E \to \mathbf{R} \cup \{+\infty\}$ inherits various properties from the function G. It is convex, non-negative, lower semi-continuous, positively homogeneous of degree two. It satisfies the following estimates (compare with (14) and (15)):

$$\forall v \in E_m,\ K(v) \leqq (4\gamma)^{-1}\|v\|_E^2 , \tag{35}$$

$$\forall v \in E,\ (4\beta)^{-1}\|v\|_E^2 \leqq K(v) \tag{36}$$

where $v = (y, q)$ belongs to E_m if and only if:

$$y_{m+1}(t) = \cdots = y_n(t) = 0 = g_{m+1}(t) = \cdots = g_n(t) = 0 \qquad \text{for all} \quad t .$$

Using condition (16), we get another estimate for G:

$$\forall v \in E,\ K(v) \leqq (4\alpha)^{-1}\|v\|_E^2 < +\infty \ . \tag{38}$$

Since K is lower semi-continuous, convex, and finite everywhere, it must be continuous. It follows that $\partial K(v) \neq \varnothing$ at every point v; this subdifferential is easily computed:

LEMMA 2. *The subdifferential $\partial K(v)$ of K at v is the set of all $w \in E$ such that there exists $u^0 \in \mathbf{R}^{2n}$ with:*

$$-\sigma\dot{w}(t) + u^0 \in \partial G(-\sigma\dot{v}(t)) \qquad \text{a.e.} \quad t \ . \tag{39}$$

Proof. We define a closed subspace E_1 of $L^2(0, 1; \mathbf{R}^{2n})$ as follows:

$$u \in E_1 \Longleftrightarrow \int_0^1 u(t)dt = 0 \ . \tag{40}$$

Let F be the $2n$-dimensional subspace of L^2 spanned by constant vectors. Clearly E_1 and F are orthogonal subspaces.

The Hilbert spaces E and E_1 are isometric, through $v \to \dot{v}$ (restricted to $[0, 1]$). Through this identification, the functional K gives rise to a functional K_1 on E_1 defined by restriction of K_2 to E_1:

$$K_2(u) = \int_0^1 G(-\sigma u(t))dt \qquad \forall u \in L^2(0, 1; \mathbf{R}^n) \ . \tag{42}$$

The subdifferential of this functional in L^2 is known to be (see [10])

$$\partial K_2(u) = \{w_2 \in L^2 \mid w_2(t) \in \sigma\partial G(-\sigma u(t)) \text{ a.e.}\} \ . \tag{43}$$

Let K_1 be the restriction of K_2 to E_1. Its subdifferential must be the projection on E_1 of the subdifferential of K_1 in L^2:

$$\begin{aligned}\partial K_1(u) &= \{w_1 \in E \mid \exists u_1 \in F\colon w_1 + u^1 \in \partial K''(u)\} \\ &= \{w_1 \in E \mid \exists u_1 \in F\colon -\sigma w_1 - \sigma u^1 \in \partial G(-\sigma u)\} \ .\end{aligned}$$

Going back to E, setting $w_1 = w$ and $-\sigma u^1 = u^0$, we get the desired result. ∎

D. *The functional I*

We now use J, K, and a constant $a > 0$ to define a new functional I on E:

$$I(v) = J(v) + (K(v) + a)K(v) \ . \tag{45}$$

The whole proof hinges on the particular choice of the constant a. It has to satisfy the condition:

$$\beta \leqq 2\pi a < 2\gamma \ . \tag{46}$$

That such a constant exists is exactly assumption (2). The functional $(K + a)K$ is convex, non-negative and continuous, because K is. It follows that it is subdifferentiable everywhere, although it need not be differentiable. We want to compute the subdifferential $\partial[(K + a)K]$ in terms of ∂K.

LEMMA 3. *Let φ: $[0, +\infty) \to \mathbf{R}$ be an increasing convex function, with $\varphi(0) = 0$. Assume φ is C^1 on $(0, \infty)$, with $\varphi'(s) > 0$ for $s > 0$. Then $\varphi \circ K$ is continuous and convex, with:*

$$K(v) \neq 0 \Longrightarrow \partial(\varphi \circ K)(v) = (\varphi' \circ K(v))\partial K(v) . \tag{47}$$

Proof. Take any $u \in \partial K(v)$. We have, for all $w \in E$:

$$\begin{aligned}\varphi \circ K(w) - \varphi \circ K(v) &\geqq \varphi' \circ K(v)[K(w) - K(v)] \\ &\geqq (\varphi' \circ K(v)u, w - v) .\end{aligned}$$

Hence $\varphi' \circ K(v)u \in \partial(\varphi \circ K)$. We have proved that the right-hand side of equation (47) is contained in the left-hand side. Now for the converse.

Take any v where $K(v) > 0$, and any $u \in \partial(\varphi \circ K)(v)$. It will suffice to prove that, for any direction $w \in F$, we have, for $s > 0$:

$$K(v + sw) \geqq K(v) + \frac{1}{\varphi' \circ K(v)}(sw, u) + o(s) . \tag{49}$$

The function $s \to K(v + sw)$ is continuous on the interval $[0, s_0]$. Choose s_0 so small that $K(v + sw) > 0$ on this interval, and apply the mean value theorem to φ:

$$\varphi \circ K(v + sw) - \varphi \circ K(v) = \varphi'(K_s)[K(v + sw) - K(v)] , \tag{50}$$

where K_t is some number between $K(v)$ and $K(v + sw)$. Note that $K_s \to K(v)$ when $s \to 0$. Using this, and the definition of u, we get the desired result:

$$K(v + sw) - K(v) \geqq \frac{1}{\varphi' \circ K(v)}(sw, u) + \varepsilon(s)$$

with

$$\frac{\varepsilon(s)}{s} = \left|\frac{1}{\varphi' \circ K(v)} - \frac{1}{\varphi'(K_s)}\right| (w, u) \to 0 \qquad \text{when} \quad s \to 0 . \qquad \blacksquare$$

Setting $\varphi(s) = s^2 + as$, we get, for any $v \in E$:

$$\partial[(K + a)K](v) = (2K(v) + a)\partial K(v) . \tag{52}$$

Now for the functional I itself. It is neither convex nor differentiable in general. It satisfies the following estimate:

LEMMA 4. *There are constants c_1, $c_2 > 0$ and $k > 0$ such that*

$$\forall v \in E,\ I(v) \geqq c_1 + c_2\|v\|_E^2 , \tag{53}$$

$$\textit{the function} \quad v \to I(v) + k\|v\|_E^2 \quad \textit{is convex continuous.} \tag{54}$$

Proof. Using the definition (45) of I, and plugging in the estimates (34) and (36), we get, for all $v \in E$:

$$I(v) \geqq \frac{-1}{4\pi} \|\dot{v}\|_2^2 + \left(a + (4\beta)^{-1}\|\dot{v}\|_2^2\right)(4\beta)^{-1}\|\dot{v}\|_2^2 \ . \tag{55}$$

Formula (53) follows by choosing c_1 and $c_2 > 0$ such that

$$\left(\frac{a}{4\beta} - \frac{1}{4\pi}\right) s^2 + (4\beta)^{-2}s^4 \geqq c_1 + c_2 s^2 \qquad \text{for all} \quad s \in \mathbf{R} \ .$$

To get relation (54) we take any $k > (4\pi)^{-1}$. Then $J(v) + k\|v\|_E^2$ is a quadratic form, and it is positive definite by relation (34), so that it is a convex functional. It follows that:

$$I(v) + k\|v\|_E^2 = \left[J(v) + k\|v\|_E^2\right] + G(v) \tag{57}$$

is a convex functional. It is obviously continuous. ∎

The *subdifferential* of I at $v \in E$ is defined to be the set:

$$\begin{aligned} \partial I(v) &= J'(v) + \partial[(K + a)K](v) \\ &= J'(v) + \left(2K(v) + a\right)\partial K(v) \ . \end{aligned} \tag{56}$$

The *critical points of* I are defined to be those points $v \in E$ where $0 \in \partial I(v)$. Now J' is given by formula (33) and ∂K by Lemma 2. Putting the pieces together, we see that $v \in E$ is a critical point of I if and only if there exists $u^0 \in \mathbf{R}^{2n}$ such that

$$[2K(v) + a]^{-1}v(t) + u^0 \in \partial G\left(-\sigma\dot{v}(t)\right) \quad \text{a.e.} \tag{57}$$

We now proceed to find critical points of I and later on, we will relate them to periodic trajectories of system (H).

E. *Critical points of I*

This step of the argument uses a general result on functionals invariant by an S^1-action. For the sake of clarity, we have postponed the proof of this result until Section III, where it appears as Theorem 2.

Recall what is known about the functional I. It is everywhere finite and continuous. It is bounded from below and $I(v) + k\|v\|_E^2$ is convex for some $k > 0$. It also satisfies the following condition of Palais-Smale type:

LEMMA 5. *If (v_k, u_k) are sequences in $E \times E$, with $u_k \in \partial I(v_k)$, such that:*

$$\operatorname{Sup}_k I(v_k) < +\infty \ , \tag{58}$$

$$u_k \to 0 \qquad in \quad E \quad (\text{i.e.: } \|u_k\| \to 0) \ , \tag{59}$$

then there is a subsequence v_{k_i} which converges in E.

