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ON THE NUMERICAL RANK OF THE OFF-DIAGONAL BLOCKS OF SCHUR COMPLEMENTS OF DISCRETIZED ELLIPTIC PDEs*

S. CHANDRASEKARAN†, P. DEWILDE‡, M. GU§, AND N. SOMASUNDERAM†

Abstract. It is shown that the numerical rank of the off-diagonal blocks of certain Schur complements of matrices that arise from the finite-difference discretization of constant coefficient, elliptic PDEs in two spatial dimensions is bounded by a constant independent of the grid size. Moreover, in three-dimensional problems the Schur complements are shown to have off-diagonal blocks whose numerical rank is a slowly growing function.

Key words. Schur complements, elliptic PDEs, fast algorithms, numerical rank

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1. Introduction. In this paper we investigate the numerical ranks of the off-diagonal blocks of the Schur complements of some matrices that arise from the finite-difference discretization of constant coefficient, elliptic partial differential equations (PDEs). In particular we prove that for two-dimensional problems, under some assumptions about the domain and boundary conditions, the numerical rank of the off-diagonal blocks of certain Schur complements is bounded by a constant that is independent of the grid size. We also show that for three-dimensional problems, the Schur complements exhibit a "Hierarchically semiseparable structure" [3], under an appropriate ordering of the grid. This in turn implies that fast direct numerical solvers can be built for these problems, though the details of the construction are outside the scope of this paper. We emphasize that we are aiming at showing a matrix property in this paper rather than deriving a new algorithm. We exhibit results for both Dirichlet- and Neumann-type problems.

2. The problem. Consider the model problem in two dimensions:

$$\frac{\partial}{\partial x}\left(p(x,y)\frac{\partial}{\partial x}u(x,y)\right)+\frac{\partial}{\partial y}\left(q(x,y)\frac{\partial}{\partial y}u(x,y)\right)-r(x,y)u(x,y)=f(x,y)$$

for $(x,y)\in\Omega$, where $\Omega=[0,1]\times[0,1]$ with homogeneous Dirichlet boundary conditions. The standard five-point or nine-point discretization of this problem on an

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†Department of Electrical and Computer Engineering, University of California, Santa Barbara, CA 93106 (shiv@ece.ucsb.edu, naveen@umail.ucsb.edu). The first author's research was supported by the National Science Foundation under grant 0515320. The fourth author's research was supported by the National Science Foundation under grant 0830604.
‡Faculty of Electrical Engineering, TU Delft, 2628 CD Delft, The Netherlands (p.dewilde@ewi.tudelft.nl).
§Department of Mathematics, University of California, Berkeley, CA 94720 (mgu@math.berkeley.edu).

$n \times m$ uniform grid leads to a system of linear algebraic equations of the form

$$Gu = \begin{pmatrix} A_0 & B_1 & & & \\ C_1 & A_1 & B_2 & & \\ & C_2 & \ddots & \ddots & \\ & & \ddots & \ddots & B_{m-1} \\ & & & C_{m-1} & A_{m-1} \end{pmatrix} \begin{pmatrix} u_0 \\ u_1 \\ \vdots \\ \vdots \\ u_{m-1} \end{pmatrix} = \begin{pmatrix} f_0 \\ f_1 \\ \vdots \\ \vdots \\ f_{m-1} \end{pmatrix},$$

where we have assumed that u_i are the discretized unknowns along the ith column of the $n \times m$ grid. In this case each of A_i, B_i, and C_i is an $n \times n$ matrix, while G itself is an $nm \times nm$ matrix. Furthermore each of the A_i, B_i, and C_i are tridiagonal matrices.

One way to find the unknowns u_i is to do sparse Gaussian elimination on the coefficient matrix G, assuming that the associated LU factorization exists. Of course, in this naive column ordering of the unknowns, when we eliminate the block entry C_1 we get in the position occupied by A_1 the new block $S_1 = A_1 - C_1 A_0^{-1} B_1$. Observe that even though all the individual matrices on the right-hand side of the expression are tridiagonal matrices, A_0^{-1} is not, and hence S_1 is a dense (nonsparse) matrix.

In the next step of Gaussian elimination we would use S_1 as the pivot block to eliminate C_2. Now in the position occupied by the block A_2 we would get the matrix $S_2 = A_2 - C_1 S_1^{-1} B_1$. Again, since S_1 is a dense matrix, in general S_1^{-1} will be a dense matrix, and therefore S_2 will also be a dense matrix.

What this implies is that during LU factorization of the sparse matrix G, we will produce fill-in quickly that causes us to compute the inverses (and hence LU factorizations) of dense $n \times n$ matrices. If we assume that these dense matrices have no structure, then we would need $O\left(n^3\right)$ flops for that operation alone. (Here we are assuming that we will not use any of the fast matrix-matrix multiplication techniques (like those of Strassen [15]) to speed up the LU factorization of dense matrices.) Therefore it follows that one would require at least $O\left(n^3\right)$ flops to compute sparse LU factorization of G. Though this argument is not rigorous, as we do not account for the different possible orderings of the unknowns and the equations, a more thorough analysis has been carried out by Lipton, Rose, and Tarjan [10], who show that this lower bound result is essentially correct.

Now, if one wishes to find an algorithm to factor G in linear time ($O(nm)$ flops), then one possible approach is to find some hidden structure in the dense matrices S_1, S_2, etc., that is produced during LU factorization, and to exploit this structure to speed up the calculation of the inverses of these matrices, and the application of these same inverses to other matrices. This is precisely the strategy suggested by many researchers, including Hackbusch [8], Gohberg, Kailath, and Koltracht [6], and Greengard and Rokhlin [7].

In fact it has been conjectured that if one looks at the off-diagonal blocks of these matrices (S_1, S_2, etc.), then their ϵ-rank is going to be small. This conjecture has been justified by the fact that, for example, S_2^{-1}can be viewed approximately (especially in the limit as n becomes large) as a subblock of the discretized Green's function of the original PDE. It is known from the theory of elliptic PDEs that under some mild constraints the Green's function will be smooth away from the diagonal singularity (see, for example, Folland [5]). This in turn has been taken to imply that the numerical ranks of the off-diagonal blocks of S_2^{-1} would be small. This conjecture has been observed to be experimentally true by us and many other researchers in

a wide variety of cases. Some related theoretical results can be found in work by Hackbusch and Bebendorf [1, 2].

In this paper we try to prove the conjecture for the constant coefficient case (for an earlier attempt see [4]). In particular, we consider off-diagonal blocks that touch the diagonal. The theory of Hackbush and Bebendorf shows the existence of low rank off-diagonal blocks away from the diagonal. Although their theory is applicable to a wider class of matrices than that considered in this paper, it produces weaker bounds on the rank of the off-diagonal blocks.

To be more precise let us first make the following definitions:

$$\begin{aligned} S_0 &= A_0, \\ S_{i+1} &= A_{i+1} - C_i S_i^{-1} B_i. \end{aligned}$$

We call S_i the *Schur complements* of the matrix G.

Furthermore, if B is a square matrix and

$$B = \begin{array}{c} \\ n_1 \\ n_2 \end{array} \begin{array}{c} \begin{array}{cc} n_1 & n_2 \end{array} \\ \begin{pmatrix} B_{00} & B_{01} \\ B_{10} & B_{11} \end{pmatrix} \end{array}$$

is a block 2×2 partitioning of B such that both B_{00} and B_{11} are also square matrices, then we call B_{01} and B_{10} *Hankel blocks* of B. In brief, a Hankel block of a square matrix is an off-diagonal block that touches the main diagonal and stretches all the way to either the top-right or bottom-left corner of the matrix.

In the case of three-dimensional problems, we cannot hope to look at just the off-diagonal blocks, since the rank of such blocks would be expected to grow linearly in the grid size along one spatial dimension. The standard discretization of the three-dimensional Laplacian on an $n \times m \times k$ uniform grid leads to a system of the form

$$G = \begin{pmatrix} \tilde{A}_0 & B_1 & & & \\ C_1 & \tilde{A}_1 & B_2 & & \\ & C_2 & \ddots & \ddots & \\ & & \ddots & \ddots & B_{k-1} \\ & & & C_{k-1} & \tilde{A}_{k-1} \end{pmatrix},$$

where $\tilde{A}_i$ are block tridiagonal matrices. The Schur complements are given by

$$\begin{aligned} S_0 &= \tilde{A}_0, \\ S_{i+1} &= \tilde{A} - C_i S_i^{-1} B_i. \end{aligned}$$

The Schur complements are of size $nm \times nm$, and their inversion would require $\mathrm{O}(n^3 m^3)$ flops at each step. However, we aim to show that these Schur complements have a much finer structure under an appropriate ordering of the grid. To this effect, we define the following. A *strip-row (column) Hankel block* of a matrix A is any row (column) block of A that excludes the diagonal block. For example, let A be a block matrix of the form

$$\begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix}.$$

Then a strip-row Hankel block of A is

$$\begin{pmatrix} A_{21} & A_{23} \end{pmatrix},$$

and a strip-column Hankel block of A is

$$\begin{pmatrix} A_{12} & A_{32} \end{pmatrix}^T.$$

One more definition. The *ϵ-rank* of a matrix A is defined to be the number of singular values of A that are bigger than ϵ. We also note that $\|A\|$ will always denote the largest singular value of A, and $\kappa(A)$ will denote the product of the largest and smallest singular values of A.

With these definitions we can state the main goals of this paper. In the two-dimensional case, we would like to investigate the ϵ-rank of the Hankel blocks of S_i for large values of i when n itself is very large. In the three-dimensional case, we shall consider the strip-row and strip-column Hankel blocks of the Schur complements in the limit of large i and n.

3. Simpler problem. In the interest of clarity we begin by first considering the simpler problem of the two-dimensional Laplacian with Dirichlet conditions. In particular we make the assumption that the grid is $n \times \infty$; that is, it is infinite in one direction. As already stated we will be assuming that the PDE has constant coefficients. So in this section we are going to assume that G looks like this:

$$G = \begin{pmatrix} A_0 & B & & \\ C & A & B & \\ & C & A & \ddots \\ & & \ddots & \ddots \end{pmatrix}.$$

Note that G is an infinite matrix (on one end) now. The Schur complements that we intend to look at now are

$$\begin{aligned} S_0 &= A_0, \\ S_{i+1} &= A - CS_i^{-1}B. \end{aligned}$$

The question we are interested in investigating in this section is, what is the ϵ-rank of the Hankel blocks of S_∞?

It turns out that S_∞ is not always guaranteed to exist in the usual sense. We therefore make the following additional assumptions.

Assumption 1. Every leading principal submatrix of G is Hermitian positive-definite.

Assumption 2. A_0, A, and B commute with each other. We will denote their common unitary eigenvector matrix by V.

