On the Numerical Solution of Integral-Equations.

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§ 1. *Introductory.*

The present communication is concerned with integral-equations of Abel's type

$$\int_0^x \phi(s)\,\kappa(x-s)\,ds = f(x), \tag{1}$$

and of Poisson's type

$$\phi(x) + \int_0^x \phi(s)\,\kappa(x-s)\,ds = f(x), \tag{2}$$

where $\kappa(x)$ is a given function called the *nucleus*, $f(x)$ is also a given function, and $\phi(x)$ is the unknown function which is to be determined. The object of the work is to obtain solutions of these equations in forms which can be made the basis of numerical calculation.

Theoretical solutions of both these equations, in the form of infinite series, are well known, and have been fully discussed by Volterra* and others. But in these solutions the nth term of the series is a multiple integral involving $(n-1)$ integrations with variable limits: and although such series are valuable for the light they throw on the general properties of the solution, it is obvious that they cannot, except in very special cases, be used in order to compute values of the solution numerically. The only case, so far as I am aware, in which a solution of an equation of one of the above types has been obtained in a form adapted for practical ends is Abel's original special form of equation (1),

$$\int_0^x \frac{\phi(s)\,ds}{(x-s)^p} = f(x), \qquad (0 < p < 1, \qquad f(0) = 0),$$

for which he gave† the solution

$$\phi(x) = \frac{1}{\pi}\sin p\pi \int_0^x \frac{f'(s)\,ds}{(x-s)^{1-p}}.$$

* 'Torino Atti,' vol. 31, pp. 311, 400, 557, 693 (1896). For fuller references *cf.* H. Bateman, "Report on the Theory of Integral Equations," 'Brit. Assoc. Report,' 1910.

† 'Œuvres,' (ed. 1881), p. 11 (1823) and p. 97 (1826). The fundamental meaning of Abel's result is most clearly seen if the integrals which occur in it are interpreted as in the theory of generalised differentiation: if $\psi(x)$ is written for $\Gamma(1-p)\,\phi(x)$, Abel's formula reduces to the simple statement that if

$$\left(\frac{d}{dx}\right)^{p-1}\psi(x) = f(x),$$

then

$$\psi(x) = \left(\frac{d}{dx}\right)^{-p} f'(x).$$

When $f(x)$ is given, the values of $f'(s)$ and of the integral last written may be obtained without difficulty by the ordinary processes of interpolation and numerical integration.*

Recently, two friends, one a seismologist and the other an actuary, have enquired of me whether the integral-equations (1) and (2), which had occurred in their researches, could be solved in such a way as to obtain numerical results when the functions $\kappa(x)$ and $f(x)$ are known (tabulated) functions. It was under the stimulus of these enquiries that the methods of solution which occupy the following pages were devised. It will be seen that I have departed altogether from the customary methods of solution by infinite series whose terms are multiple integrals, and on this account the new solutions, which are formulated in Theorems 1–5 below, and by which the unknown function may be determined numerically, may perhaps be found to be not without interest from the standpoint of pure theory.

§ 2. *Solution of Integral-Equations of Abel's Type.*

Considering first the generalised Abel's integral-equation,

$$\int_0^x \phi(s)\,\kappa(x-s)\,ds = f(x), \qquad (1)$$

we need only consider the case when the nucleus $\kappa(x)$ becomes infinite at $x = 0$, for in the simpler case when the nucleus is finite at $x = 0$, the equation (1) may be reduced immediately, as we shall see later, to the type (2), and so may be dealt with by the methods which are given subsequently in the paper.

We shall, then, suppose $\kappa(x)$ to be such that $x^p\kappa(x)$ is finite and not zero at $x = 0$, where p lies between 0 and 1.

Now if the nucleus $\kappa(x)$, which is supposed to be given by a numerical Table, has this character, so that it becomes infinite like x^{-p} at $x = 0$, but is finite for other values of x within the range of integration, we can in general (by use of Newton's or some other interpolation-formula) represent the function $x^p\kappa(x)$ over the range in question by a polynomial in x; the degree of this polynomial will depend on the nature of $\kappa(x)$ and the order of accuracy to which the work is to be carried. We may, then, assume for the nucleus $\kappa(x)$ an analytical expression of the form

$$\kappa(x) = x^{-p}(a_0 + a_1x + a_2x^2 + \ldots + a_nx^n), \qquad (0 < p < 1). \qquad (3)$$

* It is possible that the elegant solution in power-series which was given by Sonine, 'Acta Math.,' vol. 4, p. 171 (1884), for his generalised form of Abel's integral-equation, may also be utilised for numerical calculation.

Now assume* a solution of the form

$$\phi(x) = \int_0^x f'(s) \, K(x-s) \, ds, \tag{4}$$

where $K(x)$ is called the *solving function.* We have on eliminating $\phi(x)$ between this equation and (1)

$$f(x) = \int_0^x \kappa(x-s) \left\{ \int_0^s f'(t) \, K(s-t) \, dt \right\} ds,$$

or, inverting the order of integration,

$$\int_0^x f'(t) \, dt = \int_0^x f'(t) \left\{ \int_t^x \kappa(x-s) \, K(s-t) \, ds \right\} dt.$$

Since $f'(t)$ is an arbitrary function, this gives

$$\int_t^x \kappa(x-s) \, K(s-t) \, ds = 1.$$

Writing $s = t+u$, this becomes

$$\int_0^{x-t} \kappa(x-t-u) \, K(u) \, du = 1,$$

or

$$\int_0^a \kappa(a-u) \, K(u) \, du = 1,$$

where a is arbitrary. From this equation the solving function $K(x)$ is to be determined.