Proof. Comparing condition (58) with estimate (53), we see that the sequence v_k is bounded in E. It follows from condition (38) that the sequence $K(v_k)$ is bounded in $\mathbf{R}$.

The condition $u_k \in \partial I(v_k)$ means that (see (56)):

$$(2K(v_k) + a)^{-1}(u_k - J'(v_k)) \in \partial K(v_k) \, . \tag{60}$$

By Lemma 2, this means that there exists some constant $u_k^0 \in \mathbf{R}^{2n}$ such that

$$(2K(v_k) + a)^{-1}(v_k(t) - \sigma \dot{u}_k(t)) + u_k^0 \in \partial G(-\sigma \dot{v}_k(t)) \quad \text{a.e.} \tag{61}$$

The sequences u_k and v_k are bounded in E, so that $\dot{u}_k$ and $\dot{v}_k$ are bounded in L^2. Because of estimate (18), the right-hand side of equation (61) is bounded in L^2, and so are the v_k because of estimate (26). It follows that the sequence u_k^0 is bounded in $\mathbf{R}^{2n}$.

We now take subsequences so that:

$$v_{k_i} \to \bar{v} \quad \text{weakly in} \quad E \, , \tag{62}$$

$$K(v_{k_i}) \to \rho \quad \text{in} \quad \mathbf{R} \, , \tag{63}$$

$$u_k^0 \to \bar{u}^0 \quad \text{in} \quad \mathbf{R}^{2n} \, . \tag{64}$$

It follows that $v_{k_n} \to \bar{v}$ in L^2; we already know that $u_k \to 0$ in L^2. This implies that the left-hand side of equation (61) converges in L^2 towards

$$\bar{w} = (2\rho + a)^{-1}\bar{v} + u^0 \, . \tag{65}$$

But equation (61) can be inverted, using Fenchel's reciprocity formula (13), to read:

$$\dot{v}_k(t) = \sigma H'[(2K(v_k) + a)^{-1}(v_k(t) - \sigma \dot{u}_k(t)) + u_k^0] \quad \text{a.e.} \tag{66}$$

Now, because of estimate (19), the nonlinear map $w \to \sigma H' \circ w$ sends $L^2(0, 1; \mathbf{R}^{2n})$ into itself. By a theorem of Krasnoselskii ([13]; see [10] for a shorter proof), it follows that it is continuous. This enables us to pass to the limit in formula (66):

$$\dot{v}_k \to \sigma H'(\bar{w}) \qquad \text{in} \quad L^2(0, 1; \mathbf{R}^n) \, . \tag{67}$$

This is the desired result. We can see also now that v_{k_i} converges in E to $\bar{v}$ so that $\rho = K(\bar{v})$. ∎

The time has now come to introduce the natural action of the group $S^1 = \mathbf{R}/\mathbf{Z}$ on the Hilbert space E. To each $\theta \in S^1$ we associate the time translation $L(\theta)\colon E \to E$ defined by:

$$L(\theta)v\colon t \to v(t + \theta) \, . \tag{68}$$

Note that this is a group action: $L(0)$ is the identity, and $L(\theta_1 + \theta_2)$ is $L(\theta_1)L(\theta_2)$. The map $(\theta, v) \to L(\theta)v$ is continuous on $S^1 \times E$. The maps $L(\theta)$ are linear isometries of E:

$$\forall v \in E, \; \|L(\theta)\|_E = \|v\|_E \, . \tag{69}$$

The functionals J and K, being integrals over one period, clearly are invariant by this S^1-action. So is the functional I:

$$I \circ L(\theta) = I\,, \qquad \text{for all} \quad \theta \in S^1\,. \tag{70}$$

Let Ω be the set of points where $I < 0$.

$$\Omega = \{v \in E \,|\, I(v) < 0\}\,. \tag{71}$$

It is an open invariant subset of E, and does not contain the origin. It has moreover two very important properties with respect to the S^1-action: they are crucial to our proof of Theorem 1, and this is where the two-sided inequality (46) comes in.

LEMMA 6. *Take any v in Ω. Then its first Fourier coefficient is non-zero:*

$$\left[\left(v = \textstyle\sum_{k\neq 0} v_k e^{2i\pi kt}\right) \in \Omega\right] \Longrightarrow |v_1| \neq 0\,. \tag{72}$$

Proof. Assume $v_1 = 0$. We then have $||v||_2 \leqq (4\pi)^{-1}||\dot{v}||_2$, by estimate (29). Writing this into the formula (30) for J, we get:

$$J(v) \geqq -\frac{1}{2}||\sigma v||_2 ||\dot{v}||_2 \geqq -(8\pi)^{-1}||\dot{v}||_2^2\,. \tag{73}$$

Writing this into the definition (45) of I, and using estimate (36), we get:

$$\begin{aligned} I(v) &\geqq J(v) + aK(v)\,, \\ &\geqq -(8\pi)^{-1}||v||_E^2 + a(4\beta)^{-1}||v||_E^2\,, \\ &= \frac{1}{4}\left(\frac{a}{\beta} - \frac{1}{2\pi}\right)||v||_E^2\,. \end{aligned} \tag{74}$$

This is non-negative, since a was chosen in formula (46) to be greater than $\beta/2\pi$. Hence $I(v) \geqq 0$, contradicting the assumption that v belongs to Ω. ∎

It follows from Lemma 6 that all functions v in Ω have minimal period one: they cannot be k^{-1}-periodic, for any integer $k \geqq 2$. In other words, *the group S^1 acts* <u>*freely*</u> *on Ω.*

LEMMA 7. *There lies in Ω an invariant* $(2m-1)$ *sphere on which the action of S^1 is the usual Hopf fibration.*

Proof. Imbed the euclidian $(2m-1)$ sphere S^{2m-1} in $\mathbf{R}^{2n}$:

$$S^{2m-1} = \left\{(\xi, \eta) \in \mathbf{R}^n \times \mathbf{R}^n \,\middle|\, \begin{matrix} \Sigma\xi_i^2 + \Sigma\eta_i^2 = 1 \\ \xi_i = 0 = \eta_i \quad \text{for} \quad m+1 \leqq i \leqq n \end{matrix}\right\}. \tag{75}$$

This is the unit sphere. Of course, rS^{2m-1}, with $r > 0$, will be the sphere of radius r.

With any $(\xi, \eta) \in \mathbf{R}^n \times \mathbf{R}^n$, we associate a function $V(\xi, \eta) = \big(Y(\xi, \eta), Q(\xi, \eta)\big)$ by the formulas:

$$Y(\xi, \eta)(t) = \xi \cos 2\pi t + \eta \sin 2\pi t \tag{76}$$

$$Q(\xi, \eta)(t) = \eta \cos 2\pi t - \xi \sin 2\pi t \ . \tag{77}$$

Note the relation:

$$Q(\xi, \eta)(t) = Y(\xi, \eta)\left(t - \frac{1}{4}\right) . \tag{78}$$

It follows that the linear map $V\colon \mathbf{R}^{2n} \to E$ is injective, so that $r\Sigma = V(rS^{2m-1})$ is diffeomorphic to S^{2m-1}. It follows also from relation (78) that $r\Sigma$ is invariant. More precisely, setting $\zeta = \xi + i\eta \in \mathbf{C}^m$, and writing equations (76) and (77) as $V(\zeta)(t) = \zeta e^{-2i\pi t}$, we get the relation:

$$L(\theta) V(\zeta)(t) = \zeta e^{-2i\pi(t+\theta)} , \qquad \forall \theta \in S^1 \tag{79}$$

$$L(\theta) V(\zeta) = V(e^{-2i\pi\theta}\zeta) , \qquad \forall \theta \in S^1 . \tag{80}$$

This means that the action of S^1 on $r\Sigma$ pulls back through v^{-1} as the action $\zeta \to e^{-2i\theta}\zeta$ on S^{2m-1}, the usual Hopf fibration.

We will now show that $r > 0$ can be chosen so small that $r\Sigma$ is contained in Ω. This will conclude the proof.

Take any $v \in r\Sigma$ and compute $I(v) = J(v) + \big(K(v) + a\big)K(v)$. We first compute $J(v)$ by plugging equations (76) and (77) in formula (30):

$$\begin{aligned} J(v) &= \int_0^1 (2\pi\xi \sin 2\pi t - 2\pi\eta \cos 2\pi t, \eta \cos 2\pi t - \xi \sin 2\pi t) dt \\ &= -\pi(\xi^2 + \eta^2) = -\pi r^2 . \end{aligned} \tag{81}$$

Since the components v_i are identically zero for $m + 1 \leqq i \leqq n$, we can use estimate (35):

$$\begin{aligned} K(v) &\leqq (4\gamma)^{-1} \| v \|_E^2 = (4\gamma)^{-1} \| \dot{v} \|_2^2 \\ &= (4\gamma)^{-1}(4\pi^2)(\xi^2 + \eta^2) = \gamma^{-1}\pi^2 r^2 . \end{aligned} \tag{82}$$

Hence we have an estimate for $I(v)$:

$$\begin{aligned} I(v) &\leqq -\pi r^2 + (\gamma^{-1}\pi^2 r^2 + a)(\gamma^{-1}\pi^2 r^2) \\ &= -\pi r^2 \left(1 - \frac{a\pi}{\gamma}\right) + \frac{\pi^4}{\gamma^2} r^4 . \end{aligned} \tag{83}$$

Since a has been chosen in formula (46) to be strictly less than γ/π, the second-order term in (83) is strictly negative, and takes precedence over the fourth-order term near zero. It follows that $I(v) < 0$ for small enough $r > 0$, and hence $r\Sigma \subset \Omega$ as desired. ■

We now apply Theorem III. 2 to the functional I. All the assumptions have been checked, so the conclusion holds: Ω *contains at least m distinct S^1-orbits consisting of critical points for I.*

F. *Periodic trajectories of* (H)

The time has now come to return to the initial problem of finding periodic trajectories of Hamilton's equation (H) on the energy level $H = 1$. We first show that we can associate with each critical point of I in Ω a periodic solution of Hamilton's equation.