First, observe that under these assumptions $C = B^H$ and hence commutes with A_0 and B. Furthermore, it follows that the S_i are Hermitian positive-definite matrices that also commute with A and B. Next we make an important observation.

THEOREM 1. *$A^2 - 4B^HB$ is Hermitian positive-definite.*

To prove this theorem we need a very standard result about tridiagonal Toeplitz matrices (see Carl Meyer's book [11], for example).

THEOREM 2. *If A is an $n \times n$ tridiagonal Toeplitz matrix with*

$$A = \begin{pmatrix} a & b & & & \\ c & a & b & & \\ & c & \ddots & \ddots & \\ & & \ddots & \ddots & b \\ & & & c & a \end{pmatrix},$$

then the eigenvalues of A are given by

$$\lambda_j = a + 2b\sqrt{\frac{c}{b}} \cos\left(\frac{j\pi}{n+1}\right),$$

where $1 \le j \le n$, and the corresponding eigenvector is given by

$$v_j = \begin{pmatrix} \left(\frac{c}{b}\right)^{1/2} \sin\left(\frac{1j\pi}{n+1}\right) \\ \left(\frac{c}{b}\right)^{2/2} \sin\left(\frac{2j\pi}{n+1}\right) \\ \left(\frac{c}{b}\right)^{3/2} \sin\left(\frac{3j\pi}{n+1}\right) \\ \vdots \\ \left(\frac{c}{b}\right)^{n/2} \sin\left(\frac{nj\pi}{n+1}\right) \end{pmatrix}.$$

Now we can present the proof of Theorem 1. Since every leading principal submatrix of G is positive-definite it follows that

$$H = \begin{pmatrix} A & B & & & \\ B^H & A & B & & \\ & B^H & \ddots & \ddots & \\ & & \ddots & \ddots & B \\ & & & B^H & A \end{pmatrix}$$

is always positive-definite. Now, utilizing the fact that A and B are diagonalized by V, we can compute the eigenvalues of H using Theorem 2. Doing the straightforward calculation, we find that the eigenvalues of the $np \times np$ matrix H are given by

$$\lambda_{n(i-1)+j}(H) = \lambda_i(A) + 2\lambda_i(B) \cos\left(\frac{j\pi}{p+1}\right), \quad 1 \le j \le p, \quad 1 \le i \le n.$$

Since H is positive-definite for all finite $p > 1$, we have that

$$\lambda_i(A) > 2|\lambda_i(B)|,$$

which establishes Theorem 1. Note that another way to compute the eigenvalues of H is to just observe that since A and B commute with each other, they essentially behave like scalars (except that a nonzero B is not necessarily invertible) and that the proof of Theorem 2 carries over to this case.

Model problems. We next observe that Assumptions 1 and 2 are true when the PDE is just Poisson's equations. If we discretize the Laplacian with the five-point stencil, then we obtain

$$A_0 = A = \begin{pmatrix} 4 & -1 & & & \\ -1 & 4 & -1 & & \\ & -1 & \ddots & \ddots & \\ & & \ddots & \ddots & -1 \\ & & & -1 & 4 \end{pmatrix}$$

and $B = I$. The truth of Assumption 1 is well known and follows from the explicit calculation of the eigenvalues of H in the proof of Theorem 1. The truth of Assumption 2 follows from the standard result in Theorem 2.

What about the case when we discretize the Laplacian using a nine-point stencil on a uniform grid with finite-difference? In this case

$$A_0 = A = \begin{pmatrix} 20 & -4 & & & \\ -4 & 20 & -4 & & \\ & -4 & \ddots & \ddots & \\ & & \ddots & \ddots & -4 \\ & & & -4 & 20 \end{pmatrix},$$

while

$$B = \begin{pmatrix} -4 & -1 & & & \\ -1 & -4 & -1 & & \\ & -1 & \ddots & \ddots & \\ & & \ddots & \ddots & -1 \\ & & & -1 & -4 \end{pmatrix}.$$

The validity of Assumptions 1 and 2 follows as before from Theorem 2.

Now we can state the main theorem of this section.

THEOREM 3. *Under the stated assumptions, S_∞ exists and the ϵ-ranks of the Hankel blocks of S_∞ are bounded by*

$$r\left(1 + 8\ln^4\left(\frac{3\|A\|}{\epsilon}\right)\right),$$

where r is the maximum Hankel block rank of A, B, and C.

The proof of this theorem will occupy the rest of this section. Observe that for five-point and nine-point stencil discretizations $r = 1$.

We begin by looking at the operator M that produces S_{i+1} from S_i:

$$S_{i+1} = M(S_i) = A - CS_i^{-1}B.$$

We observe that since $A_0 = S_0$, A, B, and C share the same eigenvector matrix and are diagonalizable, it follows that S_i is also diagonalizable and shares the same eigenvector matrix. Therefore it follows that A, B, C, and S_i all commute with each other. Since these matrices commute with each other, and since they are the only

ones that appear in our expressions, it helps to think of them as scalar quantities. The only significant way in which they differ from scalars in our algebraic expressions is that a nonzero expression may not be invertible, while that is not possible for a scalar. Since that case will never arise for us, we will switch to a more scalar-like expression for writing inverses of matrices that more clearly expresses the mutual commutativity of our expressions. Hence we write

$$S_{i+1} = M(S_i) = \frac{AS_i - CB}{S_i}.$$

In this form it is clear that M is a Möbius transform (see Needham's book [12] for more information), and it is well known how to analyze discrete dynamical systems whose iterator is a Möbius transform. We proceed in the standard way. For each S_i we define two matrices P_i and Q_i that are diagonalizable by V, and such that

$$S_i = \frac{P_i}{Q_i}$$

and

$$\begin{aligned} P_0 &= A_0, \\ Q_0 &= I, \\ \begin{pmatrix} P_{i+1} \\ Q_{i+1} \end{pmatrix} &= \begin{pmatrix} A & -CB \\ I & 0 \end{pmatrix} \begin{pmatrix} P_i \\ Q_i \end{pmatrix}. \end{aligned}$$

We first observe that the P_i and the Q_i are well defined. To establish that indeed $S_i = P_i/Q_i$ we need to establish that Q_i is invertible for all finite i. To see this, first observe that $Q_{i+1} = P_i$. So we need to establish that P_i remains invertible for all finite i. This follows immediately from induction and the formula

$$P_{i+1} = S_{i+1}P_i.$$

We refer to the iterator matrix representation of M as T, where

$$T = \begin{pmatrix} A & -CB \\ I & 0 \end{pmatrix}.$$

It is obvious now that the question of finding S_∞ reduces to the question of finding P_∞ and Q_∞, which is much easier as we only need to analyze a linear iterator. We proceed in the standard way and first compute the eigendecomposition of T. This is not as hard as it looks since the entries in the block 2×2 representation of T are essentially diagonal matrices. Define the two matrices

$$\begin{aligned} \Xi_+ &= \frac{A + \sqrt{A^2 - 4CB}}{2}, \\ \Xi_- &= \frac{A - \sqrt{A^2 - 4CB}}{2}. \end{aligned}$$

Now we make some crucial observations about these two matrices. Observe that the eigenvector matrix for these two is still V and that they are clearly diagonalizable. Next, since the eigenvalues of $A^2 - 4CB$ and A are strictly positive real numbers, it follows that the eigenvalues of Ξ_+ are bigger in magnitude than the corresponding eigenvalues (belonging to the same eigenvector) of Ξ_-.

By direct computation one can verify the following block eigendecomposition of the iterator matrix T:

$$T = \begin{pmatrix} \Xi_+ & \Xi_- \\ I & I \end{pmatrix} \begin{pmatrix} \Xi_+ & \\ & \Xi_- \end{pmatrix} \begin{pmatrix} \Xi_+ & \Xi_- \\ I & I \end{pmatrix}^{-1}.$$

Since

$$\begin{pmatrix} P_i \\ Q_i \end{pmatrix} = T^i \begin{pmatrix} P_0 \\ Q_0 \end{pmatrix},$$

is just the well-known power iteration, we see that P_∞ and Q_∞ will be a block multiple of one of the block eigenvectors of T. Since we have already established that Ξ_+ has the larger eigenvalues in magnitude, we only need to check that the starting block vector

$$\begin{pmatrix} P_0 \\ Q_0 \end{pmatrix} = \begin{pmatrix} A_0 \\ I \end{pmatrix}$$

has a nonzero oblique projection on the corresponding block eigenvector

$$\begin{pmatrix} \Xi_+ \\ I \end{pmatrix},$$

which is true. Hence we can say immediately that

$$S_\infty = \frac{P_\infty}{Q_\infty} = \frac{\Xi_+}{I} = \frac{A + \sqrt{A^2 - 4CB}}{2}. \tag{3.1}$$

We now make some trivial observations.

LEMMA 1. *Let*

$$A = \begin{array}{c} \\ n_1 \\ n_2 \end{array} \begin{array}{c} \begin{array}{cc} n_1 & n_2 \end{array} \\ \begin{pmatrix} A_{00} & A_{01} \\ A_{10} & A_{11} \end{pmatrix} \end{array} \quad \textit{and} \quad B = \begin{array}{c} \\ n_1 \\ n_2 \end{array} \begin{array}{c} \begin{array}{cc} n_1 & n_2 \end{array} \\ \begin{pmatrix} B_{00} & B_{01} \\ B_{10} & B_{11} \end{pmatrix} \end{array},$$

$$C = AB = \begin{array}{c} \\ n_1 \\ n_2 \end{array} \begin{array}{c} \begin{array}{cc} n_1 & n_2 \end{array} \\ \begin{pmatrix} C_{00} & C_{01} \\ C_{10} & C_{11} \end{pmatrix} \end{array},$$

and

$$D = A + B = \begin{array}{c} \\ n_1 \\ n_2 \end{array} \begin{array}{c} \begin{array}{cc} n_1 & n_2 \end{array} \\ \begin{pmatrix} D_{00} & D_{01} \\ D_{10} & D_{11} \end{pmatrix} \end{array};$$

then $\operatorname{rank}(C_{01}) \le \operatorname{rank}(A_{01}) + \operatorname{rank}(B_{01})$ *and* $\operatorname{rank}(D_{01}) \le \operatorname{rank}(A_{01}) + \operatorname{rank}(B_{01})$.

Using this lemma we see that in order to get an upper bound on the ϵ-rank of the Hankel blocks of S_∞ we need a theorem that connects the ϵ-rank of the Hankel block of the square root of a matrix with the rank of the Hankel blocks of the matrix itself. Here is one such theorem.

THEOREM 4. *Let A be a diagonalizable positive-definite matrix and let $B = \sqrt{A}$. Furthermore, suppose that the rank of every Hankel block of A is at most r. Then the* ϵ-rank *of any Hankel block of B is at most* $2r \ln^4 \left(\frac{3\sqrt{\|A\|}}{\epsilon}\right)$.