By (3) this may be written

$$\int_0^a (a-u)^{-p} \{a_0 + a_1(a-u) + a_2(a-u)^2 + \ldots + a_n(a-u)^n\} \, K(u) \, du = 1. \tag{5}$$

Now let $\int_0^u K(u) \, du$ be denoted by $K_1(u)$; let $\int_0^u K_1(u) \, du$ be denoted by $K_2(u)$, and so on. Then integrating by parts we have

$$\int_0^a (a-u)^{1-p} \, K(u) \, du = (1-p) \int_0^a (a-u)^{-p} \, K_1(u) \, du,$$

$$\int_0^a (a-u)^{2-p} \, K u \, (du) = (2-p) \int_0^a (a-u)^{1-p} \, K_1(u) \, du,$$

$$= (2-p)(1-p) \int_0^a (a-u)^{-p} \, K_2(u) \, du,$$

and so on. Thus equation (5) may be written

$$\int_0^a (a-u)^{-p} \{a_0 K(u) + (1-p) a_1 K_1(u) + (1-p)(2-p) a_2 K_2(u) + \ldots$$
$$+ (1-p)(2-p)\ldots(n-p) a_n K_n(u)\} \, du = 1. \tag{6}$$

* The legitimacy of this may be inferred at once from Volterra's theory.

Now if $\int_0^a (a-u)^{-p} h(u)\,du = 1$, where a is arbitrary and $h(u)$ does not involve a, we have by writing $u = as$

$$a^{p-1} = \int_0^1 (1-s)^{-p} h(as)\,ds,$$

and, since a is arbitrary, this shows that $h(u) = Cu^{p-1}$, where C is independent of u.

Substituting in the last equation, we have

$$1 = C\int_0^1 (1-s)^{-p} s^{p-1}\,ds = \frac{C\pi}{\sin p\pi}$$

and therefore

$$h(u) = \frac{\sin p\pi}{\pi} \cdot u^{p-1}.$$

Applying this result to equation (6), we have

$$a_0 K(u) + (1-p)\,a_1 K_1(u) + (1-p)(2-p)\,a_2 K_2(u) + \dots$$
$$+ (1-p)(2-p)\dots(n-p)\,a_n K_n(u) = \frac{\sin p\pi}{\pi} \cdot u^{p-1},$$

or, writing $y(u)$ for $K_n(u)$

$$a_0 \frac{d^n y}{du^n} + (1-p)\,a_1 \frac{d^{n-1}y}{du^{n-1}} + (1-p)(2-p)\,a_2 \frac{d^{n-2}y}{du^{n-2}} + \dots$$
$$+ (1-p)(2-p)\dots(n-p)\,a_n y = \frac{\sin p\pi}{\pi} \cdot u^{p-1}. \quad (7)$$

This is a linear differential equation in y, and $K_n(u)$ is that solution of it which vanishes, together with its first $(n-1)$ differential coefficients, when u vanishes.

Now let the polynomial

$$a_0 x^n + (1-p)\,a_1 x^{n-1} + (1-p)(2-p)\,a_2 x^{n-2} + \dots + (1-p)(2-p)\dots(n-p)\,a_n$$

be denoted by $F(x)$: and let its n roots (supposed for the present to be distinct) be $\alpha, \beta, \gamma, \dots, \nu$. Then it is known from the general theory of linear differential equations that the solution of (7) which vanishes, together with its first $(n-1)$ differential coefficients, when u vanishes is

$$K_n(u) = \int_0^u \frac{\sin p\pi}{\pi} \cdot t^{p-1} \cdot h(u-t)\,dt \quad (8)$$

where
$$h(x) = \frac{e^{\alpha x}}{F'(\alpha)} + \frac{e^{\beta x}}{F'(\beta)} + \dots + \frac{e^{\nu x}}{F'(\nu)}. \quad (9)$$

From (8) we have by successive differentiation

$$K_{n-1}(u) = \frac{\sin p\pi}{\pi}\int_0^u t^{p-1}h'(u-t)\,dt, \qquad \text{since } h(0) = 0,$$

$$K_{n-2}(u) = \frac{\sin p\pi}{\pi}\int_0^u t^{p-1}h''(u-t)\,dt, \qquad \text{since } h'(0) = 0,$$

....

....

$$K_1(u) = \frac{\sin p\pi}{\pi}\int_0^u t^{p-1}h^{(n-1)}(u-t)\,dt, \qquad \text{since } h^{(n-2)}(0) = 0,$$

$$K(u) = \frac{\sin p\pi}{\pi}\left\{\frac{u^{p-1}}{a_0} + \int_0^u t^{p-1}h^{(n)}(u-t)\,dt\right\}, \quad \text{since } h^{(n-1)}(0) = \frac{1}{a_0}.*$$

Substituting for $h(x)$ from (9) we have $K(x) = (1/\pi)\sin p\pi\, L(x)$, where

$$L(x) = \frac{x^{p-1}}{a_0} + \frac{\alpha^n}{F'(\alpha)}e^{\alpha x}\int_0^x t^{p-1}e^{-\alpha t}\,dt + \frac{\beta^n}{F'(\beta)}e^{\beta x}\int_0^x t^{p-1}e^{-\beta t}\,dt + \dots + \frac{\nu^n}{F'(\nu)}e^{\nu x}\int_0^x t^{p-1}e^{-\nu t}\,dt.$$

Now
$$\int_0^x t^{p-1}e^{-\alpha t}\,dt = \alpha^{-p}\int_0^{\alpha x} s^{p-1}e^{-s}\,ds;$$

and thus if the function
$$e^x\int_0^x s^{p-1}e^{-s}\,ds$$

(which is well known under the name of the *Incomplete Gamma-Function*) be denoted by $\gamma_p(x)$, we have

$$L(x) = \frac{x^{p-1}}{a_0} + \frac{\alpha^{n-p}}{F'(\alpha)}\gamma_p(\alpha x) + \frac{\beta^{n-p}}{F'(\beta)}\gamma_p(\beta x) + \dots + \frac{\nu^{n-p}}{F'(\nu)}\gamma_p(\nu x).$$

