On the Numerical Solution of a Differential-Difference Equation Arising in Analytic Number Theory

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1. Abstract and Introduction. In the January 1962 issue of this journal R. Bellman and B. Kotkin published a short paper under the same title as this one (cf. [1]). In that paper Bellman and Kotkin presented some of their results concerning the numerical computation of the continuous function $y(x)$, defined by

$$y(x) = 1 \qquad (0 \leqq x \leqq 1)$$

$$y'(x) = -\frac{1}{x} \cdot y(x-1) \qquad (x > 1)\,.$$

Tables of $y(x)$ were given for $x = 1(0.0625)6$ and $x = 6(1)20$. In the process of extending these tables beyond $x = 20$ we discovered that the second table was rather inaccurate for all values of $x > 9$. Bellman and Kotkin found, for example, that $y(20) = 0.149 \cdot 10^{-8}$, whereas the actual value of $y(20)$ can be shown to be smaller than 10^{-20}. Moreover, in view of the method used by Bellman and Kotkin, one may expect that it would be quite time consuming to compute $y(x)$ for values of x up to say $x = 1{,}000$.

In this paper we describe a different method which enables us to compute $y(x)$ for values of x up to about "as far as one would like." ■

2. The Main Formula and Some of its Consequences. The function $y(x)$ defined in the introduction satisfies the following fundamental

LEMMA 1.

$$x \cdot y(x) = \int_{x-1}^{x} y(t)dt \qquad (x \geqq 1)\,.$$

Proof. Cf. de Bruijn [2].

A simple consequence of this lemma is

LEMMA 2.

$$y(x) > 0 \qquad (x \geqq 0).$$

As an easy consequence of this lemma and the definition of $y(x)$ we find that $y(x)$ is monotonically decreasing on $x \geqq 1$.

LEMMA 3. *$y(x)$ is concave on $x \geqq 1$.*

Proof. From the definition of $y(x)$ it follows that

$$y\,(x) = 1 - \log x \qquad (1 \leqq x \leqq 2),$$

so that

$$y(x) \text{ is concave on } 1 \leqq x \leqq 2.$$

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Also from the definition of $y(x)$ it is easily seen that $y(x)$ is twice differentiable on $x > 2$, whereas $y(x)$ is precisely once differentiable at $x = 2$. On $x > 2$ we have

$$y''(x) = \frac{d}{dx}\left(-\frac{1}{x}\cdot y(x-1)\right) = \frac{1}{x^2}\cdot y(x-1) + \frac{-1}{x}\cdot\frac{-1}{x-1}\cdot y(x-2) > 0 .$$

Since $y(x)$ is concave on the intervals $1 \leqq x \leqq 2$ and $x > 2$ and differentiable at $x = 2$, we may conclude that $y(x)$ is concave on $x \geqq 1$.

LEMMA 4.

$$y(x) < \frac{1}{2x-1}\cdot y(x-1) , \qquad (x \geqq 2) .$$

Proof. On $x \geqq 2$ we have by Lemma 3 that

$$x\cdot y(x) = \int_{x-1}^{x} y(t)dt < \frac{1}{2}\{y(x-1) + y(x)\}$$

and consequently

$$y(x) < \frac{1}{2x-1}\cdot y(x-1) .$$

From Lemma 4 one easily deduces by induction that

$$y(n) < \frac{1}{3\cdot 5\cdot 7\cdot \cdots (2n-1)} = \frac{2^n\cdot n!}{(2n)!}, \qquad (n = 2, 3, 4, \cdots) .$$

Hence, for example,

$$y(20) < \frac{2^{20}\cdot 20!}{40!} = \frac{2^{20}}{21\cdot 22\cdot 23\cdot \cdots 40} < \frac{2^{20}}{20^{20}} = 10^{-20} .$$

This rough upper bound for $y(20)$ shows that the value of $y(20)$ given by Bellman and Kotkin is not even of the proper order.