This is the key technical theorem in this paper. Its proof in turn depends on a highly acclaimed result of Newman (see Petrushev and Popov [14]), which goes as follows.

THEOREM 5 (Newman [13]). *For every integer $p \geq 5$ there is a rational function of order p which approximates the function $\sqrt{x}$ to an accuracy better than $3e^{-\sqrt{p}}$ for all $0 \leq x \leq 1$.*

Note that the interval of approximation includes the singular point 0. An elementary proof of this theorem can be found in the aforementioned book by Petrushev and Popov.

We also need a known result that links the rank of the Hankel blocks of a matrix to that of its inverse.

LEMMA 2. *Let*

$$A = \begin{array}{c} \\ n_1 \\ n_2 \end{array} \begin{array}{c} \begin{array}{cc} n_1 & n_2 \end{array} \\ \begin{pmatrix} A_{00} & A_{01} \\ A_{10} & A_{11} \end{pmatrix} \end{array}$$

be an invertible matrix and let

$$A^{-1} = \begin{array}{c} \\ n_1 \\ n_2 \end{array} \begin{array}{c} \begin{array}{cc} n_1 & n_2 \end{array} \\ \begin{pmatrix} B_{00} & B_{01} \\ B_{10} & B_{11} \end{pmatrix}. \end{array}$$

Then $\operatorname{rank}(A_{01}) = \operatorname{rank}(B_{01})$.

We will indicate a short proof of Lemma 2 now. First observe that if A_{00} and A_{11} were invertible, then, using the LU factorization of A, we could prove that both B_{00} and B_{11} would be invertible too. Therefore, in this case, we can easily establish the lemma from the equation

$$A_{00}B_{01} + A_{01}B_{11} = 0.$$

Now, by using continuity, we can finish the proof in the general case.

At this point we can establish Theorem 4. From Newman's theorem it follows that to approximate the square-root function uniformly on the interval $[0, \|A\|]$ to an accuracy of ϵ or better, we can use a rational function of order p, where

$$p = \ln^2 \left(\frac{3\sqrt{\|A\|}}{\epsilon} \right).$$

Call this rational function s. Now we can use s to approximate the square root of the matrix A as $s(A) \approx \sqrt{A}$. Since the rational function will contain, in the worst case, a polynomial of degree p in the numerator and another in the denominator, we see, using Lemmas 1 and 2, that the rank of the Hankel blocks of $s(A)$ will be at most $2rp^2$. Now substituting the value of p we obtain the proof of Theorem 4 since $\|s(A) - \sqrt{A}\| \leq \epsilon$.

Now, we can apply Theorem 4 to finish the proof of Theorem 3. We apply it, along with Lemma 1, to the expression given in (3.1) to obtain the following bound on the ϵ-rank of every Hankel block of S_∞:

$$r \left(1 + 8 \ln^4 \left(\frac{3\|A\|}{\epsilon} \right) \right),$$

where r is the maximum Hankel block rank of A, B, and C. This finishes the proof of Theorem 3.

4. The $n \times m$ grid. We are now ready to study the ϵ-rank of the Hankel blocks of S_m when the underlying PDE has been discretized on a uniform $n \times m$ grid, with the unknowns ordered in column-major order. Again we will concentrate only on the constant-coefficient case. Now we must explicitly keep track of the size of each matrix. To facilitate this we adopt the convention of denoting the $n \times n$ matrix A as ${}_nA$; that is, we put the size as a left subscript on the name of the matrix.

With this notation we consider the matrix

$$ {}_{nm}G = \begin{pmatrix} {}_nA_0 & {}_nB & & & \\ {}_nC & {}_nA & {}_nB & & \\ & {}_nC & {}_nA & \ddots & \\ & & \ddots & \ddots & {}_nB \\ & & & {}_nC & {}_nA \end{pmatrix}_{nm \times nm}. $$

Note that each of the subblocks is an $n \times n$ matrix, but that ${}_{nm}G$ is itself a block $m \times m$ matrix. Usually for PDEs m and n will be linearly related to each other. Therefore, we will assume that there is a constant l such that $m = ln$. As before we define the $n \times n$ Schur complements of ${}_{nm}G$ as

$$ {}_nS_{i+1} = {}_nA - {}_nC\, {}_nS_i^{-1}\, {}_nB, $$

with ${}_nS_0$ depending upon the boundary conditions. Now, we can state the main concern of this paper. What is the ϵ-rank of the Hankel blocks of ${}_nS_m$ as n approaches infinity? Note that as n approaches infinity m also approaches infinity. However, ${}_\infty S_\infty$ is not a sensible object to look at anymore. The problem is that as n gets bigger the size of ${}_nS_m$ also gets bigger. Hence, we no longer have the luxury of studying the limit point of a sequence of matrices in a fixed space. Instead, the best we can hope for is some kind of asymptotic result.

With that in mind we make the following definitions:

$$ {}_n\Xi_+ = \frac{{}_nA + \sqrt{{}_nA^2 - 4\,{}_nC\,{}_nB}}{2}, $$
$$ {}_n\Xi_- = \frac{{}_nA - \sqrt{{}_nA^2 - 4\,{}_nC\,{}_nB}}{2}. $$

Note that these are just the fixed points of the Schur complements if we let m approach infinity while holding n fixed, as described in section 3. Furthermore, as shown in that section the ϵ-ranks of the Hankel blocks of ${}_n\Xi_+$ are at most

$$ r_n \left(1 + 8 \ln^4 \left(\frac{3 \| {}_nA \|}{\epsilon} \right) \right), $$

where r_n is the maximum rank of any of the Hankel blocks of ${}_nA$ and ${}_nB$.

For the five-point stencil discretization of the Laplacian $r_n = 1$ and $\| {}_nA \| \leq 6$. For the nine-point stencil discretization of the Laplacian $r_n = 1$ and $\| {}_nA \| \leq 28$. Thus, in the model problems, the ϵ-ranks of the Hankel blocks of ${}_n\Xi_+$ are bounded by a constant as n approaches infinity.

What we will show in the rest of this paper is that under assumptions that hold true for the model problems, ${}_nS_m$ approaches closer and closer to ${}_n\Xi_+$ as n gets bigger and bigger.

Also, define ${}_nD_A$, ${}_nD_B$, ${}_nD_C$, ${}_nD_m$ to be diagonal matrices of the eigenvalues of ${}_nA$, ${}_nB$, ${}_nC$, and ${}_nS_m$. Let

$$
\begin{aligned}
{}_nX_+ &= \frac{{}_nD_A + \sqrt{{}_nD_A{}^2 - 4_nD_C\,{}_nD_B}}{2},\\
{}_nX_- &= \frac{{}_nD_A - \sqrt{{}_nD_A{}^2 - 4_nD_C\,{}_nD_B}}{2}.
\end{aligned}
$$

First we make our assumptions explicit.

4.1. Assumptions.

Assumption 3. ${}_nS_0$, ${}_nA$, ${}_nB$, and ${}_nC$ commute with each other. We will denote their common unitary eigenvector matrix by ${}_nV$.

Assumption 4. There is a constant $W < \infty$ such that $\|{}_nA\| \le W$, $\|{}_nB\| \le W$, $\|{}_nC\| \le W$, $\|{}_nV\| \le W$, and $\|{}_nV^{-1}\| \le W$ for all n.

Assumption 5. There is a constant $r < \infty$ such that the rank of the off-diagonal blocks of ${}_nA_0$, ${}_nA$, and ${}_nB$ are all bounded by r for all n.

Assumption 6. ${}_nA$, ${}_nC{}_nB$, and ${}_nA^2 - 4_nC{}_nB$ are positive-definite.

Assumption 7. There is a constant $\mu > 0$ such that ${}_nS_0 > \mu{}_n\Xi_+ + (1-\mu){}_n\Xi_-$.

Assumption 8. There is a constant $0 \le \gamma < 1$ such that $\lim_{n\to\infty} \|{}_n\Gamma^{ln}\| \le \gamma$, where

$$
{}_n\Gamma = \frac{{}_nX_-}{{}_nX_+}.
$$

Note that Assumption 6 implies that

$$
{}_nX_+ - {}_nX_- > 0 \tag{4.1}
$$

and Assumption 7 and (4.1) imply

$$
{}_nD_0 - {}_nX_- > 0. \tag{4.2}
$$

By Assumptions 4 and 6, we have that ${}_nX_+$ and ${}_nX_-$ are bounded, and ${}_n\Gamma$ is bounded between 0 and 1.

LEMMA 3. *With* ${}_n\Gamma$ *defined as above, we have*

$$
\lim_{n\to\infty} \left\| {}_n\Gamma^{ln}(I - {}_n\Gamma) \right\| = 0.
$$

This lemma is an elementary consequence of the fact that ${}_n\Gamma$ has nonnegative eigenvalues between 0 and 1. To see this, observe that

$$
0 \le x^{ln}(1-x) \le x^n(1-x) \le \left(\frac{n}{n+1}\right)^n \frac{1}{n+1}, \qquad 0 \le x \le 1.
$$

Therefore the function $x^{ln}(1-x)$ approaches zero uniformly on $[0,1]$ as $n \to \infty$. Now observe that the eigenvalues of ${}_n\Gamma^{ln}(I - {}_n\Gamma)$ are of the form ${}_n\lambda^{ln}(1 - {}_n\lambda)$ with $0 \le {}_n\lambda \le 1$, and the proof follows immediately.

From earlier discussions it is clear that Assumptions 3, 4, 5, and 6 hold for the model problems of finite-difference discretization of the Laplacian with Dirichlet conditions using five-point and nine-point stencils. We just have to verify Assumptions 7 and 8.

4.2. Five-point stencil. For this case with ${}_nS_0 = {}_nA$, Assumption 7 is true by picking $\mu = 0.5$. Next let us verify the validity of Assumption 8 for the five-point stencil case. In this case the eigenvalues of ${}_nA$ are given by

$$\lambda_j({}_nA) = 4 - 2\cos\left(\frac{j\pi}{n+1}\right) \quad \text{for } j = 1, \ldots, n, \tag{4.3}$$

while ${}_nB = {}_nC = I$. The eigenvalues of ${}_nA \pm \sqrt{{}_nA^2 - 4I}$ are

$$\lambda_j({}_nA) \pm \sqrt{\lambda_j^2({}_nA) - 4}.$$

Let

$$f(x) = \frac{x - \sqrt{x^2 - 4}}{x + \sqrt{x^2 - 4}}.$$

Then

$$\lambda_j({}_n\Gamma) = f(\lambda_j({}_nA)). \tag{4.4}$$

We observe that the eigenvalues of ${}_nA$ are arranged in increasing order in the interval $(2, 6)$. Since the function f is monotonically decreasing in the interval $(2, \infty)$, it follows that $\lambda_j({}_n\Gamma)$ is decreasing with increasing j.