Combining our results, we have

THEOREM 1.—*The solution of the integral-equation*

$$\int_0^x \phi(s)\,\kappa(x-s)\,ds = f(x)$$

where the nucleus $\kappa(x)$ is supposed to be given numerically and to have been expressed by the ordinary methods of interpolation in the form

$$\kappa(x) = x^{-p}(a_0 + a_1x + a_2x^2 + \dots + a_nx^n), \qquad (0 < p < 1)$$

is
$$\phi(x) = \frac{\sin p\pi}{\pi}\int_0^x f'(s)\,L(x-s)\,ds, \tag{10}$$

where
$$L(x) = \frac{x^{p-1}}{a_0} + \frac{\alpha^{n-p}}{F'(\alpha)}\gamma_p(\alpha x) + \frac{\beta^{n-p}}{F'(\beta)}\gamma_p(\beta x) + \dots + \frac{\nu^{n-p}}{F'(\nu)}\gamma_p(\nu x), \tag{11}$$

* For $h^{(n-1)}(0)$ = sum of residues of $t^{n-1}/F(t)$ at its singularities $\alpha, \beta, \dots, \nu$
= −(residue of this function at ∞) = $1/a_0$.

and where $\alpha, \beta, \ldots, \nu$, are *the roots of the algebraic equation*

$$F(x) \equiv a_0 x^n + (1-p) a_1 x^{n-1} + (1-p)(2-p) a_2 x^{n-2} + \ldots + (1-p)(2-p)\ldots(n-p) a_n = 0, \quad (12)$$

and $\gamma_p(x)$ *denotes the incomplete Gamma-function*

$$\gamma_p(x) = e^x \int_0^x s^{p-1} e^{-s} ds.$$

This may be regarded as a direct extension of Abel's original formula, which may be derived from it by taking $n = 0$. It expresses the solution of the integral-equation *in a finite form* in terms of the incomplete Γ-function. The incomplete Γ-functions which occur in the solution all have the same parameter p, and are, therefore, really all the same function, with different values of the argument; its values may be tabulated from any of the expansions which were given for it by Legendre,* such as the absolutely convergent expansion

$$\gamma_p(x) = \frac{x^p}{p} + \frac{x^{p+1}}{p(p+1)} + \frac{x^{p+2}}{p(p+1)(p+2)} + \ldots,$$

or (for large positive values of x) the asymptotic expansion

$$\gamma_p(x) = \Gamma(p) e^x - x^{p-1} \left\{ 1 + \frac{p-1}{x} + \frac{(p-1)(p-2)}{x^2} + \ldots \right\}.$$

When this function has been tabulated, and the algebraic equation (12) solved,† the function $L(x)$ may readily be tabulated from equation (11), and then the required function $\phi(x)$ is given at once by quadrature from equation (10). In this way Theorem 1 yields a numerical solution of the generalised Abel's equation.

It is obvious that when the polynomial $F(x)$ has a pair of equal roots certain modifications must be made in the above solution, but it does not seem necessary to set these forth in detail here.

It may be remarked that by making the degree n of the polynomial in (3) increase indefinitely, we obtain in the limit the solving-function of the integral equation, with an arbitrary function as nucleus, in the form of an infinite series of incomplete Γ-functions of the same parameter. In connection with such a solution it would be necessary to discuss convergence, etc., and it is not proposed to undertake this in the present paper; but the matter seems worthy to be mentioned in passing, as series of incomplete Γ-functions have not (so far as I know) presented themselves hitherto in

* 'Exerc. de Calc. Int.,' vol. 1, pp. 338–343 (1811).

† For this Newton's method will probably be found in most cases the most convenient, if $n > 3$; or if some of the roots are complex, the Lobachevsky-Graeffe method.

analysis. They may evidently be regarded as an extension of *Dirichlet's series, i.e.*, series of the type

$$F(x) = ae^{-\lambda x} + be^{-\mu x} + ce^{-\nu x} + \ldots,$$

which have received much attention in recent years.

§ 3. *Solution of Equations of Poisson's Type.*

When the nucleus $\kappa(x)$ of equation (1), instead of being infinite at $x = 0$ as we have supposed hitherto, possesses a finite differential coefficient over the range of integration, we have by differentiating equation (1)

$$\phi(x)\kappa(0) + \int_0^x \phi(s)\kappa'(x-s)\,ds = f'(x)$$

which is an integral equation included in Poisson's type (2).* We shall, therefore, now pass on to consider integral-equations of Poisson's type, which we shall take in the form

$$\phi(x) + \int_0^x \phi(s)\kappa(x-s)\,ds = f(x). \tag{2}$$

Since the nucleus $\kappa(x)$ is supposed to be specified by a table of finite numerical values over the range of values of x considered, we may apply Prony's method of interpolation by exponentials in order to represent it analytically in the form of a sum of μ exponentials

$$\kappa(x) = Pe^{px} + Qe^{qx} + Re^{rx} + \ldots + Ve^{vx}, \tag{13}$$

where $(P, Q, R, \ldots, V, p, q, r, \ldots, v)$ are constants which are chosen so as to give the closest possible representation of the given numerical values.

Although Prony's method is more than a century old, it does not appear to be widely known or to have found its way into any text-book; and, as his original paper is perhaps not accessible to many readers, I may be justified in giving here a brief notice of it.

Suppose that $\kappa(x)$ is given numerically for a certain range of values of x. Take any set of values of x equally spaced within this range, say $x = 0, \omega, 2\omega, 3\omega, 4\omega, \ldots$, and let the corresponding values of $\kappa(x)$ be $\kappa_0, \kappa_1, \kappa_2, \kappa_3, \ldots$. Now if $\kappa(x)$ could be represented *exactly* in the form of a sum of μ exponentials, say,

$$Pe^{px} + Qe^{qx} + Re^{rx} + \ldots + Ve^{vx},$$

then $\kappa(x)$ would satisfy a linear difference-equation of the form

$$A\kappa_{n+\mu} + B\kappa_{n+\mu-1} + C\kappa_{n+\mu-2} + \ldots\ldots + M\kappa_n = 0,$$

where the roots of the algebraic equation

$$Ax^{\mu} + Bx^{\mu-1} + Cx^{\mu-2} + \ldots\ldots + M = 0$$

would be

$$e^{p\omega},\ e^{q\omega},\ e^{r\omega},\ \ldots\ldots e^{v\omega}.$$

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* It need scarcely be said that if $\kappa(0)$ vanishes we differentiate again.