LEMMA 8. *Let $v \in E$ be a critical point of I in Ω. Recall that there is some constant $u^0 \in \mathbf{R}^{2n}$ such that*:

$$[2K(v) + a]^{-1}v(t) + u^0 \in \partial G(-\sigma\dot{v}(t)) \quad \text{a.e.} \tag{84}$$

Set:

$$\omega = a + 2K(v) , \tag{85}$$

$$\rho = K(v)^{-1/2} , \tag{86}$$

$$u(t) = \rho(\omega^{-1}v(\omega t) + u^0) . \tag{87}$$

Then u is a C^1 periodic solution of equation (H):

$$\dot{u} = \sigma H'(u) \tag{88}$$

lying on the energy level 1:

$$H(u(t)) = 1 , \qquad \text{for all} \quad t \tag{89}$$

and with minimal period T:

$$T = \omega^{-1} , \tag{90}$$

$$\pi\beta^{-1} < T \leqq 2\pi\beta^{-1} . \tag{91}$$

Proof. Formula (84) is just (57) rewritten. By Lemma 6, the function v has minimal period 1. It is clear from its definition (87) that the function u will have minimal period $\omega^{-1} = T$. We now have to prove that it is indeed a solution of Hamilton's equation, and to check relations (89) and (91).

Comparing (84) and (87), we have:

$$\rho^{-1}u(t) \in \partial G(-\sigma\dot{v}(t)) \quad \text{a.e.} \tag{92}$$

Using Fenchel's reciprocity formula (13) or (17), this becomes:

$$-\sigma\dot{v}(\omega t) = H'(\rho^{-1}u(t)) \quad \text{a.e.} \tag{93}$$

We have seen in Lemma 1 that H' is positively homogeneous of degree one, so that (93) becomes:

$$-\sigma\rho\dot{v}(\omega t) = H'(u(t)) \quad \text{a.e.} \tag{94}$$

But it follows from formula (87) that $\dot{u}(t) = \rho\dot{v}(t)$. Writing this into (94), we get:

$$-\sigma\dot{u}(t) = H'(u(t)) \quad \text{a.e.} \tag{95}$$

Since H' is continuous, u is C^1, and is a classical solution to Hamilton's equation (H). Let us find its energy level.

Since H is positively homogeneous of degree two, we have $2H(u) = (u, H'(u))$ (Euler's formula). Writing this into Fenchel's reciprocity formula (13), we get:

$$H(u) = (u, H'(u)) - G \circ H'(u) = 2H(u) - G \circ H'(u) \tag{96}$$

$$H(u) = G \circ H'(u) , \qquad \text{for all} \quad u \in \mathbf{R}^{2n} . \tag{97}$$

Applying this to formula (93), we get:

$$\begin{aligned} H(u(t)) &= \rho^2 H(\rho^{-1}u(t)) \\ &= \rho^2 G(-\sigma\dot{v}(\omega t)) . \end{aligned} \tag{98}$$

Let h be the energy level: $H(u(t)) = h$ for all t. Integrating the preceding equation, we get:

$$\begin{aligned} h &= \omega \int_0^{\omega^{-1}} H(u(t))dt \\ &= \omega\rho^2 \int_0^{\omega^{-1}} G(-\sigma\dot{v}(\omega t))dt \\ &= \rho^2 \int_0^1 G(-\sigma\dot{v}(s))ds \\ &= \rho^2 K(v) = 1 . \end{aligned} \tag{99}$$

Only estimate (95) is left. We start off by writing formula (97) in a different way:

$$G \circ H'(u) = \frac{1}{2}(u, H'(u)) , \qquad \text{for all} \quad u \in \mathbf{R}^{2n} . \tag{100}$$

Comparing with equation (93), we get:

$$G(-\sigma\dot{v}(\omega t)) = \frac{1}{2}(\rho^{-1}u(t), -\sigma\dot{v}(\omega t)) . \tag{101}$$

With equation (87), this becomes:

$$G(-\sigma\dot{v}(\omega t)) = \frac{1}{2}(\omega^{-1}v(\omega t) + u^0, -\sigma\dot{v}(\omega t)) . \tag{102}$$

Integrating over one period, the constant disappears, and we are left with:

$$\omega \int_0^{\omega^{-1}} G(-\sigma\dot{v}(\omega t))dt = \frac{1}{2}\int_0^{\omega^{-1}} (v(\omega t), -\sigma\dot{v}(\omega t))dt , \tag{103}$$

$$K(v) = -\omega^{-1}J(v) . \tag{104}$$

Hence:

$$T = \omega^{-1} = -\frac{K(v)}{J(v)} .$$

With use of estimates (34) and (36), this implies that $T \geqq \pi\beta^{-1}$. The other inequality (91) we get simply by writing:

$$\omega^{-1} = \left(2K(v) + a\right)^{-1} < a^{-1} < 2\pi\beta^{-1} . \tag{106}$$ ■

We now prove that the m distinct S^1-orbits of critical points for I which we have found in Ω give rise to distinct periodic trajectories for the system (H) (we refer to subsection A for what we mean by distinct trajectories).

LEMMA 9. *Let v_1 and v_2 be two critical points of I in Ω, and let u_1 and u_2 be the periodic solutions of* (H) *associated with v_1 and v_2 by Lemma* 8. *If v_1 and v_2 belong to different S^1-orbits in E, then u_1 and u_2 describe different trajectories in $\mathbf{R}^{2n}$.*

Proof. If u_1 and u_2 described the same trajectory in $\mathbf{R}^{2n}$, there would be some $t_0 \in \mathbf{R}$ such that:

$$u_1(t) = u_2(t + t_0) , \qquad \text{for all} \quad t . \tag{107}$$

It follows that u_1 and u_2 have the same minimal period: $T_1 = T_2$. Using Lemma 8, we get $\omega_1 = \omega_2$, then $K(v_1) = K(v_2)$, then $\rho_1 = \rho_2$, then, by formula (93):

$$\begin{aligned} -\sigma\dot{v}_1(t) &= H'\left(\rho_1^{-1}u_1(t)\right) \\ &= H'\left(\rho_2^{-1}u_2(t + t_0)\right) \\ &= -\sigma\dot{v}_2(t + t_0) \qquad \text{for all} \quad t . \end{aligned} \tag{108}$$

Hence $v_1(t) = v_2(t + t_0)$ for all t. This means that $v_1 = L(t_0)v_2$, so that v_1 and v_2 belong to the same S^1-orbit in E. ■

This concludes the proof of Theorem 1: we have found m distinct critical orbits of I in Ω (see conclusion of subsection E), which, by Lemma 9, have to correspond to m distinct periodic trajectories of (H) in $\mathbf{R}^{2n}$.

III. S^1-Actions and invariant functions on Hilbert space

In this section, we prove two abstract theorems about critical points of functionals invariant by an S^1-action. It is self-contained and can be read independently of the rest of the paper. Its main result, Theorem III. 2, was used in the preceding section to prove Theorem II. 1.

Theorems III. 1 and III. 2 differ only by the regularity assumptions on the functional F, which is assumed to be C^1 in Theorem 1, while in Theorem 2 we assume there is some $k \geqq 0$ such that $F(v) + k\|v\|^2$ is a convex function of v. The proof of Theorem 1 will be obtained by a topological argument of Liusternik-Schnirelman type, and Theorem 2 will follow by a regularization argument.

Let us first set the stage. Throughout, E will be a Hilbert space on which the group S^1 acts through isometries. In other words, with every $\theta \in S^1$, we associate a linear operator $L(\theta) \in \mathcal{L}(E)$ such that:

$$\forall \theta \in S^1, \forall u \in E, \|L(\theta)u\| = \|u\| , \qquad L(0) = \mathrm{Id} ,$$
$$\forall (\theta_1, \theta_2), L(\theta_1 + \theta_2) = L(\theta_1)L(\theta_2) .$$

We will assume also that the map $(\theta, u) \to L(\theta)u$ is continuous on $S^1 \times E$. Recall that a subset Ω of E is invariant if $L(\theta)u \in \Omega$ for all $\theta \in S^1$ and $u \in \Omega$. We say that S^1 acts *freely* on Ω if $\theta \neq \theta'$ implies $L(\theta)u \neq L(\theta')u$ for all $u \in \Omega$.

