1978MNRAS.183..765K

Mon. Not. R. astr. Soc. (1978) **183**, 765–769

On the number of unstable modes of an equilibrium

J. Katz* *Institute of Astronomy, Madingley Road, Cambridge*

Received 1977 November 25; in original form 1977 October 31

Summary. The number of unstable modes of an equilibrium may, in a variety of cases of astrophysical interest, be deduced from topological properties of continuous series of equilibria without having to solve an eigenvalue equation.

Downloaded from https://academic.oup.com/mnras/article/183/4/765/952714 by guest on 20 August 2022

1 Introduction

Poincaré invented a powerful method for separating stable from unstable equilibria. It is based on series of equilibrium configurations. The method has been used by Poincaré (1885) and many others (from Jeans 1932, new edition 1961, to Ledoux 1958), to find stable equilibria of rotating liquid masses and rotating systems of rigid bodies. Wheeler (in Harrisson *et al.* 1965) rederived some of Poincaré's results and found stable configurations of star models at zero temperature (see also Thorne 1966). Bardeen, Thorne & Meltzer (1966) described a method, similar to that of Wheeler, and which applies to hot isentropic stellar models. Lynden-Bell & Wood (1968) found stable configurations of isothermal spheres by applying the method to thermodynamic systems.

All these examples come from astronomy and astrophysics. The method is, however, applicable to very different types of physical systems (see e.g. Thompson & Hunt 1977).

In brief, the method works as follows: a change of stability may occur only where two or more series of equilibria have one equilibrium configuration in common or where two or more series merge into each others. When this happens, stable equilibria may turn into unstable ones; reciprocally, unstable ones may become either stable or more unstable. The nature of this change in stability or instability is usually known by solving an eigenvalue equation. There are, however, known exceptions where the eigenvalue equation need not be solved. For instance, Wheeler (in Harrisson *et al.* 1965) has noted that for zero temperature stars the series of equilibria configurations gives also the number of unstable modes of each equilibrium. A similar situation exists for hot isentropic stellar models (Bardeen, Thorne & Meltzer 1966).

We want to point out that in many more cases than these two, the number of unstable modes may be deduced solely from the topological properties of series of equilibria. General conditions under which this is possible are given below. The method is illustrated with two new examples.

* Permanent address: The Racah Institute of Physics, Hebrew University of Jerusalem, Jerusalem, Israel.

© Royal Astronomical Society • Provided by the NASA Astrophysics Data System

1978MNRAS.183..765K

2 Parameters and conjugate parameters of linear series

We shall very briefly review the basis of Poincaré's analysis so as to formulate our problem precisely. Let the system be described by n variables x^i (i, j only $= 1, 2, \ldots, n$). Let F be a function of the forces such that if the system is in equilibrium, F is at an extremum:

$$\partial_i F = 0, \tag{1}$$

and that an equilibrium is stable if this extremum is a maximum:

$$\partial_{ij} F \, \delta x^i \, \delta x^j < 0, \quad \delta x^i \text{ arbitrary.} \tag{2}$$

Suppose F contains a real parameter, say s, so that all equilibrium configurations are functions of s. Let

$$x^i = X_a^i(s), \quad a, b \text{ only} = 1, 2, \ldots, N, \tag{3}$$

be the multiplicity of solutions of equation (1). The following conditions of continuity will be enough for our purpose: (i) the function F and its derivatives of order 1 and 2 are continuous functions of x^i and s; (ii) the solutions X_a^i are continuous functions of s. These $X_a^i(s)$ generate N 'linear series' or N lines in the $(n+2)$-dimensional space with coordinates (F, s, x^i). Some of these lines may cross each other and have a common point; this is a point of bifurcation. Lines may also merge into each other at some point which is then called a limit point.