Prony's method, which is based on this fact, is to write down a set of linear equations,

$$\begin{aligned}
A\kappa_{\mu} + B\kappa_{\mu-1} + C\kappa_{\mu-2} + \ldots\ldots + M\kappa_0 &= 0 \\
A\kappa_{\mu+1} + B\kappa_{\mu} + C\kappa_{\mu-1} + \ldots\ldots + M\kappa_1 &= 0 \\
A\kappa_{\mu+2} + B\kappa_{\mu+1} + C\kappa_{\mu} + \ldots\ldots + M\kappa_2 &= 0 \\
A\kappa_{\mu+3} + B\kappa_{\mu+2} + C\kappa_{\mu+1} + \ldots + M\kappa_3 &= 0 \\
\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots
\end{aligned}$$

where the quantities κ_0, κ_1, κ_2, κ_3, ..., are known, since $\kappa(x)$ is a known tabulated function, and by the ordinary method of Least Squares to find the values of A, B, C, ..., M, which most nearly satisfy the equations; then with these values of A, B, C, ..., M, to form the algebraic equation

$$Ax^{\mu} + Bx^{\mu-1} + Cx^{\mu-2} + \ldots + M = 0,$$

and find its roots; these roots will be $e^{p\omega}$, $e^{q\omega}$, $e^{r\omega}$, ... $e^{v\omega}$, and thus p, q, r, ..., v, are determined. Knowing p, q, r, ..., v, we have a set of linear equations to determine the coefficients P, Q, R, ..., V, and these also are to be solved by the method of Least Squares.

Taking then this form (13) for the nucleus $\kappa(x)$, we shall show that the integral-equation (2) may be satisfied by a solution of the form

$$\phi(x) = f(x) - \int_0^x K(x-s) f(s)\, ds, \tag{14}$$

where the solving function $K(x)$ is also a sum of μ exponentials, say

$$K(x) = Ae^{\alpha x} + Be^{\beta x} + Ce^{\gamma x} + \ldots + Ne^{\nu x}. \tag{15}$$

To prove this, we remark first that the existence-theorems established by Volterra justify us in assuming for the solution the form (14), where $K(x)$ is now the function to be determined.

In (14) put $\kappa(x)$ for $f(x)$: thus

$$\phi(x) = \kappa(x) - \int_0^x K(x-s)\, \kappa(s)\, ds,$$

which gives the value of $\phi(x)$ corresponding to this value of $f(x)$.

Putting $(x-s)$ for s in the integral, we have

$$\phi(x) = \kappa(x) - \int_0^x K(s)\, \kappa(x-s)\, ds.$$

Comparing this with the integral-equation (2), after replacing $f(x)$ by $\kappa(x)$ in the latter, we have

$$\phi(x) = K(x),$$

and therefore the pair of functions

$$\phi(x) = K(x), \qquad f(x) = \kappa(x),$$

satisfy the integral-equation: that is to say,

$$K(x) + \int_0^x K(s)\, \kappa(x-s)\, ds = \kappa(x). \tag{16}$$

In this equation substitute the value (13) for $\kappa(x)$ and the value (15) for K(x). Thus we have

$$Ae^{\alpha x}+Be^{\beta x}+Ce^{\gamma x}+\ldots+Ne^{\nu x}$$

$$+\int_0^x \{Ae^{\alpha s}+Be^{\beta s}+Ce^{\gamma s}+\ldots+Ne^{\nu s}\}\{Pe^{px-ps}+Qe^{qx-qs}+\ldots+Ve^{vx-vs}\}\,ds$$

$$=Pe^{px}+Qe^{qx}+Re^{rx}+\ldots+Ve^{vx}.$$

Equating coefficients of $e^{\alpha x}$ on the two sides of this equation, we have

$$\frac{P}{\alpha-p}+\frac{Q}{\alpha-q}+\ldots+\frac{V}{\alpha-v}+1=0.$$

Similarly by equating coefficients of $e^{\beta x}$

$$\frac{P}{\beta-p}+\frac{Q}{\beta-q}+\ldots+\frac{V}{\beta-v}+1=0,$$

and so on: these equations show that α, β, γ, ..., ν, are the roots of the algebraic equation in x

$$\frac{P}{x-p}+\frac{Q}{x-q}+\frac{R}{x-r}+\ldots+\frac{V}{x-v}+1=0. \tag{17}$$

This enables us to determine α, β, γ, ...

Next equating coefficients of e^{px} on the two sides of the equation, we have

and similarly

$$\left.\begin{aligned}
&\frac{A}{\alpha-p}+\frac{B}{\beta-p}+\frac{C}{\gamma-p}+\ldots+\frac{N}{\nu-p}+1=0\\
&\frac{A}{\alpha-q}+\frac{B}{\beta-q}+\frac{C}{\gamma-q}+\ldots+\frac{N}{\nu-q}+1=0\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
&\frac{A}{\alpha-v}+\frac{B}{\beta-v}+\frac{C}{\gamma-v}+\ldots+\frac{N}{\nu-v}+1=0
\end{aligned}\right\}. \tag{18}$$

Since $(\alpha, \beta, \gamma, \ldots, \nu)$ and $(p, q, r, \ldots, v)$ are known, these equations (18) enable us to determine A, B, C, ..., N, and we see that if the constants $(\alpha, \beta, \gamma, \ldots, \nu)$ and (A, B, C, ..., N) are determined by equations (17) and (18), the equation (16) is satisfied by the value (15) of K(x).

