

ON THE NUMBER OF SUMS AND PRODUCTS

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ABSTRACT

A new lower bound on $\max\{|A + A|, |A \cdot A|\}$ is given, where A is a finite set of complex numbers.

1. Introduction

Let A be a finite subset of complex numbers. The *sum-set* of A is $A + A = \{a + b : a, b \in A\}$, and the *product-set* is $A \cdot A = \{a \cdot b : a, b \in A\}$. Erdős and Szemerédi [7] proved the inequality

$$\max(|A + A|, |A \cdot A|) \geq c|A|^{1+\varepsilon}$$

for a small but positive ε , where A is a subset of integers. They conjectured that

$$\max(|A + A|, |A \cdot A|) \geq c|A|^{2-\delta}$$

for any positive δ . (In this paper, c stands for the general constant. Some authors use the $n \ll m$ or $n \gg m$ notation instead of our $n \leq cm$ or $n \geq cm$.)

After improvements given in [9], [8], and [3], the best bound so far has been obtained by Elekes [4], who showed that $\varepsilon \geq 1/4$ if A is a set of real numbers. His result was extended to complex numbers in [13] and [11]. For further results and related problems, we refer the reader to [1] and [5].

In this paper, we prove the following theorem.

THEOREM 1. *There is a positive absolute constant c such that, for every n -element set A ,*

$$\frac{cn^{14}}{\log^3 n} \leq |A + A|^8 \cdot |A \cdot A|^3,$$

whence $cn^{14/11}/\log^{3/11} n \leq \max\{|A + A|, |A \cdot A|\}$.

Nathanson and Tenenbaum [10] proved that the product set should be large, namely $|A|^{2-\varepsilon}$, if the sumset is at most $3|A| - 4$. Chang [2], and independently Elekes and Ruzsa [6], proved a similar bound if the sumset is at most $c|A|$. As a consequence of Theorem 1, we obtain the following corollary.

COROLLARY 1. *If $|A| = n$ and $|A + A| \leq Cn$, then $|A \cdot A| \geq cn^2/\log n$.*

2. Proof

Our proof is based on the following estimates of the number of incidences between lines and points.

THEOREM 2 (Szemerédi and Trotter [12]). *The maximum number of incidences between n points and m straight lines of the real plane is $O(n^{2/3}m^{2/3} + n + m)$.*

COROLLARY 2 (Szemerédi and Trotter [12]). *Given a set of n points on the real plane, the number of k -rich lines (that is, lines incident to at least k points) is $O(n^2/k^3 + n/k)$.*

In the proof of Theorem 1 we use Theorem 2 and Corollary 2 on Cartesian products only; similar statements are easy to prove for complex lines in the complex plane. (The general case has recently been solved by Tóth [13].) The following lemma has been proved but not published by the author.

LEMMA 1. *Given two sets of complex numbers S_1 and S_2 with sizes $|S_1| = n_1$ and $|S_2| = n_2$, let $S = S_1 \times S_2$ be the Cartesian product. The maximum number of incidences between S and m complex lines of the complex plane is $O((n_1n_2)^{2/3}m^{2/3} + n_1n_2 + m)$.*

Proof of Theorem 1. If $|A \cdot A| = t$, then the number of pairs $(a_i, a_j), (a_u, a_v)$ such that $a_i \cdot a_j = a_u \cdot a_v$ (where $a_i, a_j, a_u, a_v \in A$) is at least cn^4/t . Then the number of pairs $(a_i, a_v), (a_u, a_j) \in A \times A$, where $a_i/a_v = a_u/a_j$, is at least cn^4/t as well. Let us partition the elements of $A \times A$ into classes (lines) L_1, L_2, \dots, L_k using the relation $(a_i, a_j) \sim (a_u, a_v)$ if and only if $a_i/a_j = a_u/a_v$. Each class is a collection of collinear points, and the line through them contains the origin $(0, 0)$. If l_i denotes the size of L_i , then

$$\sum_{i=1}^k \binom{l_i}{2} \geq \frac{cn^4}{t}.$$

We partition these lines into sets C_1, C_2, \dots, C_s ($s \leq \log n^2$) with respect to their ‘squared’ sizes. Then $L_i \in C_j \iff 2^{2(j-1)} < \binom{l_i}{2} \leq 2^{2j}$. There are at most $\log n^2$ sets, so there is at least one set, C_j , which covers many elements. Let X_j be a set of all pairs $((a_\nu, a_\mu), (a_\rho, a_\rho))$ such that there exists L_i in C_j with (a_ν, a_μ) and (a_ρ, a_ρ) both in L_i . Then at least one of the sets X_j is large. Also,