We shall now assume that $(-\partial_{ij}F)_a$ has a non-degenerate spectrum of eigenvalues, say $k_{1a} < k_{2a} < \ldots < k_{ha} < \ldots < k_{na}$. In that case one and only one eigenvalue at a time, e.g. k_{ha}, may be equal to zero. A change of stability corresponds to a change of sign of k_{ha}. It may be shown (see Thompson 1977 and references therein) that a change of stability can only happen at a point of bifurcation or at a limit point.

What we shall show is, that in certain cases, the sign of $k_{ha}(s)$, in the vicinity of $s = s_0$ where $k_{ha}(s_0) = 0$, may be deduced from the form of the projection of the line $X_a^i(s)$ on the plane (F, s). The equation of this projection is

$$F_a(s) \equiv F[X_a^i(s), s]. \tag{4}$$

F_a is a continuous function whose physical relevance will become clear in the examples given below. Let $\dot{F}_a(s)$, the derivative of (4), define a conjugate parameter of s, with respect to a given series a. Our conditions of continuity on F and X_a^i imply that

$$\dot{F}_a(s) = (\partial_s F)_a \quad \text{for} \quad s \neq s_0. \tag{5}$$

Thus $\dot{F}_a$ is a continuous function. Its derivative

$$\ddot{F}_a(s) = (\partial_{ss} F)_a + (\partial_{si} F)_a \, \dot{X}_a^i \quad \text{for} \quad s \neq s_0, \tag{6}$$

is also continuous but $\ddot{F}_a$ may, like $\dot{X}_a^i$ itself, be discontinuous at $s = s_0$, as may the left and right derivatives of $\dot{F}_a$ be different at s_0 when they exist. If we take $(-\partial_{ij}F)_a$ to be diagonal, which we may almost always do, equation (6) may be put in the following form with the help of (1):

$$\ddot{F}_a(s) = (\partial_{ss} F)_a + \sum_{i=1}^{n} (\partial_{si} F)_a^2 / k_{ia}(s) \quad \text{for} \quad s \neq s_0. \tag{7}$$

Both $(\partial_{ss}F)_a$ and $(\partial_{si}F)_a$ are continuous functions of s by our assumptions. Equation (7) shows that if there is a bifurcation or a limit point at s_0, where $k_{ha}(s_0) = 0$, the slope of $\dot{F}_a(s_0)$ *may*, but will not necessarily be, infinite. If it is infinite, the value of $\ddot{F}_a$ in the vicinity of s_0 is essentially that of the dominant term from the sum of equation (7); it is of the form

© Royal Astronomical Society • Provided by the NASA Astrophysics Data System

Downloaded from https://academic.oup.com/mnras/article/183/4/765/952714 by guest on 20 August 2022

1978MNRAS.183..765K

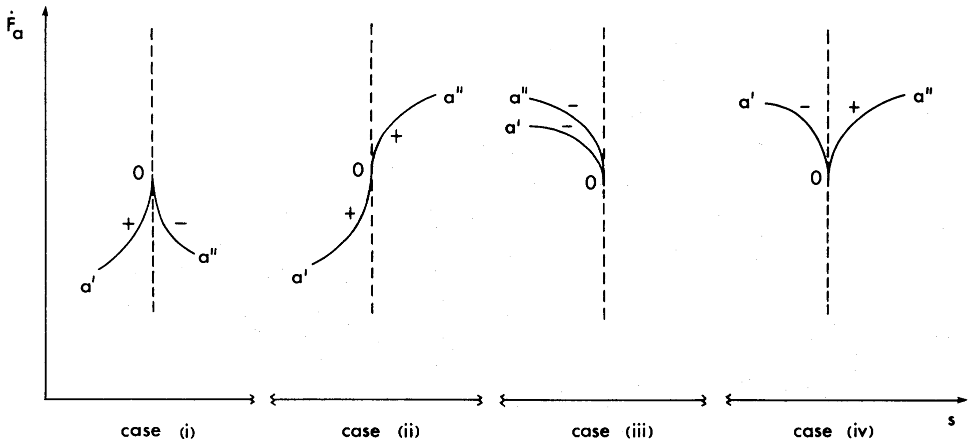

Figure 1. Possible forms of $\dot{F}_a(s)$ near a bifurcation point for the case where the linear series a has a vertical tangent. The line b, which cuts a at point O is not drawn. The sign of the eigenvalue $k_{ha}(s)$, which is equal to zero at point O, is indicated by $a+$ and $a-$ on each branch of a. More general situations may exist in which only one branch a' or a'' has a vertical tangent.