We now compute the limit of the kth largest eigenvalue of ${}_n\Gamma^m$ as n approaches infinity. First observe from (4.3) that

$$\lambda_k({}_nA) = 2 + \frac{k^2\pi^2}{(n+1)^2} + O(n^{-4}),$$

where k is assumed constant in this expression. Using this expansion in (4.4), we can get an expansion for the corresponding eigenvalues of ${}_n\Gamma^{ln}$:

$$\lambda_k({}_n\Gamma^{ln}) = \left(\frac{1 - \frac{k\pi}{n+1} + O(n^{-2})}{1 + \frac{k\pi}{n+1} + O(n^{-2})}\right)^{ln}.$$

Taking the limit, we obtain

$$\lim_{n\to\infty} \lambda_k({}_n\Gamma^{ln}) = e^{-2kl\pi}.$$

Therefore we see that γ can be taken to be $e^{-l\pi/2}$ in this case for Assumption 8 to hold.

4.3. Nine-point stencil. What about the nine-point stencil case? In this case, too, we have ${}_nS_0 = {}_nA$, and Assumption 7 is true by picking $\mu = 0.5$. The eigenvalues of ${}_nA$ and ${}_nB$ are

$$\lambda_j({}_nA) = 20 - 8\cos\left(\frac{j\pi}{n+1}\right)$$

and

$$\lambda_j({}_nB) = -4 - 2\cos\left(\frac{j\pi}{n+1}\right) \quad \text{for } j = 1, \ldots, n.$$

Note that

$$\frac{\lambda_j({}_nA)}{|\lambda_j({}_nB)|} = \frac{20 - 8\cos\left(\frac{j\pi}{n+1}\right)}{4 + 2\cos\left(\frac{j\pi}{n+1}\right)}$$

is increasing with the index j. Since

$$\lambda_j({}_n\Gamma) = f\left(\frac{\lambda_j({}_nA)}{|\lambda_j({}_nB)|}\right),$$

it follows that $\lambda_j({}_n\Gamma)$ is decreasing with increasing j. For fixed k and large values of n we have the expansion

$$\frac{\lambda_k({}_nA)}{|\lambda_k({}_nB)|} = \frac{12 + 4\frac{k^2\pi^2}{(n+1)^2} + O(n^{-4})}{6 - \frac{k^2\pi^2}{(n+1)^2} + O(n^{-4})} = 2 + \frac{k^2\pi^2}{(n+1)^2} + O(n^{-4}).$$

Therefore we again find that

$$\lim_{n\to\infty} \lambda_k({}_n\Gamma^{ln}) = e^{-2kl\pi},$$

and hence we can take γ in Assumption 8 to be $e^{-l\pi/2}$.

4.4. Mixed conditions. Also consider the case of a mixed problem with a Neumann condition on the leftmost side of the grid. The five-point discretization of the Laplacian with such mixed conditions leads to a system of the type

$${}_{nm}G = \begin{pmatrix} {}_nA & -2I & & & \\ -I & {}_nA & -I & & \\ & -I & \ddots & \ddots & \\ & & \ddots & \ddots & -I \\ & & & -I & {}_nA \end{pmatrix},$$

with

$${}_nA = \begin{pmatrix} 4 & -1 & & & \\ -1 & 4 & -1 & & \\ & -1 & \ddots & \ddots & \\ & & \ddots & \ddots & -1 \\ & & & -1 & 4 \end{pmatrix}.$$

Note that since we have a $-2I$ on the first row, we simply start the recursion from the second row. Hence the starting Schur complement is ${}_nS_0 = {}_nA - 2{}_nA^{-1}$ in this case. Assumptions 3, 4, 5, 6, and 8 are the same as in the five-point Dirichlet case. We just need to verify Assumption 7 by some simple algebra. Since

$$\begin{aligned} \mu\,{}_n\Xi_+ + (1-\mu)\,{}_n\Xi_- &= \mu\frac{{}_nA + \sqrt{{}_nA^2 - 4I}}{2} + (1-\mu)\frac{{}_nA + \sqrt{{}_nA^2 - 4I}}{2} \\ &= \frac{{}_nA}{2} + \frac{2\mu - 1}{2}\sqrt{{}_nA^2 - 4I}, \end{aligned}$$

with $\mu = 0.5$ we get

$$\mu_n \Xi_+ + (1-\mu)_n \Xi_- = \frac{{}_nA}{2}.$$

Since ${}_nA^2 - 4I > 0$, it implies that

$$\frac{{}_nA^2 - 2I}{{}_nA} > \frac{{}_nA}{2}$$

and

$$\begin{aligned} {}_nS_0 &= {}_nA - 2{}_nA^{-1} \\ &= \frac{{}_nA^2 - 2I}{{}_nA}. \end{aligned}$$

Therefore, ${}_nS_0 > \frac{{}_nA}{2}$, which proves Assumption 7.

Now consider a mixed condition such as Neumann on the top and left sides of the grid. In that case we get a system of the form

$${}_{nm}G = \begin{pmatrix} {}_nA & -2I & & & \\ -I & {}_nA & -I & & \\ & -I & \ddots & \ddots & \\ & & \ddots & \ddots & -I \\ & & & -I & {}_nA \end{pmatrix},$$

with

$${}_nA = \begin{pmatrix} 4 & -2 & & & \\ -1 & 4 & -1 & & \\ & -1 & \ddots & \ddots & \\ & & \ddots & \ddots & -1 \\ & & & -1 & 4 \end{pmatrix}.$$

Again, we have a $-2I$ on the first row, and we start the recursion from the second row. So the starting Schur complement is ${}_nS_0 = {}_nA - 2{}_nA^{-1}$. The eigenvalues of ${}_nA$ are given by

$$\lambda_j({}_nA) = 4 - 2\cos\left(\frac{(2j+1)\pi}{2n}\right) \quad \text{for } j = 0, \ldots, n-1.$$

Note that the eigenvalues of ${}_nA$ are bounded between 2 and 6. Also, ${}_nA$ is no longer symmetric, but ${}_nA$ can be made into a symmetric matrix by a diagonal matrix of the form

$$R = \begin{pmatrix} \sqrt{2} & & & & \\ & 1 & & & \\ & & \ddots & & \\ & & & \ddots & \\ & & & & 1 \end{pmatrix}.$$

Since $R^{-1}{}_nAR$ is symmetric it has orthogonal eigenvectors. Suppose we denote this eigenvector matrix by T. Then we can pick the eigenvector matrix ${}_nV$ of ${}_nA$ such that ${}_nV = RT$. So we have

$$\begin{aligned}\|{}_nV\| &= \|RT\| \\ &\le \|R\| \\ &= \sqrt{2}\end{aligned}$$

and

$$\begin{aligned}\|{}_nV^{-1}|| &= \|T^TR^{-1}\| \\ &\le \|R^{-1}\| \\ &= 1.\end{aligned}$$

Therefore, $\|{}_nV\|\|{}_nV^{-1}\| \le \sqrt{2}$. This proves Assumption 4. As in the earlier case, take $\mu = 0.5$ to satisfy Assumption 7. We need to verify Assumption 8. To this end note that the largest eigenvalue of ${}_n\Gamma$ occurs with the very first eigenvalue of ${}_nA$, i.e., $\lambda_0({}_nA) = 4 - 2\cos\left(\frac{\pi}{2n}\right)$. So we have

$$\lambda_0({}_n\Gamma^{ln}) = \left(\frac{1 - \frac{\pi}{2n} + O(n^{-2})}{1 + \frac{\pi}{2n} + O(n^{-2})}\right)^{ln}.$$

Taking the limit we obtain

$$\lim_{n\to\infty} \lambda_0({}_n\Gamma^{ln}) = e^{-l\pi}.$$

4.5. ${}_nS_{ln}$. Now that we have confirmed that our assumptions hold in the model cases of interest, we can continue with our analysis of ${}_nS_{ln}$. The analysis proceeds almost exactly as before, with some important changes. We proceed briskly through the similar parts. The Schur complements are given by

$${}_nS_{m+1} = {}_nA - \frac{{}_nC\,{}_nB}{{}_nS_m}.$$

We can write this as a recursion in the eigenvalues

$${}_nD_{m+1} = {}_nD_A - \frac{{}_nD_C\,{}_nD_B}{{}_nD_m},$$

where ${}_nD_m$, ${}_nD_A$, ${}_nD_B$, ${}_nD_C$ are diagonal matrices of the eigenvalues of ${}_nS_m$, ${}_nA$, ${}_nB$, and ${}_nC$. We can consider this as the power iteration

$$\begin{aligned}\begin{pmatrix} {}_nP_{m+1} \\ {}_nQ_{m+1}\end{pmatrix} &= \begin{pmatrix} {}_nD_A & -{}_nD_C\,{}_nD_B \\ I & 0\end{pmatrix}\begin{pmatrix} {}_nP_m \\ {}_nQ_m\end{pmatrix} \\ &= \begin{pmatrix} {}_nD_A & -{}_nD_C\,{}_nD_B \\ I & 0\end{pmatrix}^{m+1}\begin{pmatrix} {}_nP_0 \\ {}_nQ_0\end{pmatrix},\end{aligned}$$

where

$${}_nD_m = \frac{{}_nP_m}{{}_nQ_m}.$$

We now write this as the eigendecomposition

$$\begin{pmatrix} {}_nP_m \\ {}_nQ_m \end{pmatrix} = \begin{pmatrix} {}_nX_+ & {}_nX_- \\ I & I \end{pmatrix} \begin{pmatrix} {}_nX_+ & 0 \\ 0 & {}_nX_- \end{pmatrix}^m \begin{pmatrix} {}_nX_+ & {}_nX_- \\ I & I \end{pmatrix}^{-1} \begin{pmatrix} {}_nP_0 \\ {}_nQ_0 \end{pmatrix}.$$

Noting that

$$\begin{pmatrix} {}_nX_+ & {}_nX_- \\ I & I \end{pmatrix}^{-1} = \frac{I}{{}_nX_+ - {}_nX_-} \begin{pmatrix} I & -{}_nX_- \\ -I & {}_nX_+ \end{pmatrix}$$

is well defined by (4.1), we get

$$\begin{aligned} {}_nD_m &= \frac{\left({}_nX_+^{m+1} - {}_nX_-^{m+1}\right) {}_nP_0 + \left({}_nX_-^{m+1}{}_nX_+ - {}_nX_+^{m+1}{}_nX_-\right) {}_nQ_0}{\left({}_nX_+^{m} - {}_nX_-^{m}\right) {}_nP_0 + \left({}_nX_-^{m}{}_nX_+ - {}_nX_+^{m}{}_nX_-\right) {}_nQ_0} \\ &= \frac{{}_nX_+^{m+1}\left({}_nP_0 - {}_nX_-{}_nQ_0\right) + {}_nX_-^{m+1}\left({}_nX_+{}_nQ_0 - {}_nP_0\right)}{{}_nX_+^{m}\left({}_nP_0 - {}_nX_-{}_nQ_0\right) + {}_nX_-^{m}\left({}_nX_+{}_nQ_0 - {}_nP_0\right)} \\ &= \frac{{}_nX_+^{m+1}\left({}_nD_0 - {}_nX_-\right) + {}_nX_-^{m+1}\left({}_nX_+ - {}_nD_0\right)}{{}_nX_+^{m}\left({}_nD_0 - {}_nX_-\right) + {}_nX_-^{m}\left({}_nX_+ - {}_nD_0\right)}. \end{aligned}$$