The value of K(x) may be obtained in a more explicit form in the following manner. If we eliminate A, B, C, ..., N, determinantly from the equations (15) and (18), we have

$$K(x)\begin{vmatrix} \frac{1}{\alpha-p} & \frac{1}{\beta-p} \cdots \frac{1}{\nu-p} \\ \frac{1}{\alpha-q} & \frac{1}{\beta-q} \cdots \frac{1}{\nu-q} \\ \cdots & \cdots \\ \cdots & \cdots \\ \frac{1}{\alpha-v} & \frac{1}{\beta-v} \cdots \frac{1}{\nu-v} \end{vmatrix} = -e^{\alpha x}\begin{vmatrix} 1 & \frac{1}{\beta-p} \cdots \frac{1}{\nu-p} \\ 1 & \frac{1}{\beta-q} \cdots \frac{1}{\nu-q} \\ \cdots & \cdots \\ \cdots & \cdots \\ 1 & \frac{1}{\beta-v} \cdots \frac{1}{\nu-v} \end{vmatrix} - e^{\beta x}\begin{vmatrix} \frac{1}{\alpha-p} & 1 \cdots \frac{1}{\nu-p} \\ \frac{1}{\alpha-q} & 1 \cdots \frac{1}{\nu-q} \\ \cdots & \cdots \\ \cdots & \cdots \\ \frac{1}{\alpha-v} & 1 \cdots \frac{1}{\nu-v} \end{vmatrix} - \ldots$$

The determinants which occur in this equation are of the kind known as *alternants,* and may be factorised by known methods.* Performing the factorisation, we have

$$K(x) = -\frac{(\alpha-p(\alpha-q)(\alpha-r)\ldots(\alpha-v)}{(\alpha-\beta)(\alpha-\gamma)\ldots(\alpha-\nu)}e^{\alpha x}$$
$$-\frac{(\beta-p)(\beta-q)(\beta-r)\ldots(\beta-v)}{(\beta-\alpha)(\beta-\gamma)\ldots(\beta-\nu)}e^{\beta x}-\ldots$$
$$\ldots-\frac{(\nu-p)(\nu-q)(\nu-r)\ldots(\nu-v)}{(\nu-\alpha)(\nu-\beta)\ldots(\nu-\mu)}e^{\nu x}.$$

Combining our results we have

THEOREM 2.—*The solution of the integral-equation*

$$\phi(x)+\int_0^x \phi(s)\,\kappa(x-s)\,ds = f(x),$$

where the nucleus $\kappa(x)$ *is supposed to be given numerically, and to have been expressed by Prony's method of interpolation in the form*

$$\kappa(x) = \mathrm{P}e^{px}+\mathrm{Q}e^{qx}+\mathrm{R}e^{rx}+\ldots+\mathrm{V}e^{vx},$$

is

$$\phi(x) = f(x)-\int_0^x \mathrm{K}(x-s)f(s)\,ds,$$

where

$$K(x) = -\frac{(\alpha-p)(\alpha-q)\ldots(\alpha-v)}{(\alpha-\beta)(\alpha-\gamma)\ldots(\alpha-\nu)}e^{\alpha x}-\frac{(\beta-p)(\beta-q)\ldots(\beta-v)}{(\beta-\alpha)(\beta-\gamma)\ldots(\beta-\nu)}e^{\beta x}-\ldots$$
$$\ldots-\frac{(\nu-p)(\nu-q)\ldots(\nu-v)}{(\nu-\alpha)(\nu-\beta)\ldots(\nu-\mu)}e^{\nu x}, \quad (19)$$

and where $\alpha, \beta, \gamma, \ldots, \nu$, *are the roots of the algebraic equation in* x

$$\frac{\mathrm{P}}{x-p}+\frac{\mathrm{Q}}{x-q}+\frac{\mathrm{R}}{x-r}+\ldots+\frac{\mathrm{V}}{x-v}+1 = 0.$$

* The evaluation of alternants of this type is due to Cauchy, 'Exercices d'An.,' vol. 2, p. 151 (1841).

It is obvious from the last equation that if $p, q, r, \ldots, v$, are arranged in ascending order of magnitude, and, if P, Q, R, ..., V, are all positive, then the lowest root α is less than p, the next root β is between p and q, and so on.

If the number of exponential terms in (13) is supposed to increase indefinitely, the representation of the nucleus $\kappa(x)$ becomes a Dirichlet's series, and by (19) the solving function $K(x)$ is then also a Dirichlet's series, formed with exponentials $e^{\alpha x}$, $e^{\beta x}$, $e^{\gamma x}$, ..., whose exponents $\alpha, \beta, \gamma, \ldots$, are the roots of equation (17), which now becomes a transcendental equation. A rigorous examination of convergence, etc., in this limit-process would be necessary to establish the theorem which appears to be indicated, namely, that *in the solution of a Poisson's integral-equation whose nucleus is expressible as a Dirichlet's series, the solving-function is also expressible as a Dirichlet's series, but with a different set of exponents for the exponentials.*

§ 4. *An Alternative Solution of Integral-Equations of Poisson's Type.*

Theorem 2 above supplies what I think will be found to be in general the most convenient method of solving integral-equations of Poisson's type numerically. But in certain cases the nucleus $\kappa(x)$ may be given as a polynomial, or it may happen, when $\kappa(x)$ is given in the form of a numerical table, that it is preferred for some reason to apply ordinary interpolation and express $\kappa(x)$ approximately as a polynomial, rather than to apply Prony's method of interpolation and express $\kappa(x)$ as a sum of exponentials.* We shall therefore now consider the problem of integrating an integral-equation of Poisson's type

$$\phi(x)+\int_0^x \phi(s)\,\kappa(x-s)\,ds = f(x) \tag{2}$$

when the nucleus $\kappa(x)$ is expressed as a polynomial.

The solution of this problem may be deduced as a limiting case from that solved in § 3. For, in equation (13), suppose that $p, q, r, \ldots, v$ (being n in number), each tends to zero, while P, Q, R, ..., V, increase indefinitely in such a way that

$$\left.\begin{aligned} P+Q+R+\ldots+V &= \kappa_0 \\ Pp+Qq+Rr+\ldots+Vv &= \kappa_1 \\ &\cdots\cdots \\ &\cdots\cdots \\ Pp^{n-1}+Qq^{n-1}+Rr^{n-1}+\ldots+Vv^{n-1} &= \kappa_{n-1} \end{aligned}\right\} \tag{20}$$

* One great advantage of Prony's method is that each exponential term involves *two* disposable constants (*e.g.*, the term Pe^{px} involves P and p), whereas a term of a polynomial involves only *one* disposable constant (*e.g.*, px^3 involves only p), and therefore it is in general possible to obtain as high a degree of accuracy in approximating with n exponential terms as with a polynomial of $2n$ terms.

where $\kappa_0, \kappa_1, \kappa_2, \ldots, \kappa_{n-1}$, are finite. Then from equation (13) we have

$$\kappa(x) = \mathrm{P}\left(1+px+\frac{p^2x^2}{2!}+\ldots\right)+\mathrm{Q}\left(1+qx+\frac{q^2x^2}{2!}+\ldots\right)+\ldots$$
$$+\mathrm{V}\left(1+vx+\frac{v^2x^2}{2!}+\ldots\right)$$
$$=\kappa_0+\kappa_1 x+\frac{\kappa_2 x^2}{2!}+\ldots+\frac{\kappa_{n-1}x^{n-1}}{(n-1)!},$$

the terms in higher powers of x vanishing.