1. *Invariant C^1 functions*

Our purpose in this subsection is to prove the following:

THEOREM 1. *Let $F: E \to \mathbf{R}$ be a C^1 function, bounded from below and satisfying condition* (C-): *if u_n is a sequence in E such that* $\mathrm{Sup}_n F(u_n) < 0$ *and $F'(u_n) \to 0$, then u_n has a convergent subsequence. Assume that F is invariant*:

$$\forall \theta \in S^1, F \circ L(\theta) = F \tag{1}$$

and S^1 acts freely on the subset:

$$\Omega = \{u \mid F^{-1}(u) < 0\} .$$

If Ω contains an invariant $(2n - 1)$ sphere Σ, and the action of S^1 on Σ is the usual Hopf fibration, then there are at least n S^1-orbits within Ω consisting of critical points of F. ∎

Note that condition (C-) is of Palais-Smale type. The strict inequality in $\mathrm{Sup}_n F(u_n) < 0$ makes it easier to check and comes in handy in various problems (see [4], [18]). Note also that the requirement that S^1 act freely on Ω implies that Ω does not contain the origin. In particular, $F(0)$ cannot be negative: $F(0) \geqq 0$.

To prove this theorem, we need the cohomological index theory of Fadell and Rabinowitz. Let us just recall the relevant features and refer the reader to their paper [11] for full details and proofs in the general case of a compact Lie group acting on E.

With every paracompact space A on which S^1 acts freely, they associate an integer, its index, with the following properties:

$$\mathrm{Index}\, \varnothing = 0 ; \tag{1}$$

$$1 \leqq \mathrm{Index}\, A \leqq \frac{1 + \dim A}{2} \qquad \text{for} \quad A \neq \varnothing ; \tag{2}$$

$$(3)\qquad \begin{cases}\text{Monotonicity: if there is an equivariant continuous map}\\ \varphi\colon A \to B, \text{ then Index } A \leqq \text{Index } B\end{cases}$$

$$(4)\qquad \text{Subadditivity: } \text{Index}(A \cup B) \leqq \text{Index } A + \text{Index } B$$

$$(5)\qquad \begin{cases}\text{Continuity: if } K \text{ is a closed invariant subset of } A, \text{ there is a closed}\\ \text{invariant neighborhood } N \text{ of } K \text{ with Index } N = \text{Index } K.\end{cases}$$

Note that the dimension formula (2) is misprinted in Section 5 of the original paper. For the sake of completeness, we quote the definition of Index A: it is the greatest integer k such that $f^*(\varphi^{k-1})$ does not vanish, where f is a classifying map of the fibre bundle $A \to A/S^1$ into the universal S^1-bundle $\mathbf{C}P_\infty$, and $\varphi \in H^2(\mathbf{C}P^\infty, \mathbf{Z})$ is the first universal Chern class. It follows from this definition that the index of S^{2n-1}, with the usual S^1-action (Hopf fibration), is n.

We now proceed to the study of the function F. It is C^1, so that its gradient is a continuous vector field on Ω. However, it need not be Lipschitzian, so that we cannot integrate it. For this purpose, we will use a pseudo-gradient, defined on an open subset $\mathcal{U}$ of E.

Definition 2. A vector field $\Phi\colon \mathcal{U} \to E$ is a pseudo-gradient for F if it is locally Lipschitzian and, for all $u \in \mathcal{U}$:

$$(6)\qquad \big(F'(u), \Phi(u)\big) \geqq \|F'(u)\|^2 ,$$

$$(7)\qquad 2\|F'(u)\| \geqq \|\Phi(u)\| . \qquad \blacksquare$$

From now on, the arguments are standard, and can be found for instance in Palais [17] or Rabinowitz [18]. We first state that there exists on $\mathcal{U} = \{u \in \Omega \mid F'(u) \neq 0\}$ a pseudo-gradient, equivariant with respect to S^1:

LEMMA 3. *There is a vector field* $\Phi\colon E \to E$ *satisfying conditions* (6), (7) *and*

$$(8)\qquad \forall \theta \in S^1,\ \Phi \circ L(\theta) = L(\theta) \circ \Phi .$$

Proof. The existence of a pseudo-gradient ψ is proved in [17] or [18], using a partition of unity. We then define a new vector field $\Phi\colon \Omega \to E$ by:

$$(9)\qquad \forall u \in E,\ \Phi(u) = \int_{S^1} L(-\theta)\psi\big(L(\theta)u\big)d\theta \qquad \left(\int_{S^1} d\theta = 1\right).$$

We check successively conditions (6), (7) and (8), using the fact that the $L(\theta)$ are isometries, and that F is invariant, $F\big(L(\theta)u\big) = F(u)$. If we differentiate with respect to u, this yields $F'\big(L(\theta)u\big) = L(\theta)F'(u)$ and hence $\|F'\big(L(\theta)u\big)\| = \|F'(u)\|$. We have:

$$\begin{aligned}\big(F'(u), \Phi(u)\big) &= \int \big(F'(u), L(-\theta)\psi(L(\theta)u)\big)d\theta\\ &= \int \big(L(-\theta)F'(L(\theta)u), L(-\theta)\psi(L(\theta)u)\big)d\theta\end{aligned}$$

$$
\begin{aligned}
&= \int (F'(L(\theta)u), \psi(L(\theta)u))d\theta \\
&\geqq \int \|F'(L(\theta)u)\|^2 d\theta = \|F'(u)\|^2 . \\
\|\Phi(u)\| &\leqq \int \|L(-\theta)\psi(L(\theta)u)\| d\theta \\
&= \int \|\psi(L(\theta)u)\| d\theta \\
&\leqq 2 \int \|F'(L(\theta)u)\| d\theta \\
&= 2\|F'(u)\| . \\
\Phi(L(\tau)u) &= \int L(-\theta)\psi(L(\theta)L(\tau)u)d\theta \\
&= \int L(\tau)L(-\tau - \theta)\psi(L(\tau + \theta)u)d\theta \\
&= L(\tau)\Phi(u) . \qquad \blacksquare
\end{aligned}
$$

We then give a deformation lemma, the proof of which can be found in the same references. Here we denote by Ω_c the set of points u such that $F(u) \leqq c$. It is a closed invariant subset of Ω.

LEMMA 4. *Take $c < 0$ and let $K = \{x \mid F'(x) = 0, F(x) = c\}$, which is a compact invariant subset of Ω. For any invariant neighborhood N of K, there exists a constant $\varepsilon > 0$ and a continuous map:*

$$\varphi: [0, 1] \times (\Omega_{c+\varepsilon} \backslash N) \longrightarrow \Omega_{c+\varepsilon} \backslash K \tag{10}$$

with the following properties:

$$\varphi_0(x) = x , \qquad \text{for all} \quad x \in \Omega_{c+\varepsilon} \backslash N , \tag{11}$$

$$\varphi_1 \circ L(\theta) = L(\theta) \circ \varphi_1 , \qquad \text{for all} \quad \theta \in S^1 , \tag{12}$$

$$\varphi_1(\Omega_{c+\varepsilon} \backslash N) \subset \Omega_{c-\varepsilon} . \tag{13}$$

If $K = \emptyset$, the same statement holds with $N = \emptyset$.

Proof. Let η and ε be given, with $-c > \eta > \varepsilon > 0$. Consider the disjoint invariant sets:

$$A = \{x \in \Omega \mid F(x) \geqq c + \eta \quad \text{or} \quad F(x) \leqq c - \eta\} , \tag{14}$$

$$B = \{x \in \Omega \mid c - \varepsilon \leqq F(x) \leqq c + \varepsilon\} . \tag{15}$$

Define functions $f: \Omega \to \mathbf{R}$ and $\alpha: \mathbf{R}_+ \to \mathbf{R}$ by:

$$f(u) = [\operatorname{dist}(u, A) + \operatorname{dist}(u, B)]^{-1} \operatorname{dist}(u, A) , \tag{16}$$

$$\alpha(t) = \begin{cases} 1 & \text{if} \quad 0 \leqq t \leqq 1 \\ t^{-1} & \text{if} \quad t \geqq 1 . \end{cases} \tag{17}$$

It follows from the definition that f is invariant. We now consider the vector field V on $\Omega \backslash K$ defined by:

$$V(u) = -f(u)\alpha(\|\Phi(u)\|)\Phi(u) \tag{18}$$

where $\Phi(u)$ is an equivariant pseudo-gradient for F.