In order for the iteration to be well defined, we need to ensure that ${}_nD_m$ is not zero at any point in the iteration. If ${}_nD_m = 0$, we get the condition

$${}_nX_+^{m+1}\left({}_nD_0 - {}_nX_-\right) + {}_nX_-^{m+1}\left({}_nX_+ - {}_nD_0\right) = 0,$$

which leads to

$$\begin{aligned} {}_nD_0 &= \left(\frac{{}_nX_+^{m+1}{}_nX_- - {}_nX_+{}_nX_-^{m+1}}{{}_nX_+{}^{m+1} - {}_nX_-^{m+1}}\right) \\ &= \left(\frac{{}_nX_+^{m} - {}_nX_-^{m}}{{}_nX_+^{m+1} - {}_nX_-^{m+1}}\right) {}_nX_+{}_nX_- \\ &= \left(\frac{I - {}_n\Gamma^m}{I - {}_n\Gamma^{m+1}}\right) {}_nX_- \\ &< {}_nX_-. \end{aligned}$$

Since by (4.2), ${}_nD_0 > {}_nX_-$, ${}_nD_m$ is well defined. We can now write the above equation as

$$\begin{aligned} {}_nD_m &= {}_nX_+ \left(\frac{I - \left(\frac{{}_nD_0 - {}_nX_+}{{}_nD_0 - {}_nX_-}\right) {}_n\Gamma^{m+1}}{I - \left(\frac{{}_nD_0 - {}_nX_+}{{}_nD_0 - {}_nX_-}\right) {}_n\Gamma^{m}}\right) \\ &= {}_nX_+ \left(\frac{I - {}_nK\,{}_n\Gamma^{m+1}}{I - {}_nK\,{}_n\Gamma^{m}}\right), \end{aligned}$$

where

$${}_nK = \frac{{}_nD_0 - {}_nX_+}{{}_nD_0 - {}_nX_-}.$$

Now we have

$$\begin{aligned}\|{}_nS_m - {}_n\Xi_+\| &= \left\| {}_nV \left({}_nX_+ \left(\frac{I - {}_nK\,{}_n\Gamma^{m+1}}{I - {}_nK\,{}_n\Gamma^m} - I \right) \right) {}_nV^{-1} \right\| \\ &\le \|{}_nV\| \|{}_nV^{-1}\| \left\| {}_nX_+ \left(\frac{{}_nK\,{}_n\Gamma^m\,(I - {}_n\Gamma)}{I - {}_nK\,{}_n\Gamma^m} \right) \right\| \\ &\le \|{}_nV\| \|{}_nV^{-1}\| \|{}_nX_+\| \left\| \frac{{}_nK}{I - {}_nK\,{}_n\Gamma^m} \right\| \|{}_n\Gamma^m\,(I - {}_n\Gamma)\|.\end{aligned}$$

Assumptions 6 and 7 imply that $\left(\frac{\mu-1}{\mu}\right) < {}_nK$ and ${}_nK < 1$. Therefore, we have that

$$\|{}_nK\| < \max\left(\left| \frac{\mu - 1}{\mu} \right|, 1 \right)$$

and

$$\left\| \frac{{}_nK}{I - {}_nK\,{}_n\Gamma^m} \right\| \le \|{}_nK\|\,(1 - \gamma)^{-1}.$$

So,

$$\|{}_nS_m - {}_n\Xi_+\| \le \|{}_nV\| \|{}_nV^{-1}\| \|{}_nX_+\| \|{}_nK\|\,(1 - \gamma)^{-1}\,\|{}_n\Gamma^m\,(I - {}_n\Gamma)\|.$$

From Lemma 3 we see that this upper bound approaches zero in the limit as n approaches infinity.

4.6. Neumann problem. We will now consider the purely Neumann case and the mixed problem with Neumann conditions on three sides of the grid. These problems are different from the cases considered earlier in that ${}_nA^2 - 4{}_nC{}_nB$ is only positive-semidefinite, and Assumptions 6, 7, and 8 will no longer hold. We will proceed by breaking the analysis of ${}_nS_m$ into two parts. Suppose $\lambda({}_nA)$, $\lambda({}_nB)$, $\lambda({}_nC)$ denote the eigenvalues of ${}_nA$, ${}_nB$, and ${}_nC$. We will first consider the iterations due to those eigenvalues such that $\lambda^2({}_nA) - 4\lambda({}_nC)\lambda({}_nB) = 0$. Second, we will consider a reduced system, excluding the degenerate eigenvalues considered in the former case. This latter case will proceed as in the Dirichlet problem of section 4.5, with Assumptions 6, 7, and 8 being valid.

4.6.1. Five-point stencil. Consider the purely Neumann problem with a five-point discretization of the Laplacian. This leads to a system of the form

$${}_{nm}G = \begin{pmatrix} {}_nA & -2I & & & \\ -I & {}_nA & -I & & \\ & \ddots & \ddots & \ddots & \\ & & -I & {}_nA & -I \\ & & & -2I & {}_nA \end{pmatrix},$$

where ${}_nA$ is given by

$${}_nA = \begin{pmatrix} 4 & -2 & & & & \\ -1 & 4 & -1 & & & \\ & -1 & 4 & \ddots & & \\ & & \ddots & \ddots & \ddots & \\ & & & -1 & 4 & -1 \\ & & & & -2 & 4 \end{pmatrix}.$$

The mixed problem with Neumann on three sides of the grid with a Dirichlet condition on the rightmost side gives the same system, except that ${}_{nm}G$ would look like

$$ {}_{nm}G = \begin{pmatrix} {}_nA & -2I & & & \\ -I & {}_nA & -I & & \\ & \ddots & \ddots & \ddots & \\ & & -I & {}_nA & -I \\ & & & -I & {}_nA \end{pmatrix}. $$

The eigenvalues of ${}_nA$ are given by

$$ \lambda_j({}_nA) = 4 - 2\cos\left(\frac{\pi j}{n-1}\right) \quad \text{for } j = 0, \ldots, n-1. $$

Since we have a $-2I$ on the first row, we take the starting Schur complement to be $S_0 = {}_nA - 2{}_nA^{-1}$. Note that ${}_nA$ can be made into a symmetric matrix by a diagonal matrix of the form

$$ R = \begin{pmatrix} \sqrt{2} & & & & \\ & 1 & & & \\ & & \ddots & & \\ & & & 1 & \\ & & & & \sqrt{2} \end{pmatrix}, $$

with $R^{-1}{}_nAR$ being symmetric. Therefore, we can pick the eigenvector matrix ${}_nV$ of ${}_nA$ such that $\|{}_nV\|\|{}_nV^{-1}\| \le \sqrt{2}$. This proves Assumption 4.

Looking at the eigenvalues of ${}_nA$, we see that if we drop the first eigenvalue of 2, then Assumptions 6 and 7 are valid. Moreover, Assumption 8 is also valid since the kth eigenvalue of ${}_{n-1}\Gamma$ is given by

$$ \lambda_k({}_{n-1}\Gamma) = \frac{\lambda_k({}_nA) - \sqrt{\lambda_k^2({}_nA) - 4}}{\lambda_k({}_nA) + \sqrt{\lambda_k^2({}_nA) - 4}}, $$

and for $k > 0$

$$ \begin{aligned} \lim_{n\to\infty} \lambda_k({}_{n-1}\Gamma^{ln}) &= \lim_{n\to\infty} \left(\frac{1 - \frac{k\pi}{n-1} + O(n^{-2})}{1 + \frac{k\pi}{n-1} + O(n^{-2})} \right)^{ln} \\ &= e^{-2lk\pi} \\ &< 1. \end{aligned} $$

4.6.2. ${}_nS_{ln}$. We proceed by breaking the analysis into two parts. We will first treat the recursion due to the first eigenvalue separately.

Consider now the case when $\lambda^2({}_nA) - 4\lambda({}_nC)\lambda({}_nB) = 0$. The recursion of such an eigenvalue is then given by

$$ d_{m+1} = \lambda({}_nA) - \frac{\lambda({}_nC)\lambda({}_nA)}{d_m}, $$

where d_m is the corresponding eigenvalue of ${}_nS_m$. Define the power iteration

$$ \begin{pmatrix} p_{m+1} \\ q_{m+1} \end{pmatrix} = \begin{pmatrix} \lambda({}_nA) & -\lambda({}_nC)\lambda({}_nB) \\ 1 & 0 \end{pmatrix}^m \begin{pmatrix} p_0 \\ q_0 \end{pmatrix}. $$

We can write the above 2×2 matrix in its Jordan form, with $\alpha = \sqrt{\lambda({}_nC)\lambda({}_nB)}$, as

$$J = \begin{pmatrix} \frac{\lambda({}_nA)}{2} & 1 \\ 0 & \frac{\lambda({}_nA)}{2} \end{pmatrix},$$

$$S = \frac{1}{\alpha}\begin{pmatrix} \alpha^2 & 2\alpha \\ \alpha & 1 \end{pmatrix},$$

$$S^{-1} = \frac{1}{\alpha}\begin{pmatrix} -1 & 2\alpha \\ \alpha & -\alpha^2 \end{pmatrix},$$

with

$$SJS^{-1} = \begin{pmatrix} \lambda({}_nA) & -\lambda({}_nC)\lambda({}_nB) \\ 1 & 0 \end{pmatrix}.$$

From this we can write,

$$\begin{aligned} \begin{pmatrix} p_{m+1} \\ q_{m+1} \end{pmatrix} &= SJ^mS^{-1}\begin{pmatrix} p_0 \\ q_0 \end{pmatrix} \\ &= \frac{1}{\alpha^2}\begin{pmatrix} \alpha^2 & 2\alpha \\ \alpha & 1 \end{pmatrix}\begin{pmatrix} \alpha^m & m\alpha^{m-1} \\ 0 & \alpha^m \end{pmatrix}\begin{pmatrix} -1 & 2\alpha \\ \alpha & -\alpha^2 \end{pmatrix}\begin{pmatrix} p_0 \\ q_0 \end{pmatrix} \\ &= \begin{pmatrix} (m+1)\alpha^m & -m\alpha^{m+1} \\ m\alpha^{m-1} & -(m-1)\alpha^m \end{pmatrix}\begin{pmatrix} p_0 \\ q_0 \end{pmatrix}. \end{aligned}$$