Moreover, the equation (17), whose roots are $\alpha, \beta, \gamma, \ldots, \nu$, may be written

$$\mathrm{P}\left(\frac{1}{x}+\frac{p}{x^2}+\frac{p^2}{x^3}+\ldots\right)+\mathrm{Q}\left(\frac{1}{x}+\frac{q}{x^2}+\frac{q^2}{x^3}+\ldots\right)+\ldots+\mathrm{V}\left(\frac{1}{x}+\frac{v}{x^2}+\frac{v^2}{x^3}+\ldots\right)+1=0,$$

which by (20) becomes

$$\frac{\kappa_0}{x}+\frac{\kappa_1}{x^2}+\frac{\kappa_2}{x^3}+\ldots+\frac{\kappa_{n-1}}{x^n}+1 = 0,$$

the terms in higher powers of $(1/x)$ vanishing, so that $\alpha, \beta, \gamma, \ldots, \nu$, are now the roots of the algebraic equation

$$x^n+\kappa_0 x^{n-1}+\kappa_1 x^{n-2}+\ldots+\kappa_{n-1} = 0.$$

The equation (19) for the determination of the solving function $\mathrm{K}(x)$ now becomes

$$\mathrm{K}(x) = -\frac{\alpha^n}{(\alpha-\beta)(\alpha-\gamma)\ldots(\alpha-\nu)}e^{\alpha x}-\frac{\beta^n}{(\beta-\alpha)(\beta-\gamma)\ldots(\beta-\nu)}e^{\beta x}-\ldots$$
$$\ldots-\frac{\nu^n}{(\nu-\alpha)(\nu-\beta)\ldots(\nu-\mu)}e^{\nu x}$$

and thus, collecting our results, we have

THEOREM 3.—*The solution of the integral-equation*

$$\phi(x)+\int_0^x \phi(s)\,\kappa(x-s)\,ds = f(x),$$

where the nucleus $\kappa(x)$ is supposed to be given numerically, and to have been expressed by the ordinary methods of interpolation in the form

$$\kappa(x) = \kappa_0+\kappa_1 x+\frac{\kappa_2 x^2}{2!}+\frac{\kappa_3 x^3}{3!}+\ldots+\frac{\kappa_{n-1}x^{n-1}}{(n-1)!}, \tag{21}$$

is

$$\phi(x) = f(x)-\int_0^x \mathrm{K}(x-s)f(s)\,ds, \tag{22}$$

where

$$\mathrm{K}(x) = -\frac{\alpha^n}{(\alpha-\beta)(\alpha-\gamma)\ldots(\alpha-\nu)}e^{\alpha x}-\frac{\beta^n}{(\beta-\alpha)(\beta-\gamma)\ldots(\beta-\nu)}e^{\beta x}-\ldots$$
$$\ldots-\frac{\nu^n}{(\nu-\alpha)(\nu-\beta)\ldots(\nu-\mu)}e^{\nu x}, \tag{23}$$

and where $\alpha, \beta, \gamma, \ldots, \nu$, *are the roots of the algebraic equation in* x

$$x^n + \kappa_0 x^{n-1} + \kappa_1 x^{n-2} + \ldots + \kappa_{n-1} = 0. \tag{24}$$

If this equation (24) has a pair of equal roots, terms of the type $xe^{\lambda x}$ will occur in $K(x)$, thus, if

$$\kappa(x) = -2p + p^2 x,$$

we find

$$K(x) = -p(2+px)e^{px}.$$

An interesting expansion is obtained by making n increase indefinitely in this theorem. We then have the solving problem $K(x)$ of the general Poisson's integral-equation expressed in the form of a Dirichlet's series, or, at any rate, a series of exponentials with real or complex arguments. This appears to indicate a new field of analysis, in which Dirichlet's series present themselves naturally. But a thorough investigation of convergence would be necessary for the rigorous establishment of this result, and, indeed, we can show by simple examples that the expression obtained by making n tend to infinity will not necessarily be a Dirichlet's series, even though, for all finite values of n, it is a sum of exponentials. For instance, if

$$\kappa(x) = p + p^2 x + \frac{p^3 x^2}{2!} + \frac{p^4 x^3}{3!} + \ldots + \frac{p^n x^{n-1}}{(n-1)!},$$

so that $\kappa_0 = p, \quad \kappa_1 = p^2, \quad \kappa_2 = p^3, \quad \ldots, \quad \kappa_{n-1} = p^n,$

we see from (23) and (24) that $K(x)$ is equal to the sum of the residues of

$$-\frac{(1-p/t)\,e^{tx}}{1-p^{n+1}/t^{n+1}}$$

at its n poles, which are the $(n+1)$th roots of unity other than unity itself. This sum of residues is evidently a sum of exponentials in x, one corresponding to each pole, so long as n is finite, but, when n increases indefinitely, we have

$$K(x) = \text{coefficient of } 1/t \text{ in } -(1-p/t)\,e^{tx} = p,$$

so that when the nucleus is $\kappa(x) = pe^{px}$, the solving function is $K(x) = p$, which is not a series of exponentials.