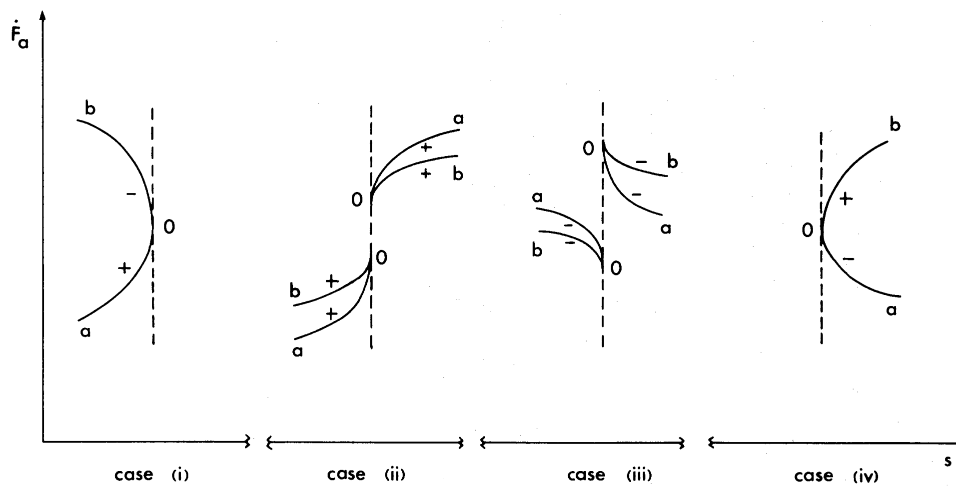

Figure 2. Possible forms of $\dot{F}_a(s)$ near a limit point for the case where two linear series a and b have a vertical tangent at point O where they merge into each other. The sign of $k_{ha}(s)$, which goes to zero at point O, is indicated by $a+$ and $a-$ along a and b. Cases (ii) and (iii) have two subcases. A more general situation may exist in which either a or b alone has a vertical tangent at point O.

A^2/k_{ha}. In that case, the sign of $k_{ha}(s)$ is thus the sign of the slope of the conjugate parameter of equilibrium $\dot{F}_a$ in the vicinity of its vertical tangent. Various possible forms of $\dot{F}_a(s)$ near s_0 are given in Fig. 1 for a bifurcation and in Fig. 2 for a limit point. The signs of $k_{ha}(s)$ are also indicated on the graphs.

3 Examples of application: isothermal spheres

Isothermal spheres with identical particles have been discussed many times; for our purpose a useful reference is Lynden-Bell & Wood (1968). There are three parameters in this system: the energy E, the total mass M and the volume V. The potential of the forces is here the entropy S. We consider only spherically symmetric configurations of regular mean fields. The eigenvalue spectrum of the quadratic form (2) is given by a one-dimensional differential equation, related to radial perturbations of the equilibria and all one-dimensional differential equations with regular eigenfunctions have non-degenerate spectra. (One may ask what the x^i variables are in the present case. The entropy may be expressed as a functional of some

Downloaded from https://academic.oup.com/mnras/article/183/4/765/952714 by guest on 20 August 2022

© Royal Astronomical Society • Provided by the NASA Astrophysics Data System

1978MNRAS.183..765K

field variable $W(x, y, z)$ (see Horwitz & Katz 1977.) Each value of W at each point (x, y, z) is one variable x^i and may vary independently between $\pm\infty$. There is thus an infinite continuous set of x^i's.) Since V is finite, the support of the eigenfunctions is compact and the spectrum of eigenvalues is then discrete. Our previous considerations are thus applicable in the present case.