Therefore, we have that

$$\begin{aligned} d_m &= \frac{p_m}{q_m} \\ &= \frac{(m+1)\alpha^m p_0 - m\alpha^{m+1}q_0}{m\alpha^{m-1}p_0 - (m-1)\alpha^m q_0} \\ &= \alpha\frac{(m+1)d_0 - m\alpha}{md_0 - (m-1)\alpha} \\ &= \alpha\frac{m(d_0-\alpha)+d_0}{m(d_0-\alpha)+\alpha}. \end{aligned}$$

Note that d_m can become zero only if $m(d_0 - \alpha) + d_0 = 0$, in which case we have

$$\begin{aligned} d_0 &= \frac{m}{m+1}\alpha \\ &< \alpha. \end{aligned}$$

Therefore, we require that $d_0 \geq \alpha$, which is true. Now consider the recursions

$${}_{n-1}D_{m+1} = {}_{n-1}D_A - \frac{{}_{n-1}D_C\,{}_{n-1}D_B}{{}_{n-1}D_m},$$

where ${}_{n-1}D_A$, ${}_{n-1}D_B$, ${}_{n-1}D_C$, and ${}_{n-1}D_m$ are diagonal matrices of the eigenvalues of ${}_nA$, ${}_nB$, ${}_nC$, and ${}_nS_m$ from $j = 1, \ldots, n-1$, excluding the first eigenvalue corresponding to $j = 0$. Then following the same procedure as in section 4.5, we can consider this as the power iteration

$$\begin{pmatrix} {}_{n-1}P_{m+1} \\ {}_{n-1}Q_{m+1} \end{pmatrix} = \begin{pmatrix} {}_{n-1}D_A & -{}_{n-1}D_C\,{}_{n-1}D_B \\ I & 0 \end{pmatrix}^m \begin{pmatrix} {}_{n-1}P_0 \\ {}_{n-1}Q_0 \end{pmatrix},$$

where in the Neumann case

$$
\begin{aligned}
{}_{n-1}P_0 &= {}_{n-1}D_0^2 - 2I,\\
{}_{n-1}Q_0 &= {}_{n-1}D_0.
\end{aligned}
$$

We can now write this as the eigendecomposition

$$
\begin{pmatrix} {}_{n-1}P_{m+1} \\ {}_{n-1}Q_{m+1} \end{pmatrix} = \begin{pmatrix} {}_{n-1}X_+ & {}_{n-1}X_- \\ I & I \end{pmatrix} \begin{pmatrix} {}_{n-1}X_+ & 0 \\ 0 & {}_{n-1}X_- \end{pmatrix}^m \begin{pmatrix} {}_{n-1}X_+ & {}_{n-1}X_- \\ I & I \end{pmatrix}^{-1} \begin{pmatrix} {}_{n-1}P_0 \\ {}_{n-1}Q_0 \end{pmatrix},
$$

from which we get

$$
{}_{n-1}D_m = {}_{n-1}X_+ \left(\frac{I - {}_{n-1}K\, {}_{n-1}\Gamma^{m+1}}{I - {}_{n-1}K\, {}_{n-1}\Gamma^m} \right),
$$

where

$$
{}_{n-1}K = \frac{{}_{n-1}D_0 - {}_{n-1}X_+}{{}_{n-1}D_0 - {}_{n-1}X_-}.
$$

Now we have

$$
{}_nS_m - {}_n\Xi_+ = {}_nV \begin{pmatrix} \alpha \frac{d_0-\alpha}{m(d_0-\alpha)+\alpha} & \\ & {}_{n-1}X_+ \left(\frac{I - {}_{n-1}K\, {}_{n-1}\Gamma^{m+1}}{I - {}_{n-1}K\, {}_{n-1}\Gamma^m} - I \right) \end{pmatrix} {}_nV^{-1}.
$$

Therefore,

$$
\|{}_nS_m - {}_n\Xi_+\| \le \|{}_nV\| \left\|{}_nV^{-1}\right\| \left\| \begin{pmatrix} \alpha \frac{d_0-\alpha}{m(d_0-\alpha)+\alpha} & \\ & {}_{n-1}X_+ \left(\frac{{}_{n-1}K\, {}_{n-1}\Gamma^m (I - {}_{n-1}\Gamma)}{I - {}_{n-1}K\, {}_{n-1}\Gamma^m} \right) \end{pmatrix} \right\|.
$$

We see that the term on the right-hand side goes to zero for large n, and therefore ${}_nS_m$ approaches ${}_n\Xi_+$ as n gets larger.

4.7. Main theorem. We have now proved that the Schur complements ${}_nS_m$ approach ${}_n\Xi_+$ in the limit of large n for the model cases of interest. Therefore, we can now state the main result of this paper.

THEOREM 6. *Under Assumptions* 3 *to* 8*, the ϵ-rank of the Hankel blocks of ${}_nS_{ln}$, in the limit of large n, is bounded by*

$$
r \left(1 + 8 \ln^4 \left(\frac{3W}{\epsilon} \right) \right).
$$

5. The three-dimensional problem. In the previous sections it was shown that the Hankel block ranks of the Schur complements that arise from finite-difference discretizations in two dimensions were bounded independent of the grid size. In this section we shall consider the three-dimensional problem for the constant coefficient Laplacian.

5.0.1. Seven-point discretization with Dirichlet conditions. The seven-point discretization of the Laplacian with Dirichlet conditions leads to a system of linear equations of the form

$$
G = \begin{pmatrix} \tilde{A} & -I & & & \\ -I & \tilde{A} & -I & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & -I \\ & & & -I & \tilde{A} \end{pmatrix}_{nmk \times nmk},
$$

where

$$\tilde{A} = \begin{pmatrix} A & -I & & & \\ -I & A & -I & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & -I \\ & & & -I & A \end{pmatrix}_{nm \times nm},$$

$$A = \begin{pmatrix} 6 & -1 & & & \\ -1 & 6 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & -1 \\ & & & -1 & 6 \end{pmatrix}_{n \times n},$$

with n, m, and k denoting the grid sizes in the x, y, and z directions. We could write $\tilde{A}$ in tensor notation as

$$\tilde{A} = I_{m \times m} \otimes A_{n \times n} + \tilde{I}_{m \times m} \otimes I_{k \times k},$$

where

$$\tilde{I} = \begin{pmatrix} 0 & -1 & & & \\ -1 & 0 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & -1 \\ & & & -1 & 0 \end{pmatrix}.$$

From the above equation, and using the fact that the eigenvalues of the above Kronecker sum are equal to the sum of the eigenvalues of A and $\tilde{I}$ [9], we see that the eigenvalues of $\tilde{A}$ are given by

$$\lambda_{\tilde{A}} = \lambda_A + \lambda_{\tilde{I}} = 6 - 2\cos\left(\frac{i\pi}{n+1}\right) - 2\cos\left(\frac{j\pi}{m+1}\right),\ 1 \le i \le n,\ 1 \le j \le m. \tag{5.1}$$

The eigenvalues of $\tilde{A}$ are therefore bounded between 2 and 10.

5.0.2. Seven-point discretization with Neumann conditions. The seven-point discretization of the Laplacian with Neumann conditions leads to a system of linear equations of the form

$$G = \begin{pmatrix} \tilde{A} & -2I & & & \\ -I & \tilde{A} & -I & & \\ & \ddots & \ddots & \ddots & \\ & & -I & \ddots & -I \\ & & & -2I & \tilde{A} \end{pmatrix}_{nmk \times nmk},$$

where

$$\tilde{A} = \begin{pmatrix} A & -2I & & & \\ -I & A & -I & & \\ & \ddots & \ddots & \ddots & \\ & & -I & \ddots & -I \\ & & & -2I & A \end{pmatrix}_{nm \times nm},$$

$$A = \begin{pmatrix} 6 & -2 & & & \\ -1 & 6 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & \ddots & -1 \\ & & & -2 & 6 \end{pmatrix}_{n \times n},$$

with n, m, and k denoting the grid sizes in the x, y, and z directions. We could write $\tilde{A}$ in tensor notation as

$$\tilde{A} = I_{m \times m} \otimes A_{n \times n} + \tilde{I}_{m \times m} \otimes I_{k \times k},$$

where

$$\tilde{I} = \begin{pmatrix} 0 & -2 & & & \\ -1 & 0 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & \ddots & -1 \\ & & & -2 & 0 \end{pmatrix}.$$

The eigenvalues of $\tilde{A}$ are given by

$$\begin{aligned} \lambda_{\tilde{A}} &= \lambda_A + \lambda_{\tilde{I}} \\ &= 6 - 2\cos\left(\frac{i\pi}{n-1}\right) - 2\cos\left(\frac{j\pi}{m-1}\right),\ 0 \le i \le n-1,\ 0 \le j \le m-1. \end{aligned} \tag{5.2}$$

5.1. The nested dissection ordering. Note that the Schur complements of $\tilde{A}$ are of size $N = mn$, which would require $O(N^3)$ flops for direct Gaussian elimination at each step. If we consider this matrix in columnwise ordering of the unknowns, the structure considered previously would give us an $O(N^2)$ algorithm at each step. However, in this case we could do better by considering the matrix obtained by a nested dissection (ND) ordering of the two-dimensional plane. This would lead to a finer hierarchically semiseparable (HSS) matrix structure that could be exploited to produce an $O(N^{1.5})$ solver for the inversion of each Schur complement [3]. Therefore, we aim to show that the Schur complements of the matrix obtained by an ND ordering have an HSS structure.

We will call the block rows (columns) excluding the diagonal blocks of the ND ordered matrix its *strip-row (column) Hankel blocks*. For example, consider the following 3×3 block matrix:

$$\begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix}.$$

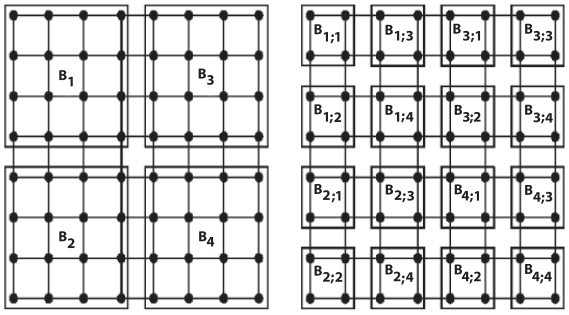

FIG. 1. *ND ordering.*

Then a strip-row Hankel block is

$$\begin{pmatrix} A_{21} & A_{23} \end{pmatrix},$$

and a strip-column Hankel block is

$$\begin{pmatrix} A_{12} & A_{32} \end{pmatrix}^T.$$

We make note of a useful fact here. Any strip-row Hankel block can be moved into the position of the upper rightmost row block, and the diagonal block can be moved into the position of the first diagonal block by means of a symmetric permutation. For example, the first row of the above matrix can be made to look like

$$\begin{pmatrix} 0 & I & 0 \\ I & 0 & 0 \\ 0 & 0 & I \end{pmatrix} \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix} \begin{pmatrix} 0 & I & 0 \\ I & 0 & 0 \\ 0 & 0 & I \end{pmatrix} = \begin{pmatrix} A_{22} & A_{21} & A_{23} \\ A_{12} & A_{11} & A_{13} \\ A_{32} & A_{31} & A_{33} \end{pmatrix}.$$

We will denote such a permutation matrix that moves any jth row to the first row in this manner as P_j.