The map φ is simply the flow associated with the locally Lipschitzian vector field V. It is shown in references [17] and [18], or in paper [4], that if η and ε are appropriately chosen, φ is well-defined and satisfies properties (11) and (13), as well as a few more. Condition (12) follows readily from the fact that the vector field V is equivariant: $V \circ L(\theta) = L(\theta) \circ V$. ■

We are now all set for the proof of Theorem 1. We start by associating with each integer i between 1 and n a family Γ_i of subsets of Ω:

$$\Gamma_i = \{B \subset \Phi \mid B \text{ is compact, invariant, Index } B \geqq i\} . \tag{19}$$

By assumption, Ω contains a compact invariant subset Σ of index n. It follows that all the Γ_i, $1 \leqq i \leqq n$, are non-empty, since all contain Σ. We define:

$$c_i = \operatorname{Inf}_{B \in \Gamma_i} [\operatorname{Max}_{u \in B} F(u)] . \tag{20}$$

We have, for all $i \in \{1, \cdots, n\}$:

$$0 > \operatorname{Max}_{u \in \Sigma} F(u) \geqq c_i \geqq \operatorname{Inf}_{u \in \Omega} F(u) > -\infty , \tag{21}$$

$$c_{i+1} \geqq c_i . \tag{22}$$

The proof now proceeds in two steps:

LEMMA 5. *Each c_i is a critical value of F.*

Proof. Assume it is not. Apply Lemma 4 with $K = \varnothing$ and $c = c_i$: there is some $\varepsilon > 0$ and some map φ satisfying conditions (11) to (13), with $N = \varnothing$. Pick some $B \in \Gamma_i$ such that:

$$\operatorname{Max}_{u \in B} F(u) < c_i + \varepsilon . \tag{23}$$

Consider $\varphi_1(B)$. It is a compact subset of Ω, because φ_1 is continuous, and it is invariant as B because φ_1 is equivariant. The group S^1 acts freely on $\varphi_1(B)$, because it acts freely on Ω. It then follows from the monotonicity condition (3) that Index $\varphi_1(B) \geqq$ Index B, so that $\varphi_1(B)$ also belongs to Γ_i. It follows from the definition (20) of c_i that:

$$\operatorname{Max}_{u \in \varphi_1(B)} F(u) \geqq c_i , \tag{24}$$

and from condition (13) that:

$$\operatorname{Max}_{u \in \varphi_1(B)} F(u) \leqq c_i - \varepsilon , \tag{25}$$

a clear contradiction. ■

If the c_i are all distinct, the proof of Theorem 1 is complete: each $F^{-1}(c_i)$ contains one of the n desired critical orbits. If some of the c_i coincide, the following lemma proves that there are infinitely many critical orbits, so that the theorem still holds.

LEMMA 6. *Assume $c_i = c_j$, with $i < j$. Call this common value c, and set:*

$$K = \{u \in \Omega \mid F(u) = c,\ F'(u) = 0\} . \tag{26}$$

Then $\operatorname{Index} K \geqq j - i + 1$.

Proof. The condition (C-) implies that K is a compact invariant subset of Ω. By the continuity property (5), there is a closed invariant neighborhood N of K with the same index. Its interior $\mathring{N}$ is still invariant, and by condition (3) or (4):

$$\operatorname{Index} K \leqq \operatorname{Index} \mathring{N} \leqq \operatorname{Index} N = \operatorname{Index} K . \tag{27}$$

Let us now apply Lemma 4 to get some $\varepsilon > 0$ and some map φ with the properties (10) to (13). Pick some $B \in \Gamma_j$ such that:

$$\operatorname{Max}_{u \in B} F(u) \leqq c + \varepsilon . \tag{28}$$

In other words, B is compact, invariant, and $B \subset \Omega_{c+\varepsilon}$. Set $C = B \backslash \mathring{N}$. By the monotonicity and subadditivity properties (3) and (4), we have:

$$\begin{aligned} \operatorname{Index} B &\leqq \operatorname{Index} (C \cup \mathring{N}) \\ &\leqq \operatorname{Index} C + \operatorname{Index} \mathring{N} . \end{aligned} \tag{29}$$

But C is compact, invariant, and condition (13) tells us that $\varphi_1(C)$ is contained in $\Omega_{c-\varepsilon}$. It follows from the definition (20) of c_i that:

$$\operatorname{Index} \varphi_1(C) < i . \tag{30}$$

Comparing inequalities (29) and (30), and remembering that $B \in \Gamma_j$, we get:

$$\begin{aligned} j &\leqq \operatorname{Index} B \leqq \operatorname{Index} C + \operatorname{Index} \mathring{N} \\ &\leqq \operatorname{Index} \varphi_1(C) + \operatorname{Index} K \\ &< i + \operatorname{Index} K . \end{aligned}$$

■

If two of the c_i coincide, the index of the corresponding K will be at least 2, and by condition (2), its dimension will be at least 3. Since the S^1-orbits have dimension 1, there must be an infinite number of them to fill up K. This concludes the proof of Theorem 1.

2. *Invariant non-smooth functions*

The space E, and the S^1-action $L(\theta)$ are as described in the beginning. As in the preceding subsection, we shall give conditions for an invariant function F on E to have n distinct critical points, but we shall no longer assume F to be smooth, or even finite-value. Instead, we assume the following:

$$\text{(31)} \qquad \begin{cases} F\colon E \to \mathbf{R} \cup \{+\infty\} \text{ is lower semi-continuous, not identically } +\infty, \\ \text{and there exists some } k \geqq 0 \text{ such that } G(v) = F(v) + k\|v\|^2 \text{ is a} \\ \text{convex function of } v. \end{cases}$$

We then can define the subgradient $\partial G(v)$ of the function F at the point v:

$$\text{(32)} \qquad \partial F(v) = \partial G(v) - 2kv\ ,$$

where ∂G stands for the subgradient of G in the sense of convex analysis (see [15], [21], [10] for instance); that is:

$$\text{(33)} \qquad \partial G(v) = \{u \in E \mid G(w) - G(v) \geqq (w - v, u)\ \ \forall w \in E\}\ .$$

Note that $\partial F(v) \neq \emptyset$ if and only if $\partial G(v) \neq \emptyset$.

From (32) and (33) it can be seen that, if $u \in \partial F(v)$, then:

$$\text{(34)} \qquad F(w) - F(v) - (w - v, u) \geqq -k\|v - w\|^2 \quad \forall w \in E\ ,$$

which certainly implies that (we use Landau's notation on the right-hand side):

$$\text{(35)} \qquad F(w) - F(v) - (w - v, u) \geqq o(\|w - v\|) \quad \forall w \in E\ .$$

Conversely, if $u \in E$ satisfies condition (35), it will imply that:

$$\text{(36)} \qquad G(w) - G(v) - (w - v, u + 2kv) \geqq o(\|w - v\|) \quad \forall w \in E\ .$$

Since the function G is convex, this will imply that $u + 2kv \in \partial G(v)$. Summing up, we see that $\partial F(v)$ is the set of vectors $u \in E$ which satisfy condition (35). It follows that the subgradient ∂F does not really depend on the particular choice of k in the decomposition $F(v) = G(v) - k\|v\|^2$, as long as k is large enough for G to be convex.

We shall also make use of another rule of calculus: if some functional F splits as $F = F_1 + F_2$, with F_1 being C^2 and F_2 convex lower semi-continuous, and if there is some $k \geqq 0$ such that $F_1''(v) + k\,\mathrm{Id}$ is positive definite for all $v \in E$, then F satisfies condition (31) and

$$\partial F(v) = F_1'(v) + \partial F_2(v)\ , \qquad \text{for all} \quad v \in E\ .$$

The proof is quite easy, using formula (35). If $u \in E$ belongs to $F_1'(v) + \partial F_2(v)$, setting $u_2 = u - F_1'(v) \in \partial F_2(v)$ we get:

$$\begin{aligned} &F(w) - F(v) - (u, w - v) \\ &\quad = \big(F_1(w) - F_1(v) - (F_1'(v), w - v)\big) + \big(F_2(w) - F_2(v) - (u_2, w - v)\big) \\ &\quad \geqq \frac{1}{2}\big(F_1''(w_0)(w - v), w - v\big)\ , \end{aligned}$$

for some w_0 on the line segment $[v, w]$. Using the assumption on F_1', we get the estimate:

$$F(w) - F(v) - (u, w - v) \geqq -\frac{k}{2}\|w - v\|^2 .$$

Hence, by formula (35), $u \in \partial F(v)$. Conversely, if $u \in \partial F(v)$, we claim that $u_2 = u - F_1'(u)$ belongs to $\partial F_2(v)$. Indeed, by formula (35), we have:

$$\begin{aligned} &F_2(w) - F_2(v) - (w - v, u_2) \\ &\quad = [F(w) - F(v) - (w - v, u)] - [F_1(w) - F_1(v) - (w - v, F_1'(u))] \\ &\quad = o(\|w - v\|) . \end{aligned}$$

The result follows as desired.

A critical point of F is a point v where:

$$\partial F(v) \ni 0 . \tag{37}$$

If the function F is S^1-invariant, then so are the functions $v \to k\|v\|^2$ (because the $L(\theta)$ are isometries) and G (because it is the sum of the two preceding ones). It follows that the critical points of F occur in S^1-orbits.

We now state the main result:

THEOREM 2. *Let $F\colon E \to \mathbf{R} \cup \{+\infty\}$ be a function, bounded from below, which satisfies conditions* (31) *and*

$$\text{(C=)} \qquad \begin{cases} \textit{If } (v_i, u_i) \textit{ are sequences in } E \times E \textit{ such that } \operatorname{Sup}_i F(v_i) < 0, \\ u_i \in \partial F(v_i) \textit{ and } u_i \to 0, \textit{ then there is a subsequence } v_{i_k} \textit{ which} \\ \textit{converges.} \end{cases}$$

Assume F is invariant:

$$\forall \theta \in S^1,\ F \circ L(\theta) = F \tag{38}$$

and S^1 acts freely on the subset:

$$\Omega = \{u \mid F(u) < 0\} . \tag{39}$$

If Ω contains an invariant $(2n - 1)$ sphere Σ, and the action of S^1 on Σ is the usual Hopf fibration, then there are at least n orbits in Ω consisting of critical points of F. ■

Note that condition (C=) is of Palais-Smale type: it is the natural extension of condition (C-) in the preceding subsection. It follows from the fact that the subgradient has closed graph as a multi-valued map from E to E that the limit of the subsequence v_{n_k} has to be a critical point. Note finally that the requirement that S^1 act freely on Ω implies that $F(0) \geqq 0$.