Let us take the energy E as parameter s; one may as well take M or V in the present example. The parameter conjugate to E is, as we know, the inverse temperature T:

$$\dot{F}_a(s) \equiv (\partial S/\partial E)_{M,V} = T^{-1} \quad \text{with Boltzmann} \quad k = 1. \tag{8}$$

The diagram of T^{-1} as a function of E may be obtained from the curves published in Lynden-Bell & Wood (1968). It is represented in Fig. 3 and is parametrized in terms of the so-called density contrast, i.e. the ratio, say h, between the density in the centre and the density on the boundary of V. The value of h increases as the curve spirals inwards. For high temperatures, $T^{-1} \to 0$, the system behaves like a collisionless gas which is stable. Thus, the branch of the curve between $h = 1$ (at infinity) and $h = 709$ is a branch of stable configurations. By comparing Fig. 3 with cases (i) and (iv) of Fig. 2, we see that between $h = 709$ and 4.5×10^4 the lowest eigenvalue is negative; then between $h = 4.5 \times 10^4$ and 6.3×10^6 the two lowest eigenvalues are negative. As the curve spirals inwards, more and more eigenvalues become negative. Thus, stable configurations exist only for $1 < h < 709$. Lynden-Bell & Wood have found that the system indeed becomes unstable at $h = 709$. The limit 709 was also found by Antonov (1962) with a more complicated technique. What we have shown here is that no stable configuration may exist with $h > 709$. A similar result may be derived by a steepest descent calculation in statistical mechanics (Horwitz & Katz 1978). The technique, which is much more sophisticated than the present one, has the advantage of being applicable to every case, including where the present method fails to work.

Fig. 3 gives us also the limit of stability when the system is in a heat bath rather than being isolated. In this case, the parameters are T^{-1}, M and V. The relevant potential is the Massieu function corresponding to Helmholtz' free energy H, that is $F \equiv (-T^{-1}H)$ and with $s \equiv T^{-1}$, the conjugate variable $\dot{F}_a \equiv (-E)$. The function $(-E)$ of T^{-1} is given by Fig. 3 rotated clockwise by 90°. We see that one stable branch exists for h between 1 and 32.1,

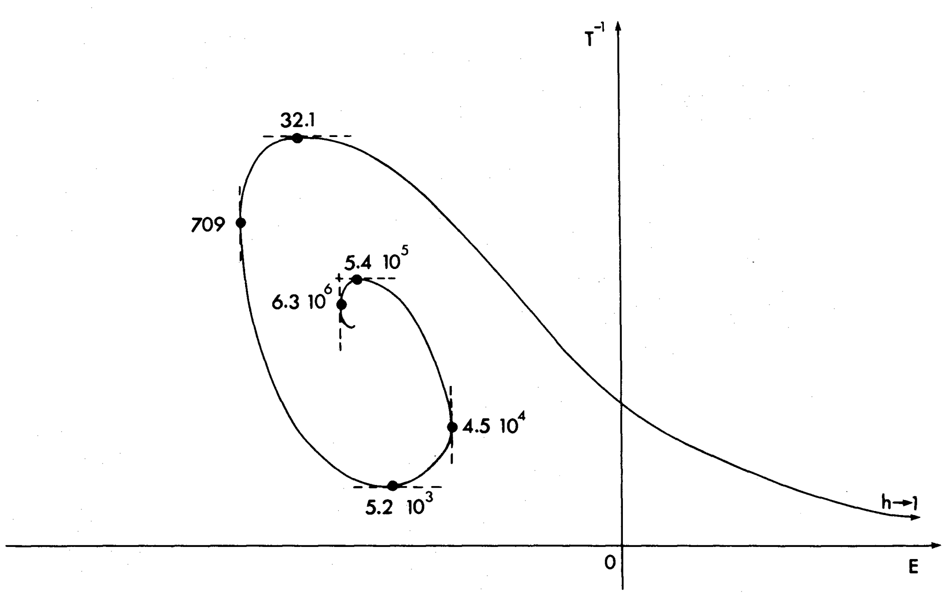

Figure 3. Form of the function $T^{-1}(E)$ for isothermal spheres of fixed mass and volume, according to Lynden-Bell & Wood (1968). Parametrization is in terms of density contrasts h. The values of h are taken from Horwitz & Katz (1978).