3. The Numerical Computation of $y(x)$. Our starting point is the result of Lemma 1

$$y(x) = 1 \qquad (0 \leqq x \leqq 1) ,$$

$$(x+1)\cdot y(x+1) = \int_{x}^{x+1} y(t)dt \qquad (x \geqq 0) .$$

We have already mentioned that

$$y(x) = 1 - \log x, \qquad (1 \leqq x \leqq 2)$$

so that we only have to compute $y(x)$ on $x > 2$.

If we approximate the integral

$$I = \int_{x_0}^{x_0+1} y(t)dt , \qquad (x_0 \geqq 1)$$

by means of the trapezoidal formula

$$\frac{1}{2n}\left\{y(x_0) + 2\sum_{k=1}^{n-1} y\left(x_0 + \frac{k}{n}\right) + y(x_0+1)\right\}$$

we obtain, because of the concavity of $y(x)$ on $x \geqq 1$, that

$$(x_0+1)y(x_0+1) = \int_{x_0}^{x_0+1} y(t)dt < \frac{1}{2n}\left\{y(x_0) + 2\sum_{k=1}^{n-1} y\left(x_0 + \frac{k}{n}\right) + y(x_0+1)\right\}.$$

It follows that

$$y(x_0+1) < \frac{1}{2n(x_0+1)-1}\left\{y(x_0) + 2\sum_{k=1}^{n-1} y\left(x_0+\frac{k}{n}\right)\right\}.$$

Thus, if one has upper bounds for $y(x)$ at the points

$$x_0 + k/n\,, \qquad (k = 0, 1, 2, \cdots, n-1),$$

one may compute an upper bound for $y(x_0+1)$.

Continuing in this way one may compute upper bounds for $y(x)$ at the points $x_0 + 1 + v/n$, $(v = 1, 2, 3, \cdots)$.

On the other hand, approximating I by

$$\frac{1}{n}\sum_{k=1}^{n} y\left(x_0 + \frac{2k-1}{2n}\right)$$

one finds, also because of the concavity of $y(x)$ on $x \geqq 1$, that

$$y(x_0+1) > \frac{1}{n(x_0+1)}\sum_{k=1}^{n} y\left(x_0 + \frac{2k-1}{2n}\right).$$

Hence, as soon as one has lower bounds for $y(x)$ at the points $x_0 + (2k-1)/2n$, $(k = 1, 2, 3, \cdots, n)$ one may compute a lower bound for $y(x_0+1)$. If one also knows lower bounds for $y(x)$ at the points $x_0 + k/n$, $(k = 1, 2, 3, \cdots, n-1)$, one can apply the same method to compute a lower bound for $y(x_0 + 1 + 1/2n)$. Repeating this process one finds lower bounds for $y(x)$ at the points $x_0 + 1 + k/2n$, $(k = 2, 3, 4, \cdots)$. As a starting point for the computations one may take of course $x_0 = 1$.

If one chooses the grid sizes in the above integral-approximating procedures small enough, one may expect that the corresponding upper and lower bounds for $y(x)$ will not differ very much. Actual computations show that this is indeed the case.

Performing the computations on the Electrologica X8 of the Mathematical Centre in Amsterdam, using an ALGOL-60 program (with grid size 0.005), we found that the corresponding upper and lower bounds for $y(x)$ were equal up to at least the first significant digit for all $x < 100$.

Using more refined integral-approximating formulae and smaller grid sizes we were able to compute $y(x)$ for values of x up to at least $x = 1{,}000$. Below we include a table for $y(x)$ with an accuracy of five or more significant figures.