§ 5. *A Further Alternative Solution of Integral-Equations of Poisson's Type.*

Hitherto we have supposed the nucleus of the integral-equation to be given by a numerical table, and to be represented, by use of the methods of the interpolation theory, as a sum of exponentials, or as a polynomial. If, however, in an equation of Poisson's type

$$\phi(x) + \int_0^x \kappa(x-s)\,\phi(s)\,ds = f(x)$$

the expansion of the nucleus $\kappa(x)$ as a Taylor's series is known, say

$$\kappa(x) = \kappa_0 + \kappa_1 x + \frac{\kappa_2 x^2}{2!} + \frac{\kappa_3 x^3}{3!} + \ldots, \tag{25}$$

then we shall show that the solving-function $K(x)$ may be written down at once as a Taylor's series. For, the solution of the integral-equation in terms of the solving-function being

$$\phi(x) \doteq f(x) - \int_0^x K(x-s) f(s)\, ds,$$

we have already in equation (16) shown that

$$K(x) + \int_0^x K(s)\, \kappa(x-s)\, ds = \kappa(x).$$

Putting $x = 0$ in this, we have

$$K_0 = \kappa_0, \tag{26}$$

where $K_0, K_1, K_2, \ldots$, are used in order to denote the values of $K(x)$ and its successive differential coefficients at $x = 0$.

Differentiating (16), we have

$$K'(x) + \kappa_0 K(x) + \int_0^x K(s)\, \kappa'(x-s)\, ds = \kappa'(x). \tag{27}$$

Putting $x = 0$ in (27)

$$K_1 + \kappa_0 K_0 = \kappa_1. \tag{28}$$

Differentiating (27)

$$K''(x) + \kappa_0 K'(x) + \kappa_1 K(x) + \int_0^x K(s)\, \kappa''(x-s)\, ds = \kappa''(x). \tag{29}$$

Putting $x = 0$ in (29),

$$K_2 + \kappa_0 K_1 + \kappa_1 K_0 = \kappa_2. \tag{30}$$

If now the linear equations (26), (28), (30), ..., be solved for $K_0, K_1, K_2, \ldots$, we obtain

$$K_0 = \kappa_0$$

$$K_1 = -\begin{vmatrix} \kappa_0 & 1 \\ \kappa_1 & \kappa_0 \end{vmatrix}$$

$$K_2 = \begin{vmatrix} \kappa_0 & 1 & 0 \\ \kappa_1 & \kappa_0 & 1 \\ \kappa_2 & \kappa_1 & \kappa_0 \end{vmatrix}, \text{ etc.}$$

Let us now consider the convergence of the series

$$K_0 + K_1 x + \frac{K_2}{2!} x^2 + \frac{K^3}{3!} x^3 + \ldots.$$

This series will evidently be absolutely convergent for all finite values of x, provided the series

$$K_0 + K_1 x + K_2 x^2 + K_3 x^3 + \ldots$$

converges within a circle of non-zero radius. But the equations (26), (28), (30),..., may be comprehended in the single formal equation

$$(\kappa_0+\kappa_1 x+\kappa_2 x^2+\ldots)(1+\kappa_0 x+\kappa_1 x^2+\kappa_2 x^3+\ldots) \equiv K_0+K_1 x+K_2 x^2+\ldots$$

or

$$K_0+K_1 x+K_2 x^2+K_3 x^3+\ldots = \frac{1}{x}\left\{1-\frac{1}{1+\kappa_0 x+\kappa_1 x^2+\kappa_2 x^3+\ldots}\right\},$$

and the expression on the right-hand side of this equatión represents a holomorphic function within a circle of non-zero radius having the origin as centre, provided the power-series

$$1+\kappa_0 x+\kappa_1 x^2+\kappa_2 x^3+\ldots$$

converges within a circle of non-zero radius and so represents a function whose singularities and zeroes are all at a finite distance from the origin. Subject to this last condition, then (which is, as a matter of fact, unnecessarily stringent), the series

$$K_0+K_1 x+K_2\frac{x^2}{2!}+K_3\frac{x^3}{3!}+\ldots$$

converges absolutely for all finite values of x. We shall assume that it converges, since the computer will not make use of this method unless the convergence of this series is so rapid as to be obvious. Then, combining our results, we have

THEOREM 4.—*The solution of the integral-equation*

$$\phi(x)+\int_0^x \phi(s)\,\kappa(x-s)\,ds = f(x)$$

where the nucleus $\kappa(x)$ *is supposed to be expansible in a Taylor series*

$$\kappa(x) = \kappa_0+\kappa_1 x+\frac{\kappa_2 x^2}{2!}+\frac{\kappa_3 x^3}{3!}+\ldots$$

is

$$\phi(x) = f(x)-\int_0^x K(x-s)\,f(s)\,ds,$$

where

$$K(x) = \kappa_0-\begin{vmatrix}\kappa_0 & 1\\ \kappa_1 & \kappa_0\end{vmatrix}x+\begin{vmatrix}\kappa_0 & 1 & 0\\ \kappa_1 & \kappa_0 & 1\\ \kappa_2 & \kappa_1 & \kappa_0\end{vmatrix}\frac{x^2}{2!}-\begin{vmatrix}\kappa_0 & 1 & 0 & 0\\ \kappa_1 & \kappa_0 & 1 & 0\\ \kappa_2 & \kappa_1 & \kappa_0 & 1\\ \kappa_3 & \kappa_2 & \kappa_1 & \kappa_0\end{vmatrix}\frac{x^3}{3!}+\ldots \quad (31)$$

For the benefit of those who may find it convenient to use this solution, it may be well to add some remarks on the computation of the determinantal coefficients which occur in it. A numerical determinant should never, or scarcely ever, be evaluated by expanding it; it should be evaluated by reducing it successively to determinants of lower order, without expanding. To do this, we first notice whether any of the elements in the determinant is unity; if not, we reduce one of the elements to unity by dividing the row or column which contains it by that element. Having done this, to evaluate, *e.g.*, the fifth-order determinant

$$\begin{vmatrix} 1 & a_2 & a_3 & a_4 & a_5 \\ b_1 & b_2 & b_3 & b_4 & b_5 \\ c_1 & c_2 & c_3 & c_4 & c_5 \\ d_1 & d_2 & d_3 & d_4 & d_5 \\ e_1 & e_2 & e_3 & e_4 & e_5 \end{vmatrix}$$

the rule is: Strike out the row and column which contain the unit element, and subtract from each of the other elements the product of the elements which are situated at the feet of the perpendiculars from this element on the deleted row and column. Thus the above determinant becomes

$$\begin{vmatrix} b_2-a_2b_1 & b_3-a_3b_1 & b_4-a_4b_1 & b_5-a_5b_1 \\ c_2-a_2c_1 & c_3-a_3c_1 & c_4-a_4c_1 & c_5-a_5c_1 \\ d_2-a_2d_1 & d_3-a_3d_1 & d_4-a_4d_1 & d_5-a_5d_1 \\ e_2-a_2e_1 & e_3-a_3e_1 & e_4-a_4e_1 & e_5-a_5e_1 \end{vmatrix}$$

and in the same way we may reduce this fourth-order determinant to a third-order determinant, and so on.