Downloaded from https://academic.oup.com/mnras/article/183/4/765/952714 by guest on 20 August 2022

© Royal Astronomical Society • Provided by the NASA Astrophysics Data System

1978MNRAS.183..765K

as already shown by Lynden-Bell & Wood. We find now that isothermal spheres in a heat bath are unstable for any density contrast $h > 32.1$. A second mode of instability sets in at $h = 5.2 \times 10^3$ and a third one at $h = 5.4 \times 10^5$.

4 Conclusions

The present technique for finding limits of stability and the number of unstable modes is remarkably simple and of wide applicability. Our examples and those described in Thorne (1966), show that the method may also be useful. The limitations of this technique are, however, clear. (i) One needs to calculate a number of equilibrium configurations and this is not always easy to do. (ii) Bifurcations, as well as limit points, have to show up through a vertical tangent of the conjugate parameter; this is of course not always the case. (iii) One stable equilibrium has to be known. This may often be the case, but for instance in spherical clusters with an energy cut-off, it is not obvious whether any equilibrium configuration is *a priori* stable or not (see Katz & Horwitz 1978). (iv) The spectrum has to be discrete and non-degenerate. This will normally not be the case in thermodynamic systems that are not gravitationally self-bound.

Acknowledgments

The author is grateful to D. Lynden-Bell for helpful discussions and illuminating remarks. He is also pleased to acknowledge a useful conversation with J. M. T. Thompson which helped in improving the present work.

References

Antonov, V. A., 1962. *Vest. Leningr. Univ.*, **7**, 135.
Bardeen, J. M., Thorne, K. S. & Meltzer, D. W., 1966. *Astrophys. J.*, **145**, 505.
Harrisson, B. K., Thorne, K. S., Wakano, M. & Wheeler, J. A., 1965. *Gravitation theory and gravitational collapse*, University Press, Chicago.
Horwitz, G. & Katz, J., 1977. *Astrophys. J.*, **211**, 226.
Horwitz, G. & Katz, J., 1978. *Astrophys. J.*, **222**.
Jeans, J. H., 1961. *Astronomy and cosmogony*, Dover, New York.
Katz, J. & Horwitz, G., 1978. *Astrophys. J.*, **222**.
Ledoux, P., 1958. Stellar stability, in *Handbuch der Physik*, **51**, 605.
Lynden-Bell, D. & Wood, R., 1968. *Mon. Not. R. astr. Soc.*, **138**, 495.
Poincaré, H., 1885. *Acta Math.*, **7**, 259.
Thompson, J. M. T., 1977. Bifurcational aspects of catastrophe theory, in *Ann. N.Y. Acad. Sci.*, in press.
Thompson, J. M. T. & Hunt, G. W., 1977. *Interdisc. Sci. Rev.*, **2**, 240.
Thorne, K. S., 1966. In *High energy astrophysics*, eds DeWitt, C., Schatzman, E. & Véron, P., Gordon and Breach, New York.

Downloaded from https://academic.oup.com/mnras/article/183/4/765/952714 by guest on 20 August 2022

© Royal Astronomical Society • Provided by the NASA Astrophysics Data System

1978MNRAS.183..765K

Downloaded from https://academic.oup.com/mnras/article/183/4/765/952714 by guest on 20 August 2022

© Royal Astronomical Society • Provided by the NASA Astrophysics Data System