The proof of Theorem 2 relies on a regularization procedure following an earlier idea of the second author. For an application of these ideas to Morse theory, the interested reader is referred to [2].

Let $\varepsilon > 0$ be given. We associate with F and ε the new function F_ε defined by:

$$F_\varepsilon(v) = \operatorname{Inf}_{w \in E} \left\{ \frac{1}{\varepsilon} \|w - v\|^2 + F(w) \right\} . \tag{40}$$

In the case where F is convex, this is a classical regularization procedure, due to Moreau and used systematically by Yosida (see [3]). Its use in the non-convex case appears to be new.

LEMMA 7. *Assume $\varepsilon^{-1} > k$ and F satisfies condition* (31). *Then F_ε is finite-valued, differentiable everywhere, and F'_ε is globally Lipschitzian. Moreover, F_ε has the following properties*:

$$\operatorname{Inf}_{w \in E} F(w) \leqq F_\varepsilon(v) \leqq F(v) , \qquad \text{for all} \quad v \in E , \tag{41}$$

$$F'_\varepsilon(v) = 0 \Longleftrightarrow 0 \in \partial F(v) \Longleftrightarrow F_\varepsilon(v) = F(v) , \tag{42}$$

$$\big(u, F'_\varepsilon(v)\big) \geqq \frac{1}{2}(1 - \varepsilon k)\|F'_\varepsilon(v)\|^2 , \quad \forall v \in E , \ u \in \partial F(v) . \tag{43}$$

If F satisfies condition (C=), *then F_ε satisfies condition* (C-). *If F is S^1-invariant, then so is F_ε.* ■

The proof of Theorem 2 follows easily from this lemma. Indeed, choose ε strictly smaller than k^{-1}, and consider F_ε. Since F satisfies (31), has the Palais-Smale property (C=) and is invariant, F_ε is C^1, satisfies condition (C-) and is invariant. Because of inequality (41), the set Ω_ε where $F_\varepsilon < 0$ contains the set Ω where $F < 0$, so that both contain Σ. By Theorem 1, there are at least n distinct S^1-orbits in Ω_ε consisting of critical points of F_ε. But if v is a critical point of F_ε in Ω_ε, we use relations (42) to get:

$$0 \in \partial F(v) \qquad \text{and} \qquad F(v) = F_\varepsilon(v) < 0 . \tag{44}$$

So v is in fact a critical point of F and lies in Ω. The n distinct S^1-orbits just found consist of critical points of F and lie in Ω; Theorem 2 follows.

Let us note that property (43) has not been used in this proof. It is a kind of pseudo-gradient property (compare with definition 2), and may be useful in other situations (for instance, to seek an analogue of the deformation Lemma 4 to give a direct proof of Theorem 2, as in [2] for Morse inequalities).

We now proceed to the proof of Lemma 7. Throughout, k and ε are fixed with $\varepsilon^{-1} > k$. Condition (31) holds for F.

Note first that F_ε is finite everywhere. Indeed, let v_0 be a point where $F(v_0) < +\infty$, and set $w = v_0$ in the definition (40) of F_ε. We get $F_\varepsilon(v) \leqq \|v - v_0\|^2\varepsilon^{-1}$.

Taking $w = v$ in the same formula (40), we get $F_\varepsilon \leqq F$. Moreover, we clearly have:

$$\frac{1}{\varepsilon}\|w - v\|^2 + F(v) \geqq \operatorname{Inf}_E F\,, \qquad \text{for all} \quad (v, w)\,. \tag{45}$$

Using formula (40) again, we get $F_\varepsilon(v) \geqq \operatorname{Inf}_E F$. The inequalities (41) are proved. The rest of the proof of Lemma 7 is not so straightforward, and will require several steps.

Step 1. There is a map $\psi: E \to E$ such that:

$$\|\psi(v_1) - \psi(v_2)\| \leqq (1 - \varepsilon k)^{-1}\|v_1 - v_2\|\,, \qquad \text{for all} \quad (v_1, v_2)\,, \tag{46}$$

$$F(\psi(v)) = F_\varepsilon(v) - \frac{1}{\varepsilon}\|v - \psi(v)\|^2 < +\infty\,, \tag{47}$$

$$\frac{2}{\varepsilon}(v - \psi(v)) \in \partial F(\psi(v))\,. \tag{48}$$

Proof. With any $v \in E$, we associate the function $G_v: E \to \mathbf{R} \cup \{+\infty\}$ defined by:

$$G_v(w) = \frac{1}{\varepsilon}\|v - w\|^2 + F(w)\,. \tag{49}$$

Because of assumption (31) on F, and the condition $\varepsilon^{-1} > k$, the function G_v is strictly convex and lower semi-continuous. Moreover, we have

$$G_v(w) \geqq \frac{1}{\varepsilon}\|v - w\|^2 + \operatorname{Inf}_E F\,. \tag{50}$$

It follows that there is a unique point $\psi(v) \in E$ where G_v attains its minimum:

$$\frac{1}{\varepsilon}\|v - \psi(v)\|^2 + F(\psi(v)) = \operatorname{Inf}_{w \in E} G_v(w)\,. \tag{51}$$

Because of formula (40), the right-hand side of formula (51) is just $F_\varepsilon(v)$. Relation (47) follows.

The point $\psi(v)$ can also be characterized as the unique solution of $0 \in \partial G_v(w)$. This can also be written $0 \in 2(w - v)\varepsilon^{-1} + \partial F(w)$, which yields relation (48). Writing formula (32) for ∂F, we get:

$$0 \in 2(w - v)\varepsilon^{-1} - 2kw + \partial G_v(w)\,. \tag{52}$$

Let $D(A)$ be the set of points w where $\partial G_v(w) \neq \varnothing$, and define a (multi-valued) map $A: D(A) \to E$ by:

$$A(w) = 2(\varepsilon^{-1} - k)w + \partial G_v(w)\,. \tag{53}$$

Now ∂G_v is a maximal monotone operator (see [3]) because it is the sub-gradient of a lower semi-continuous convex function. It is a standard fact from the theory that if B is maximal monotone and λ is strictly positive, then $\lambda I + B$ is onto, and its inverse satisfies a global Lipschitz condition with constant λ^{-1}. It follows immediately that A is onto and $A^{-1}: E \to D(A)$ is Lipschitzian with constant $(\varepsilon^{-1} - k)^{-1}2^{-1}$.

We now rewrite equation (52) as:

$$2v\varepsilon^{-1} \in A(w) \qquad \text{for} \quad w = \psi(v) \ . \tag{54}$$

It follows that $\psi(v) = A^{-1}(2v\varepsilon^{-1})$, so that ψ is Lipschitzian with constant $\varepsilon^{-1}(\varepsilon^{-1} - k)^{-1} = (1 - k\varepsilon)^{-1}$. This is relation (46). ∎

Step 2. F_ε is everywhere differentiable, F'_ε is globally Lipschitz, and condition (42) is satisfied.

Proof. Pick any two points u and v in E. From the definition (40) of F_ε, we have:

$$F_\varepsilon(u) \leqq \frac{1}{\varepsilon} \|u - \psi(v)\|^2 + F(\psi(v)) \ . \tag{55}$$

Using equation (47), this becomes:

$$F_\varepsilon(u) - F_\varepsilon(v) \leqq [\|u - \psi(v)\|^2 - \|v - \psi(v)\|^2]\varepsilon^{-1} \ . \tag{56}$$

Performing some algebra, we get from this:

$$F_\varepsilon(u) - F_\varepsilon(v) - 2(v - \psi(v), u - v)\varepsilon^{-1} \leqq \|u - v\|^2\varepsilon^{-1} \ . \tag{57}$$

Exchanging the roles of u and v, we also have:

$$F_\varepsilon(v) - F_\varepsilon(u) - 2(u - \psi(u), v - u)\varepsilon^{-1} \leqq \|u - v\|^2\varepsilon^{-1} \ . \tag{58}$$

Using the Lipschitz condition (46), we have:

$$|(u - \psi(u) - v + \psi(v), u - v)| \leqq \mu\|u - v\|^2 \ , \tag{59}$$

with $\mu = 1 + (1 - \varepsilon k)^{-1}$. It follows immediately that:

$$-(u - \psi(u), v - u) \geqq -(v - \psi(v), v - u) - \mu\|u - v\|^2 \ . \tag{60}$$

Writing this into inequality (58) yields:

$$F_\varepsilon(v) - F_\varepsilon(u) - 2(v - \psi(v), v - u) \leqq (\mu + \varepsilon^{-1})\|u - v\|^2 \ . \tag{61}$$

Let us now write inequalities (57) and (61) together:

$$|F_\varepsilon(u) - F_\varepsilon(v) - 2(v - \psi(v), u - v)\varepsilon^{-1}| \leqq (\mu + \varepsilon^{-1})\|u - v\|^2 \ . \tag{62}$$

This simply means that F_ε is differentiable at the point v, with:

$$F'_\varepsilon(v) = 2(v - \psi(v))\varepsilon^{-1} \ , \tag{63}$$

a Lipschitz map from E to itself, as announced. Now for condition (42).