Consider the two-dimensional grid in columnwise ordering. With the five-point stencil in the plane, each interior node talks to the nodes immediately above and below it, as well as to the nodes to its left and right. Now consider a division of the grid into a 2×2 partition. We choose to label the variables in each block partition locally before proceeding to the next block (i.e., in Figure 1 (left), we would first label the variables in block B_1 followed by B_2, B_3, and B_4). Each of these blocks would correspond to a strip-row (or column) of the discretization matrix. Note that with such an ordering the rank of any strip-row (column) Hankel block would just be the number of boundary nodes in that block. That is, with this above partition each strip-row (column) Hankel block in the matrix would have rank equal to the perimeter of the corresponding block in the grid. Next, we can divide each block above into a 2×2 partition as before. Again, we choose to label the variables block by block (i.e., in Figure 1 (right), we would label the blocks in the order $B_{1;1}$, $B_{1;2}$, $B_{1;3}$, $B_{1;4}$, $B_{2;1}$, etc). We can continue these recursive partitions up to some level K, where at the Kth level the strip-row (and column) Hankel blocks would have ranks equal to the perimeter of the blocks in the grid after K partitions. We will call any strip-row (column) Hankel block of the ND ordered matrix, corresponding to a block at the jth level partition of the grid, a *jth strip-row (column) Hankel block*.

The idea behind a fast HSS solver is to exploit the relative low ranks of the strip Hankel blocks as compared to their size. Therefore, the number of partition levels would depend on the relative size of the block to its rank. Now we look at the computational cost of an HSS solver. Consider a matrix that has been partitioned into K levels. We will assume an $n \times m$ grid with $m = l \times n$ for some constant l. Then the total computational cost of a fast HSS solver is bounded by [3]

$$98m^3 + 70lm^2 + m^2\left(4\left(2K + \log_2 l\right)^2 + 11\left(2K + \log_2 l\right) + 28\right).$$

Let $\tilde{A}_{\mathrm{ND}}$ be the ND ordered matrix. We aim to show that the strip Hankel block ranks of the Schur complements are bounded in terms of the strip Hankel block ranks of $\tilde{A}_{\mathrm{ND}}$ in the limit of large n. That is, we aim to show that the Schur complements also retain the HSS structure, allowing us to use the HSS solver. The effect of an ND ordering of the grid is just a multiplication of $\tilde{A}$ by permutation matrices on the left and right. As such, the Schur complements of G are multiplied from the left and right by a permutation matrix P_{ND}:

$$\begin{aligned} S_{i+1} &= \tilde{A} - S_i^{-1}, \\ \tilde{S}_{i+1} &= P_{\mathrm{ND}}^T S_{i+1} P_{\mathrm{ND}}, \end{aligned}$$

where S_i is the ith Schur complement of G, and $\tilde{S}_i$ is the ith Schur complement of the ND ordered matrix. Therefore, we first look at the asymptotic behavior of S_i.

5.2. Schur complements in the limit as $k \to \infty$. We will start by considering the Dirichlet problem when we let the grid size grow to infinity in the z direction. Therefore, the matrix G would look like

$$G = \begin{pmatrix} \tilde{A} & -I & & \\ -I & \tilde{A} & -I & \\ & -I & \tilde{A} & \ddots \\ & & \ddots & \ddots \end{pmatrix}.$$

The Schur complements of G are

$$\begin{aligned} S_0 &= \tilde{A}. \\ S_{k+1} &= \tilde{A} - S_k^{-1}. \end{aligned}$$

Note that each S_k is Hermitian since $\tilde{A}$ is Hermitian, and positive-definite since the eigenvalues of S_k are bounded below by 0.1. Moreover, each S_k commutes with $\tilde{A}$. We make the following definitions:

$$\begin{aligned} \Xi_+ &= \frac{\tilde{A} + \sqrt{\tilde{A}^2 - 4I}}{2}, \\ \Xi_- &= \frac{\tilde{A} - \sqrt{\tilde{A}^2 - 4I}}{2}. \end{aligned}$$

Following the technique used before, we can write S_k in terms of a power iteration. To that accord, we define matrices P_k and Q_k as follows:

$$\begin{pmatrix} P_{k+1} \\ Q_{k+1} \end{pmatrix} = \begin{pmatrix} \tilde{A} & -I \\ I & 0 \end{pmatrix} \begin{pmatrix} P_k \\ Q_k \end{pmatrix},$$

with $P_0 = \tilde{A}$ and $Q_0 = I$. Since P_0 and P_1 are invertible, each P_k and Q_k are also invertible. Note that $S_{k+1} = P_k Q_k^{-1}$. Let

$$T = \begin{pmatrix} \tilde{A} & -I \\ I & 0 \end{pmatrix}.$$

Then

$$\begin{pmatrix} P_{k+1} \\ Q_{k+1} \end{pmatrix} = T^k \begin{pmatrix} P_0 \\ Q_0 \end{pmatrix},$$

and the eigendecomposition of T is

$$T = \begin{pmatrix} \Xi_+ & \Xi_- \\ I & I \end{pmatrix} \begin{pmatrix} \Xi_+ & 0 \\ 0 & \Xi_- \end{pmatrix} \begin{pmatrix} \Xi_+ & \Xi_- \\ I & I \end{pmatrix}^{-1}.$$

The above power iteration converges to the eigenvector block corresponding to the largest eigenblock Ξ_+. Therefore, we have that $\lim_{k\to\infty} S_k = \Xi_+$.

$\tilde{A}_{\mathrm{ND}}$ is the ND ordering of $\tilde{A}$. There exists a permutation matrix P_{ND} such that

$$\tilde{A}_{\mathrm{ND}} = P_{\mathrm{ND}} \tilde{A} P_{\mathrm{ND}}^T.$$

Furthermore, note that any jth strip-row Hankel block of $\tilde{A}_{\mathrm{ND}}$ can be made into an upper right off-diagonal block by a symmetric permutation P_j. That is, consider B_j where

$$B_j = P_j \tilde{A}_{\mathrm{ND}} P_j^T = P_j P_{\mathrm{ND}} \tilde{A} P_{\mathrm{ND}}^T P_j^T.$$

B_j contains the jth strip-row Hankel block of $\tilde{A}_{\mathrm{ND}}$ on its upper rightmost block, and the jth diagonal block in the position of the first diagonal block . Now, Lemmas 1 and 2 apply to B_j. From this it follows that the lemmas also apply to any jth strip-row Hankel block of $\tilde{A}_{\mathrm{ND}}$. Now, we have

$$\begin{aligned} S_\infty &= \Xi_+, \\ \tilde{S}_\infty &= P_{\mathrm{ND}} \Xi_+ P_{\mathrm{ND}}^T \\ &= \frac{\tilde{A}_{\mathrm{ND}} + \sqrt{\tilde{A}_{\mathrm{ND}}^2 - 4I}}{2}. \end{aligned}$$

Applying Theorems 4 and 5, we prove the following extension to Theorem 3.

THEOREM 7. *The ϵ-rank of the strip-row (strip-column) Hankel blocks of $\tilde{S}_\infty$ at the jth level is bounded by*

$$r_j \left(1 + 8 \ln^4 \left(\frac{3\|\tilde{A}\|}{\epsilon} \right) \right),$$

where r_j is the rank of any jth strip-row (strip-column) Hankel block of $\tilde{A}_{ND}$.

5.3. An asymptotic bound as the grid grows in every direction. We now consider the case when the grid size grows in all directions. We shall assume that the grid sizes in the y and z directions are constant multiples of the grid size in the x direction. Let $m = l \times n$ and $k = q \times n$, where l, q are positive integers. We will indicate the size of a matrix as a left subscript on the name of the matrix. As before,

we first look at the asymptotic behavior of ${}_{nm}S_k$. We proceed to prove the following lemma.

LEMMA 4. *In the limit of large n, the Schur complement* ${}_{nm}S_k$ *converges in norm to* ${}_{nm}\Xi_+$.

The proof of Lemma 4 is the same as in section 4.5 for the Dirichlet case and as in section 4.6 for the Neumann case. We need to verify the validity of the key assumptions as listed in section 4.1.

5.3.1. Dirichlet problem. In this case we have ${}_{nm}S_0 = {}_{nm}\tilde{A}$, ${}_{nm}B = I$, and ${}_{nm}C = I$. Since ${}_{nm}\tilde{A}$ is symmetric and its eigenvalues are bounded, ${}_{nm}\tilde{A}$ is bounded. This verifies Assumptions 3 and 4. Looking at (5.1), we see that Assumption 6 is true. Assumption 7 is true by picking $\mu = 0.5$. We proceed to prove Assumption 8. Note that

$$\begin{aligned}\|{}_{nm}\Gamma\| &= \left\| \frac{{}_{nm}X_-}{{}_{nm}X_+} \right\| \\ &= \frac{\lambda_{min} - \sqrt{\lambda_{min}^2 - 4}}{\lambda_{min} + \sqrt{\lambda_{min}^2 - 4}},\end{aligned}$$

where $\lambda_{min} = 6 - 2\big(\cos\big(\frac{\pi}{n+1}\big) + \cos\big(\frac{\pi}{ln+1}\big)\big)$, is the minimum eigenvalue of ${}_{nm}\tilde{A}$. We note the following upper bound on $\cos(\theta)$ for $0 \le \theta \le \pi/2$:

$$\cos(\theta) \le 1 - \frac{\theta^2}{2\pi^2}.$$

From this, it follows that

$$\cos\left(\frac{\pi}{n+1}\right) + \cos\left(\frac{\pi}{ln+1}\right) < 2 - \frac{1}{(ln+1)^2}.$$

Let

$$\begin{aligned}\tilde{\lambda} &= 6 - 2\left(2 - \frac{1}{(ln+1)^2}\right) \\ &= 2\left(1 + \frac{1}{(ln+1)^2}\right).\end{aligned}$$