If the unit element is not the leading element, we must multiply the whole determinant by ± 1, according as the sum of the row-number of the deleted row and the column-number of the deleted column is even or odd.

§ 6. *A Combination of the Solution of* § 4 *with that of* § 5.

A form of solution which in some cases proves useful is obtained by combining the solution of § 4 with that of § 5. This happens when the graph of the nucleus $\kappa(x)$ in the part of it over which the integration takes place, is not very different from that of a polynomial of low degree, so that the first few of the coefficients $\kappa_0, \kappa_1, \kappa_2, \ldots$, are of preponderant importance as compared with those that succeed them. In this case it is advantageous to take out of the series (31) the terms which depend solely on these important coefficients, and, by summing them, to obtain a new form for $K(x)$ which can be more readily computed; this, as we shall now show, may be done, the extracted part of $K(x)$ being in the exponential form which was obtained in § 4.

Suppose, for instance, that κ_0 and κ_1 are important, but the succeeding coefficients $\kappa_2, \kappa_3, \ldots$ are comparatively small. The terms in (31) which depend only on κ_0 and κ_1 are evidently

$$\kappa_0 - \begin{vmatrix} \kappa_0 & 1 \\ \kappa_1 & \kappa_0 \end{vmatrix} x + \begin{vmatrix} \kappa_0 & 1 & 0 \\ \kappa_1 & \kappa_0 & 1 \\ 0 & \kappa_1 & \kappa_0 \end{vmatrix} \frac{x^2}{2!} - \begin{vmatrix} \kappa_0 & 1 & 0 & 0 \\ \kappa_1 & \kappa_0 & 1 & 0 \\ 0 & \kappa_1 & \kappa_0 & 1 \\ 0 & 0 & \kappa_1 & \kappa_0 \end{vmatrix} \frac{x^3}{3!} + \ldots \qquad (32)$$

Now if u_n denote the determinant

$$(-)^{n-1}\begin{vmatrix} \kappa_0 & 1 & 0 & 0 & 0\ldots \\ \kappa_1 & \kappa_0 & 1 & 0 & 0\ldots \\ 0 & \kappa_1 & \kappa_0 & 1 & 0\ldots \\ 0 & 0 & \kappa_1 & \kappa_0 & 1\ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots \end{vmatrix} \quad (n \text{ rows})$$

we have by expanding in terms of the elements of the first row

$$u_n = -\kappa_0 u_{n-1} - \kappa_1 u_{n-2}.$$

The solution of this difference-equation is

$$u_n = \mathrm{A}\alpha^n + \mathrm{B}\beta^n$$

where α and β are the roots of the quadratic $x^2+\kappa_0 x+\kappa_1 = 0$, and A and B are independent of n. Moreover, since

$$u_1 = \kappa_0 = -(\alpha+\beta) = -\frac{\alpha^2-\beta^2}{\alpha-\beta},$$

and

$$u_2 = -\kappa_0^2+\kappa_1 = -(\alpha^2+\alpha\beta+\beta^2) = -\frac{\alpha^3-\beta^3}{\alpha-\beta},$$

we see that

$$\mathrm{A} = -\frac{\alpha}{\alpha-\beta}, \qquad \mathrm{B} = \frac{\beta}{\alpha-\beta},$$

and therefore

$$u_n = -\frac{\alpha^{n+1}-\beta^{n+1}}{\alpha-\beta}.$$

The expression (32) may thus be written

$$-\frac{\alpha^2-\beta^2}{\alpha-\beta}-\frac{\alpha^3-\beta^3}{\alpha-\beta}x-\frac{\alpha^4-\beta^4}{\alpha-\beta}\frac{x^2}{2!}-\frac{\alpha^5-\beta^5}{\alpha-\beta}\frac{x^3}{3!}+\ldots,$$

or

$$-\frac{\alpha^2 e^{\alpha x}-\beta^2 e^{\beta x}}{\alpha-\beta}.$$

Thus, we have

THEOREM 6.—*In Theorem 4 the formula* (31) *may be replaced by*

$$\mathrm{K}(x) = -\frac{\alpha^2 e^{\alpha x}}{\alpha-\beta}-\frac{\beta^2 e^{\beta x}}{\beta-\alpha}+\kappa_2\frac{x^2}{2!}+(\kappa_3-2\kappa_0\kappa_2)\frac{x^3}{3!}+\ldots$$

where α and β are the roots of the quadratic $x^2+\kappa_0 x+\kappa_1 = 0$. This form of $\mathrm{K}(x)$ is preferable for purposes of computation when κ_0 and κ_1 are of preponderant importance as compared with $\kappa_2, \kappa_3, \ldots$.

We may in this way replace the terms of (31) to any extent by exponentials, and it is evident that when the terms thus replaced are those which involve $\kappa_0, \kappa_1, \ldots \kappa_{n-1}$ without involving $\kappa_n, \kappa_{n+1}, \ldots$, the exponential terms which are obtained are precisely those which would be obtained as the value of $\mathrm{K}(x)$ in Theorem 3, if $\kappa(x)$ were the polynomial

$$\kappa_0+\kappa_1 x+\kappa_2\frac{x^2}{2!}+\ldots+\kappa_{n-1}\frac{x^{n-1}}{(n-1)!}.$$