By formula (63), $F'_\varepsilon(v) = 0$ if and only if $v = \psi(v)$. By formula (47), this means exactly that $F(v) = F_\varepsilon(v)$. By formula (48), this also implies that $0 \in \partial F(v)$.

It only remains to prove that $0 \in \partial F(v)$ implies that $F'_\varepsilon(v) = 0$. By definition (32), $0 \in \partial F(v)$ means that $0 \in -2kv + \partial G(v)$. By formula (53), this can also be written $2\varepsilon^{-1}v \in A(v)$, or $v = A^{-1}(2\varepsilon^{-1}v)$. But the right-hand side is just $\psi(v)$, by the definition (54) of ψ, and the result follows from formula (63). ■

Step 3. Condition (43) is satisfied.

Proof. Pick any $u \in \partial F(v)$. By formula (34):

$$F(\psi(v)) - (\psi(v) - v, u) + k\|\psi(v) - v\|^2 \geqq F(v) . \tag{64}$$

By equation (47), this becomes:

$$F_\varepsilon(v) + (k - \varepsilon^{-1})\|\psi(v) - v\|^2 + (v - \psi(v), u) \geqq F(v) . \tag{65}$$

Since $F_\varepsilon \leqq F$, this implies that:

$$(v - \psi(v), u) \geqq (\varepsilon^{-1} - k)\|\psi(v) - v\|^2 . \tag{66}$$

By use of equation (63), $v - \psi(v) = \varepsilon F'_\varepsilon(v)/2$, this yields

$$\frac{\varepsilon}{2}(F'_\varepsilon(v), u) \geqq (\varepsilon^{-1} - k)\frac{\varepsilon^2}{4}\|F'_\varepsilon(v)\|^2 . \tag{67}$$

Simplifying throughout, we get condition (63). ■

Step 4. If F satisfies condition (C−), then F_ε satisfies condition (C-).

Proof. Let v_i be a sequence in E such that:

$$F_\varepsilon(v_i) \leqq -\alpha < 0 , \qquad \text{for all} \quad i ; \tag{68}$$

$$F'_\varepsilon(v_i) \longrightarrow 0 \qquad \text{when} \quad i \longrightarrow \infty . \tag{69}$$

We want to show that there is a subsequence v_{i_k} which converges in E. Set $w_i = \psi(v_i)$ and $u_i = 2(v_i - w_i)\varepsilon^{-1}$. Using formula (47), we have:

$$F(w_i) \leqq F_\varepsilon(v_i) \leqq -\alpha < 0 , \qquad \text{for all} \quad i . \tag{70}$$

Using formula (48), we also have:

$$u_i \in \partial F(w_i) , \qquad \text{for all} \quad i . \tag{71}$$

Finally, from formulas (63) and (69) we get:

$$u_i = F'_\varepsilon(v_i) \rightarrow 0 . \tag{72}$$

Since the function F satisfies condition (C−), we conclude from (70), (71) and (72) that the sequence w_i has a convergent subsequence w_{i_k}. But:

$$v_{i_k} - w_{i_k} = \frac{\varepsilon}{2}u_{i_k} \rightarrow 0 , \tag{73}$$

so that the sequence v_{i_k} itself converges. Hence the result. ■

Step 5. If F is S^1-invariant, then so is F_ε.

Proof. Pick any $\theta \in S^1$ and $v \in E$. Recall the definition (40) of F:

$$F_\varepsilon\big(L(\theta)v\big) = \operatorname{Inf}_{w \in E}\left\{\frac{1}{\varepsilon}\,||w - L(\theta)v||^2 + F(w)\right\}. \tag{74}$$

Since $L(\theta)$ is invertible, we can set $w = L(\theta)u$, and the right-hand side becomes:

$$F_\varepsilon\big(L(\theta)v\big) = \operatorname{Inf}_{u \in E}\left\{\frac{1}{\varepsilon}\,||L(\theta)u - L(\theta)v||^2 + F\big(L(\theta)u\big)\right\}. \tag{75}$$

But $L(\theta)$ is an isometry and F is invariant. The desired result follows:

$$\begin{aligned} F_\varepsilon\big(L(\theta)v\big) &= \operatorname{Inf}_{u \in E}\left\{\frac{1}{\varepsilon}\,||u - v||^2 + F(u)\right\} \\ &= F_\varepsilon(v)\,. \end{aligned} \tag{76}$$

■

IV. The general case

We now have to show how to go beyond the case where the Hamiltonian H is positively homogeneous of degree two. This is done by using a trick which we learned from Rabinowitz, but which seems to be classical in Hamiltonian mechanics.

The fact is that the trajectories of Hamilton's equations $\dot{u} = \sigma H'(u)$ on the energy surface S defined by $H(u) = h$ depend only on S and not on H. More precisely, if another Hamiltonian system also has S as a particular energy surface, it will have the same trajectories as the preceding one on that surface S. The solutions themselves, however, may not be the same for both systems, since the same trajectories may be described with different speeds all along the path.

We now substantiate these statements in the case of a convex energy surface, which is the only one we are interested in. We begin by giving a few classical definitions in convex analysis (see [21]).

Definition 1. Let C be a closed convex subset of $\mathbf{R}^{2n}$ containing the origin. Its *gauge* is the function J: $\mathbf{R}^{2n} \to \mathbf{R} \cup \{+\infty\}$ defined by:

$$J(u) = \operatorname{Inf}\{\lambda \geqq 0 \mid u \in \lambda C\}\,. \tag{1}$$

■

It is well-known that J is a lower semi-continuous convex function. If the set C is bounded, $J(u) = 0$ if and only if $u = 0$. If its interior $\mathring{C}$ contains the origin, J is finite everywhere, and hence continuous. Moreover, J is positively homogeneous of degree one, and C is exactly the set of points where $J \leqq 1$:

$$\forall \lambda \geqq 0\,, \quad J(\lambda u) = \lambda J(u) \tag{2}$$

$$u \in C \Longleftrightarrow J(u) \leqq 1\,. \tag{3}$$

Definition 2. Let C be a closed convex subset of $\mathbf{R}^{2n}$, and u a point in C. The normal cone to C at u is the set:

$$N_C(u) = \{w \in \mathbf{R}^{2n} \mid (w, v - u) \leqq 0 \quad \forall v \in C\} . \tag{4}$$ ■

It is clear that $N_C(u)$ is a closed convex cone, and that $N_C(u) = \{0\}$ if only u belongs to the interior $\mathring{C}$.

Let us assume that $0 \in \mathring{C}$, and denote by S the boundary of C. For any point u in S, the subdifferential $\partial J(u)$ of the gauge is a convex, compact, non-empty subset of $\mathbf{R}^{2n}$, not containing 0, and the following relation holds:

$$\forall u \in S, \ N_C(u) = \bigcup_{\lambda \geqq 0} \lambda \partial J(u) . \tag{5}$$

If the normal cone $N_C(u)$ reduced to a half-line, $\partial J(u)$ can be seen to be a singleton:

$$\big(N_C(u) = e\mathbf{R}_+ \quad \text{with} \quad \|e\| = 1\big) \Longrightarrow \partial J(u) = \left\{\frac{e}{(e, u)}\right\} .$$

Definition 3. Let C be a closed convex subset of $\mathbf{R}^{2n}$, with interior $\mathring{C} \neq \emptyset$ and boundary S. We shall say that it is C^1 if, for every $u \in S$, the normal cone $N_C(u)$ reduces to a half-line. ■

If C is C^1, bounded, and $0 \in \mathring{C}$, then $\partial J(u)$ is a singleton for every $u \in S$. Since J is positively homogeneous of degree one, $\partial J(\lambda u) = \partial J(u)$ for all $\lambda > 0$, so that $\partial J(v)$ will be a singleton for all v lying in $\mathbf{R}^{2n} \backslash \{0\}$. It follows that the function J is C^1 on the (open) region where it is non-zero. In other words, J is C^1 on a neighborhood of S, which enables us to consider the Hamiltonian system $\dot{u} = \sigma J(u)$ and its trajectories on S.

PROPOSITION 4. *Let $C \subset \mathbf{R}^{2n}$ be closed, convex, bounded and C^1, with 0 in its interior. Let U be some neighborhood of S and H: $U \to \mathbf{R}$ a C^1 function such that, for some constant $h \in \mathbf{R}$:*

$$S = \{u \mid H(u) = h\} . \tag{7}$$

Assume that $H'(u) \cdot n_c(u) > 0$ for all $u \in S$, where $n_c(u)$ is the exterior normal vector to C at u (the unit vector in $N_c(u)$). Then the Hamiltonian systems:

$$\dot{u} = \sigma J'(u) , \tag{8}$$

$$\dot{u} = \sigma H'(u) \tag{9}$$

have the same trajectories on S.