Then $\tilde{\lambda} < \lambda_{min}$. Consider the function $f(\lambda) = \frac{\lambda - \sqrt{\lambda^2-4}}{\lambda + \sqrt{\lambda^2-4}}$ and note that $\|{}_{nm}\Gamma\| = f(\lambda_{min})$. Since f is a monotonic decreasing function, $f(\tilde{\lambda}) > f(\lambda_{min})$ and

$$\|{}_{nm}\Gamma^{qn}\| = \|{}_{nm}\Gamma\|^{qn} = (f(\lambda_{min}))^{qn} \le (f(\tilde{\lambda}))^{qn}.$$

Now we can produce a bound on $(f(\tilde{\lambda}))^{qn}$ as follows:

$$\begin{aligned}\sqrt{\tilde{\lambda}^2 - 4} &= \frac{2\sqrt{2}}{ln+1}\sqrt{1 + \frac{1}{2(ln+1)^2}} \\ &= \frac{2\sqrt{2}}{ln+1} + o(n^{-3}).\end{aligned}$$

Hence

$$\tilde{\lambda} \pm \sqrt{\tilde{\lambda}^2 - 4} = 2\left(1 \pm \frac{\sqrt{2}}{ln+1}\right) + o(n^{-2}).$$

Therefore

$$(f(\tilde{\lambda}))^{qn} = \left[\frac{\left(1 - \frac{\sqrt{2}}{ln+1}\right) + o(n^{-2})}{\left(1 + \frac{\sqrt{2}}{ln+1}\right) + o(n^{-2})}\right]^{qn}$$

and

$$\lim_{n\to\infty} (f(\tilde{\lambda}))^{qn} = e^{-2\sqrt{2}\frac{q}{l}} < 1.$$

5.3.2. Neumann case. Here we have ${}_{nm}S_0 = {}_{nm}\tilde{A} - 2\,{}_{nm}\tilde{A}^{-1}$. The eigenvalues of ${}_{nm}\tilde{A}$ are given by (5.2). Let

$$R = \begin{pmatrix} \sqrt{2} & & & & \\ & 1 & & & \\ & & \ddots & & \\ & & & 1 & \\ & & & & \sqrt{2} \end{pmatrix}.$$

Then $(I \otimes R)^{-1}\,{}_{nm}\tilde{A}\,(I \otimes R)$ is symmetric. Therefore, we can pick the eigenvector matrix ${}_{nm}V$ such that $\|{}_{nm}V\|, \|{}_{nm}V^{-1}\| \leq \sqrt{2}$. This proves Assumptions 3 and 4.

Looking at (5.2), we can see that ${}_{nm}\tilde{A}$ has an eigenvalue at 2. Therefore, just as in section 4.6, we can analyze this iteration separately. Then Assumptions 6 and 7 hold for the reduced system obtained by disregarding this eigenvalue. We have to verify Assumption 8 for the reduced system. To this end, note that

$$\begin{aligned} \|{}_{nm-1}\Gamma\| &= \left\| \frac{{}_{nm-1}X_-}{{}_{nm-1}X_+} \right\| \\ &= \frac{\lambda_{min} - \sqrt{\lambda_{min}^2 - 4}}{\lambda_{min} + \sqrt{\lambda_{min}^2 - 4}}, \end{aligned}$$

where $\lambda_{min} = 4 - 2\cos\left(\frac{\pi}{ln-1}\right)$. So we get

$$\begin{aligned} \lim_{n\to\infty} \|{}_{nm-1}\Gamma^{qn}\| &= \lim_{n\to\infty} (f(\lambda_{min}))^{qn} \\ &\leq \lim_{n\to\infty} \left[\frac{\left(1 - \frac{\sqrt{2}}{ln-1}\right) + o(n^{-2})}{\left(1 + \frac{\sqrt{2}}{ln-1}\right) + o(n^{-2})}\right]^{qn} \\ &= e^{-2\sqrt{2}\frac{q}{l}} \\ &< 1. \end{aligned}$$

From these observations, and following the exact procedure outlined in sections 4.5 and 4.6, we complete the proof of Lemma 4. Now we can extend Theorem 6 to the seven-point Laplacian.

THEOREM 8. *The ϵ-rank of the strip Hankel blocks of ${}_{nm}\tilde{S}_k$ at any jth level partition, in the limit of large n, is bounded by*

$$r_j \left(1 + 8\ln^4\left(\frac{3\|\tilde{A}\|}{\epsilon}\right)\right),$$

where r_j is the rank of any jth strip Hankel block of $\tilde{A}_{ND}$.

6. Numerical results. We tabulated some numerical results on the ranks of the Hankel blocks for some two-dimensional model problems for different grid sizes. The first column in each table indicates the grid size, and the last two columns indicate the number of singular values that are greater than 10^{-6} and 10^{-12}.

Table 1 shows the Hankel block ranks of the Schur complement for the five-point stencil with constant coefficients and Dirichlet boundary conditions. Table 2 shows the Hankel block ranks of the Schur complement for the constant coefficient Neumann problem with a five-point stencil. Tables 3 and 4 show the ranks for the constant coefficient nine-point stencils. Note that the theoretical bounds on the ranks for these examples are of the order 10^5.

We also tabulated below the Hankel block ranks for a variable coefficient PDE of the form

$$\frac{\partial}{\partial x}\left(p(x,y)\frac{\partial}{\partial x}u(x,y)\right)+\frac{\partial}{\partial y}\left(q(x,y)\frac{\partial}{\partial y}u(x,y)\right),$$

where $p(x,y)$ and $q(x,y)$ were generated from a uniform distribution on the interval $(0,1)$. The numbers represent the maximum ranks over five trial runs. The ranks for this case over a square domain are tabulated in Tables 5 and 6.

TABLE 1
Hankel block ranks for the constant coefficient Dirichlet problem with a five-point stencil.

n	10^{-6}	10^{-12}
100	7	11
250	8	14
500	8	16
1000	9	17

TABLE 2
Hankel block ranks for the constant coefficient Neumann problem with a five-point stencil.

n	10^{-6}	10^{-12}
100	7	12
250	8	14
500	9	16
1000	10	18

TABLE 3
Hankel block ranks for the constant coefficient Dirichlet problem with a nine-point stencil.

n	10^{-6}	10^{-12}
100	7	12
250	8	14
500	9	16
1000	9	18

TABLE 4
Hankel block ranks for the constant coefficient Neumann problem with a nine-point stencil.

n	10^{-6}	10^{-12}
100	8	13
250	9	17
500	9	15
1000	10	19

TABLE 5
Hankel block ranks for the Dirichlet problem with a five-point stencil and random coefficients.

n	10^{-6}	10^{-12}
100	8	14
250	9	16
500	10	19
1000	9	18

TABLE 6
Hankel block ranks for the Neumann problem with a five-point stencil and random coefficients.

n	10^{-6}	10^{-12}
100	9	14
250	10	17
500	13	18
1000	9	19

TABLE 7
Hankel block ranks for the Dirichlet problem with random coefficients on a nonconvex polygonal domain.

n	10^{-6}	10^{-12}
100	5	8
250	6	10
500	8	14
1000	7	11

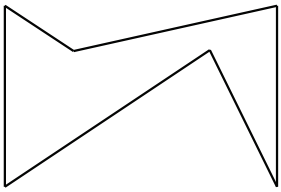

FIG. 2. *Nonconvex polygonal domain.*

TABLE 8
Hankel block ranks for the Dirichlet problem with random coefficients on a random domain.

n	10^{-6}	10^{-12}
680	7	12
608	6	11
3630	6	10
1895	7	12

Table 7 represents the Hankel block ranks for a nonconvex polygonal region as shown in Figure 2. A five-point approximation was used. We also tabulated the ranks for a case in which the size of a block on the next step of elimination is taken as a random number between ± 10 percent of the current block size. This is an interesting example in that the ratio of the difference in the sizes of two consecutive blocks relative to the grid size does not go to zero, which is not the case for any reasonably smooth domain (i.e., all the other examples considered so far). Table 8 reports these numbers corresponding to starting block sizes of $n = 100, 250, 500, 1000$. Note that the tabulated block sizes correspond to the size of the final Schur complement. Although

not covered by our present theory, we see clearly that the Schur complements exhibit low rank Hankel blocks even in these cases.

7. Conclusion. The methods of this paper are also applicable to piecewise constant coefficient PDEs. They can also be extended to handle some special indefinite problems. These matters will be published elsewhere.

REFERENCES

[1] M. Bebendorf, *Why finite element discretizations can be factored by triangular hierarchical matrices*, SIAM J. Numer. Anal., 45 (2007), pp. 1472–1494.
[2] M. Bebendorf and W. Hackbusch, *Existence of H-matrix approximants to the inverse FE-matrix of elliptic operators in L^∞-coefficients*, Numer. Math., 95 (2003), pp. 1–28.
[3] S. Chandrasekaran, M. Gu, and T. Pals, *A fast ULV decomposition solver for hierarchically semiseparable representations*, SIAM J. Matrix Anal. Appl., 28 (2006), pp. 603–622.
[4] P. Dewilde, K. Diepold, and W. Bamberger, *A semi-separable approach to a tridiagonal hierarchy of matrices with application to image flow analysis*, in Proceedings of the 17th International Symposium on Mathematical Theory of Networks and Systems (MTNS), Brussels, Belgium, 2004.
[5] G. B. Folland, *Introduction to Partial Differential Equations*, Princeton University Press, Princeton, NJ, 1995.
[6] I. Gohberg, T. Kailath, and I. Koltracht, *Linear complexity algorithms for semiseparable matrices*, Integral Equations Operator Theory, 8 (1985), pp. 780–804.
[7] L. Greengard and V. Rokhlin, *A fast algorithm for particle simulations*, J. Comput. Phys., 73 (1987), pp. 325–348.
[8] W. Hackbusch, *A sparse matrix arithmetic based on H-matrices. Part* I: *Introduction to H-matrices*, Computing, 62 (2) (1999), pp. 89–108.
[9] R. A. Horn and C. R. Johnson, *Matrix Analysis*, Cambridge University Press, Cambridge, UK, 1999.
[10] R. J. Lipton, D. J. Rose, and R. E. Tarjan, *Generalized nested dissection*, SIAM J. Numer. Anal., 16 (1979), pp. 346–358.
[11] C. D. Meyer, *Matrix Analysis and Applied Linear Algebra*, SIAM, Philadelphia, 2000.
[12] T. Needham, *Visual Complex Analysis*, Oxford University Press, New York, 1997.
[13] D. Newman, *Rational approximation to $|x|$*, Michigan Math. J., 11 (1964), pp. 11–14.
[14] P. P. Petrushev and V. A. Popov, *Rational Approximation of Real Functions*, Cambridge University Press, Cambridge, UK, 1987.
[15] V. Strassen, *Gaussian elimination is not optimal*, Numer. Math., 13 (1969), pp. 354–356.