$$|X_j| = |\{(a_\nu, a_\mu), (a_\rho, a_\rho) : (a_\nu, a_\mu) \in L_i, (a_\rho, a_\rho) \in L_i, L_i \in C_j\}| \geq \frac{cn^4}{t \log n},$$

and therefore

$$2^{2j}|C_j| \geq \frac{cn^4}{t \log n}. \tag{2.1}$$

This is the key element of the proof: every point of $A \times A$ is incident to at least $|C_j|$ lines, each of them incident to at least 2^{j-1} points of $(A + A) \times (A + A)$. Indeed, the translated lines $(a_u, a_v) + L$ with L in C_j are incident to (a_u, a_v) , and the points of the lines are points from $(A + A) \times (A + A)$ (see Figure 1). We denote the set of translated lines by \mathcal{L} , as follows:

$$\mathcal{L} = \{(a_u, a_v) + L : L \in C_j, (a_u, a_v) \in A \times A\}.$$

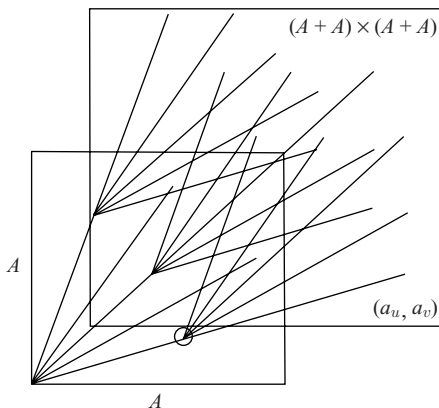


FIGURE 1. Translates of the lines of C_j .

Because of Corollary 2, the number of 2^{j-1} -rich lines (that is, lines incident to at least 2^{j-1} points) on $(A + A) \times (A + A)$ is

$$O\left(\frac{|A + A|^4}{(2^{j-1})^3} + \frac{|A + A|}{(2^{j-1})}\right).$$

The first term is always larger than the second because $|A + A| > |A|$ and $2^{j-1} \leq |A|$. Therefore,

$$|\mathcal{L}| \leq \frac{c|A + A|^4}{(2^{j-1})^3}.$$

Applying the bound from Theorem 2 to the number of incidences I between \mathcal{L} and the n^2 points of $A \times A$, we have

$$I = O(|\mathcal{L}|^{2/3}(n^2)^{2/3} + |\mathcal{L}| + n^2).$$

Therefore,

$$n^2|C_j| \leq c|\mathcal{L}|^{2/3}n^{4/3}, \tag{2.2}$$

or

$$n^2|C_j| \leq c|\mathcal{L}|, \tag{2.3}$$

or

$$n^2|C_j| \leq cn^2. \tag{2.4}$$

The right-hand side of (2.2) is always at least cn^2 , and therefore (2.2) includes case (2.4). The next step is to see that (2.2) covers case (2.3) as well. Let us suppose that, on the contrary,

$$|\mathcal{L}|^{2/3}n^{4/3} < |\mathcal{L}|.$$

Then

$$n^{4/3} < |\mathcal{L}|^{1/3} \rightarrow n^4 < |\mathcal{L}|,$$

but this is not possible, since \mathcal{L} consists of n^2 translates of less than n^2 lines.

Now we are ready for the final step of the proof. It follows from (2.1), that

$$2^{2j-2} \geq \frac{cn^4}{t \log n |C_j|}. \quad (2.5)$$

Putting (2.2) and (2.5) together, we have

$$\begin{aligned} n^2 |C_j| &\leq c |\mathcal{L}|^{2/3} n^{4/3} \leq c \left(\frac{|A + A|^4}{(2^{j-1})^3} \right)^{2/3} n^{4/3} = c \frac{|A + A|^{8/3}}{2^{2j-2}} n^{4/3} \\ &\leq c \frac{|A + A|^{8/3}}{(n^4/t \log n |C_j|)} n^{4/3}, \end{aligned}$$

which gives

$$\frac{cn^{14}}{\log^3 n} \leq |A + A|^8 \cdot t^3,$$

as stated. □

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