Proof. Let H_1, H_2: $U \to \mathbf{R}$ be two C^1 functions constant on S and satisfying:

$$H_i'(u) \cdot n_c(u) > 0 \qquad \text{for all} \quad u \in S , \quad i = 1, 2 . \tag{10}$$

We shall show that Hamiltonian systems (H_1) and (H_2) have the same trajectories on S. The proposition will follow by taking $H_1 = H$ and $H_2 = J$.

The two Hamiltonians H_1 and H_2 being constant on S with nonvanishing gradients $\big(H_i'(u) \neq 0$ for all $u \in S$, $i = 1, 2\big)$, there exists a continuous function $\alpha: S \to \mathbf{R}_+$ such that

$$H_2'(u) = \alpha(u) H_1'(\mathrm{u}) \qquad \text{for all} \quad u \in R \;, \tag{11}$$

$$0 < \alpha_0 \leqq \alpha(u) \leqq \alpha_1 \qquad \text{for all} \quad u \in S \;. \tag{12}$$

Let $\bar{u}$ be a trajectory of (H_1) on the energy level S. From the definition of a trajectory (see II. A) this means that $\bar{u}$ is the equivalence class of a solution u_1 of (H_1) on S. Let $\varphi: \mathbf{R} \to \mathbf{R}$ be the C^1 diffeomorphism defined by

$$\varphi(s) = \int_0^s \frac{dt}{\alpha\big(u_1(t)\big)} \tag{13}$$

and let $u_2 = u_1 \circ \varphi^{-1}$. We have for all $s \in \mathbf{R}$

$$\begin{aligned} u_2(s) &= \dot{u}_1\big(\varphi^{-1}(s)\big) \frac{1}{\varphi'\big(\varphi^{-1}(s)\big)} \\ &= \sigma H_1'\big(u_1(\varphi^{-1}(s))\big) \alpha\big(u_1(\varphi^{-1}(s))\big) \\ &= \sigma H_2'\big(u_2(s)\big) \;, \end{aligned} \tag{14}$$

so that u_2 is a solution of the Hamiltonian system (H_2) on S, and also belongs to the same equivalence class $\bar{u}$ as u_1 (because $u_2 = u_1 \circ \varphi^{-1}$, see II. A). So $\bar{u}$ is a trajectory for (H_2) on S.

We have seen that any trajectory of the Hamiltonian system (H_1) is a trajectory of the Hamiltonian system (H_2). Changing the roles of (H_1) and (H_2) we get that the trajectories of (H_1) and (H_2) on S are the same. ■

We can also see from the proof that a *periodic* trajectory of (H_1) on S is a *periodic* trajectory of (H_2) on S.

Hence we have the definition:

Definition 5. Let $C \subset \mathbf{R}^{2n}$ be a C^1 convex compact set, with non-empty interior. Let S be its boundary, and let u^0 be any point in its interior: $u \in C \backslash S$. Let J be the gauge of the translate $C - u^0$. We define *Hamiltonian trajectories on* S to be the trajectories of the differential system:

$$\dot{u}(t) = \sigma J'\big(u(t) + u^0\big) \qquad \text{on} \quad S \;. \tag{16}$$ ■

By the preceding considerations, this definition does not depend on the particular choice of u^0 in $C \backslash S$, and any other Hamiltonian H with non-vanishing gradient will yield the same trajectories on S.

We are now in a position to state the main result of this paper:

THEOREM 1. *Let C be a C^1 convex compact subset of $\mathbf{R}^n \times \mathbf{R}^n$ with non-empty interior. Let S be its boundary. Assume there is an m-dimensional subspace F of $\mathbf{R}^n$, a point u^0 in $C\backslash S$, and a constant $r > 0$ such that:*

$$\forall u \in S,\ \|u - u^0\| > r\ , \tag{17}$$

$$\forall u \in S,\ \|\Pi_F(u - u_0)\| \leqq r\sqrt{2}\ , \tag{18}$$

where $\Pi_F\colon \mathbf{R}^{2n} \to F \times F$ is the orthogonal projection.

Then there are at least m distinct periodic Hamiltonian trajectories on S. ∎

Proof. Let A be a rotation in $\mathbf{R}^n$ which brings the linear subspace $x_{m+1} = \cdots = x_n = 0$ onto F. Then the translation $u \to u + u_0$ and the rotation $A \times A$ in $\mathbf{R}^n \times \mathbf{R}^n$ bring the general situation to the case where:

$$u^0 = 0\ , \tag{19}$$

$$F \times F = \{u \in \mathbf{R}^{2n} \mid u_{m+1} = \cdots = u_n = 0 \quad \text{in} \quad \mathbf{R}^2\}\ . \tag{20}$$

Since these transformations are canonical, i.e., they preserve the Hamiltonian character of equations, we can assume (19) and (20), so Π_F is just Π_m (see II-(4)).

Now consider the Hamiltonian:

$$H(u) = [J(u)]^2 \tag{21}$$

with J the gauge of C. It is convex, C^1, positively homogeneous of degree two, with an isolated zero at the origin. It follows that:

$$\forall u \in \mathbf{R}^{2n},\ H(u)^{-1/2}u \in S\ . \tag{22}$$

Conditions (17) and (18) now become:

$$\forall u \in \mathbf{R}^{2n},\ \|u\| H(u)^{-1/2} > r\ , \tag{23}$$

$$\forall u \in \mathbf{R}^{2n},\ \|\Pi_m u\| \leqq r\sqrt{2}\ . \tag{24}$$

This can be made more precise, by use of the compactness of S. Define:

$$\beta = \operatorname{Max}\{\|u\|^{-2} \mid u \in S\} < r^{-2}\ , \tag{26}$$

$$\gamma = \operatorname{Min}\{\|\Pi_m u\|^{-2} \mid u \in S\} \geqq \frac{r^{-2}}{2}\ . \tag{27}$$

We have $0 < \beta < 2\gamma$, and $\gamma\|\Pi_m u\|^2 \leqq H(u) \leqq \beta\|u\|^2$. All the assumptions of Theorem II.1 are satisfied. Then so is the conclusion: there are at least m distinct periodic trajectories of $\dot{u} = \sigma H'(u)$ on S. These are the m distinct periodic Hamiltonian trajectories we were looking for. ∎

The assumptions (17) and (18) can be stated geometrically as follows. Let B be the closed ball of center u^0 and radius r, and let B_F be its intersection with $F \times F$. Then what is assumed is that S lies entirely between the ball B and the cylinder with basis $B\sqrt{2}$ and generatrices orthogonal to $F \times F$. The most interesting case is when $m = n$, which means $F = \mathbf{R}^n$:

THEOREM 2. *Let C be a C^1 compact convex subset of $\mathbf{R}^{2n}$ with interior $\mathring{C} \neq \emptyset$ and boundary S. Assume some closed ball $B(u^0; r)$ can be found with:*

$$B(u^0; r) \subset \mathring{C} \subset B(u^0; r\sqrt{2}) \ . \tag{28}$$

Then there are at least n distinct periodic Hamiltonian trajectories on S. ∎

A striking feature of condition (28) is that it is invariant by isometries—but certainly not by canonical transformations, even linear ones. Of course, the conclusion itself, the existence of n distinct periodic Hamiltonian trajectories, will hold for any compact hypersurface $S \subset \mathbf{R}^{2n}$ which can be brought to be the boundary of a convex set C satisfying (28) by a canonical transformation. We have been unable to characterize such hypersurfaces; see Weinstein ([26]) for more light on this problem.

We conclude by indicating that our global result, and Weinstein's local theorem ([24]) have peculiar features, which make them irreducible to each other: the Hamiltonians we are dealing with are C^1, and have to satisfy a geometric condition (28), whereas Weinstein has no such condition, but requires the Hamiltonian to be C^2.

This is more significant that it might seem. We could, for instance, try to prove Theorem II. 1 directly using Weinstein's theorem: since the Hamiltonian involved is positively homogeneous of degree two, periodic trajectories on small energy levels can be transported by homothety to any prescribed energy level. In other words, the global problem follows from the local one. Alas, in this particular case, the Hamiltonian is not C^2 at the origin! Indeed, it is C^2 if and only if it is exactly quadratic, so that we fall back on the linear case.

Conversely, our result will prove that there are n periodic orbits on all small energy levels, provided the Hamiltonian H is C^2 and $\omega_n < 2\omega_1$, where the $\pm i\omega_k$, $s \leqq k \leqq n$, are the eigenvalues of $\sigma H''(0)$, and $0 < \omega_1 \leqq \omega_k \leqq \omega_n$. Indeed, it is well-known that one can find a linear and canonical change of variables which brings $H''(0)$ to the form:

$$\big(H''(0)u,\, u\big) = \sum\nolimits_{i=1}^{n} \frac{\omega_i}{2} (x_i^2 + p_i^2) \ . \tag{29}$$

One can then modify H outside some neighborhood of the origin so that it becomes convex and satisfies estimate (2) of Section II. The result we get in this way is weaker than Weinstein's theorem, which lays no requirement on the ω_k. Note, however, that our condition, $\omega_n < 2\omega_1$, is not of the traditional "non-resonance" type: we could well have $\omega_j = \omega_k$ for some j and k. In other words, this approach is not deterred by "small divisors" problems.

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UNIVERSITÉ DE PARIS-DAUPHINE, FRANCE

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