Finally we will compare some of the results of Table 1 with the known asymptotic formula of de Bruijn (cf. [2])

$$y(x) \sim \frac{e^{\gamma}}{(2\pi x)^{1/2}} \exp\left\{1 - e^{\xi} + \int_0^{\xi} \frac{e^s - 1}{s}\, ds\right\} \qquad (x \to \infty)\,,$$

where ξ is the positive root of $e^{\xi} - 1 = x\xi$ and γ is Euler's constant. De Bruijn's formula can be rewritten in terms of the exponential integral $\mathrm{Ei}(t) = \int_{-\infty}^{t} (e^s/s)ds$ as

$$y(x) \sim \frac{1}{(2\pi x)^{1/2}} \cdot \frac{1}{\xi} \cdot \exp\{-x\cdot\xi + \mathrm{Ei}(\xi)\}, \qquad (x \to \infty)$$

which is somewhat more convenient for numerical computations. Writing $B(x)$ for de Bruijn's asymptotic approximation we have

TABLE 1. $y(x) = a(x)\cdot 10^{-b(x)}$

x	$a(x)$	$b(x)$	x	$a(x)$	$b(x)$	x	$a(x)$	$b(x)$
2	0.306852	0	36	0.121869	62	70	0.702809	147
3	0.486083	1	37	0.622168	65	71	0.162933	149
4	0.491092	2	38	0.307395	67	72	0.371471	152
5	0.354724	3	39	0.147112	69	73	0.833076	155
6	0.196496	4	40	0.682549	72	74	0.183819	157
7	0.874566	6	41	0.307253	74	75	0.399153	160
8	0.323206	7	42	0.134297	76	76	0.853156	163
9	0.101624	8	43	0.570381	79	77	0.179535	165
10	0.277017	10	44	0.235551	81	78	0.372043	168
11	0.664480	12	45	0.946492	84	79	0.759361	171
12	0.141971	13	46	0.370280	86	80	0.152686	173
13	0.272918	15	47	0.141120	88	81	0.302503	176
14	0.476063	17	48	0.524252	91	82	0.590640	179
15	0.758990	19	49	0.189943	93	83	0.113672	181
16	0.111291	20	50	0.671533	96	84	0.215679	184
17	0.150907	22	51	0.231788	98	85	0.403511	187
18	0.190135	24	52	0.781464	101	86	0.744510	190
19	0.223542	26	53	0.257465	103	87	0.135495	192
20	0.246178	28	54	0.829313	106	88	0.243271	195
21	0.254805	30	55	0.261272	108	89	0.430958	198
22	0.248638	32	56	0.805427	111	90	0.753402	201
23	0.229371	34	57	0.243046	113	91	0.129996	203
24	0.200549	36	58	0.718206	116	92	0.221416	206
25	0.166580	38	59	0.207907	118	93	0.372331	209
26	0.131725	40	60	0.589802	121	94	0.618228	212
27	0.993606	43	61	0.164025	123	95	0.101374	214
28	0.716213	45	62	0.447329	126	96	0.164183	217
29	0.494179	47	63	0.119673	128	97	0.262667	220
30	0.326904	49	64	0.314165	131	98	0.415161	223
31	0.207626	51	65	0.809545	134	99	0.648360	226
32	0.126782	53	66	0.204821	136	100	0.100059	228
33	0.745257	56	67	0.508958	139	200	0.983383	530
34	0.422222	58	68	0.124246	141	500	0.505734	1558
35	0.230808	60	69	0.298056	144	1000	0.458767	3463

TABLE 2

x	$y(x)$	$B(x)$	$\frac{y(x)}{B(x)}$
20	$0.246178\cdot 10^{-28}$	$0.219619\cdot 10^{-28}$	1.121
100	$0.100059\cdot 10^{-228}$	$0.090892\cdot 10^{-228}$	1.101
1000	$0.458767\cdot 10^{-3463}$	$0.422946\cdot 10^{-3463}$	1.085

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1. R. Bellman & B. Kotkin, "On the numerical solution of a differential-difference equation arising in analytic number theory," *Math. Comp.*, v. 16, 1962, pp. 473–475. MR **26** #5756.

2. N. G. de Bruijn, "The asymptotic behaviour of a function occurring in the theory of primes," *J. Indian Math. Soc.*, v. 15, 1951, pp. 25–32. MR **13**, 326.

For an extensive list of literature concerning the function $y(x)$ we refer to

3. J. van de Lune & E. Wattel, *On the Frequency of Natural Numbers m whose Prime Divisors are all Smaller than* m^α, Mathematical Centre, Amsterdam, Report ZW 1968-007, 1968.