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ON THE NUMERICAL SOLUTION OF LINEAR EVOLUTION PROBLEMS WITH AN INTEGRAL OPERATOR COEFFICIENT

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ABSTRACT. We present a method for the numerical solution of first order nonstationary problems with a pseudodifferential operator coefficient on a manifold. Using the Cayley transform we get an explicit representation of the exact solution and reduce the problem to a sequence of stationary equations which then are transformed into hypersingular integral equations of the second kind. For the numerical solution of these integral equations we use a collocation procedure based on appropriate trigonometric interpolation quadratures. Using the numerical solution of the integral equations and the explicit representation of the exact soution, we get a fully discrete approximation with respect to time and to spatial discretization parameters. In the case of a circle, the analysis of convergence and error estimates are given which show automatic dependence of the error order on the smoothness of the exact solution (the spectral property with respect to both the time and the spatial discretization parameters).

1. Introduction. Differential equations of the type

$$\frac{du}{dt} + \tilde{A}u = 0, \qquad u(0) = u_0 \tag{1.1}$$

where $u : \mathbf{R}_+ \to E$ is a vector-valued function with values in a Banach space E and $\tilde{A} : E \to E$ is an operator coefficient, describe many physical and technical processes. For example, problems of the type (1.1) arise in the theory of water and gravity waves, in the theory of viscous flows, in acoustics and elastostatics, where E is a Banach space of functions defined on a piecewise smooth curve and

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$\tilde{A}$ is a boundary integral operator [**5, 7, 19**]. In order to solve (1.1) numerically, one has to discretize the time derivative (the temporal discretization) and the operator $\tilde{A}$ (the spatial discretization). The finite element and finite difference methods are often used for the temporal discretization which implies a fixed convergence rate with respect to the temporal discretization parameter even when the exact solution is infinitely differentiable [**17**]. On the other hand, these discretizations are universal and possess a convenient algorithmical "step-by-step" structure. There are also a few spectral discretizations in time for the parabolic equation with a convergence rate automatically dependent on the temporal smoothness, but these methods are either applicable for very restrictive classes of problems or algorithmically complicated, see [**17**] and references therein. A new method of temporal discretization based on the Cayley transform (the CT-method) was recently developed in [**1, 8, 9**]. This method possesses a simple algorithmical structure and its convergence rate with respect to a temporal parameter is either exponential for the analytical solution or depends automatically on the smoothness of the exact solution (the spectral property).

Various effective methods for partial discretization of boundary integral operators were described in [**4, 12, 13, 19, 20**], among them the boundary element and boundary collocation method. It was shown, see, for example, [**4, 11**], that the collocation method for a number of integral equations based on the special quadrature rules provides exponential convergence for analytical initial data.

Our goal is to combine the CT-method from [**1, 8, 9**] with a trigonometric collocation method in order to give a fully discrete approximation for the problem (1.1) which possesses the spectral property with respect to both temporal and spatial discretization parameters. Using an explicit representation of the exact solution by the CT-method, we reduce the problem (1.1) to a sequence of stationary equations which then are transformed into hypersingular integral equations of the second kind. The last ones are discretized by a collocation procedure based on appropriate trigonometric interpolation quadratures. The method is justified for the case of a circle (Section 3). This approximation appears to be a linear combination of solutions of a sequence of linear algebraic equations with the same circular matrix which allows one to use an effective method based on the Fast Fourier Transform. A generalization for the case of an arbitrary parametrized curve is described and numerical examples are presented (Section 4).

2. A semi-discrete approximation. Let D be a bounded domain in $\mathbf{R}^2$ with a smooth boundary Γ. Further, let w be the solution of the boundary value problem

$$\Delta w(x) = 0, \quad x \in D \tag{2.1}$$

$$w(x) = v(x), \quad x \in \Gamma. \tag{2.2}$$

We define the Dirichlet to Neumann operator A by

$$(Av)(x) = \frac{\partial w(x)}{\partial \nu(x)}, \tag{2.3}$$

where ν is the exterior normal to Γ. For a fixed $\beta > 0$ the operator $\tilde{A} = A + \beta I$ is self-adjoint and positive definite in $L^2(\Gamma)$ [**15**]. We consider the following problem of the type (1.1). Find a function $u(x,t)$ defined on $\Gamma \times (0,T]$ such that

$$\frac{\partial u}{\partial t} + \tilde{A}u = 0 \quad \text{on} \quad \Gamma \times (0,T], \tag{2.4}$$

$$u(x,0) = w_0(x), \quad x \in \Gamma. \tag{2.5}$$

It is known [**15**] that, for $w_0 \in L^2(\Gamma)$, there exists a unique solution of the problem (2.4) and (2.5), which belongs to $L^2(0,T;H^{1/2}(\Gamma)) \cap L^\infty(0,T;L^2(\Gamma)) \cap L^2(\Gamma \times (0,T])$. If w_0 and Γ are smooth, then one can expect that u is also smooth. We will also characterize the smoothness of w_0 and u by a number σ such that $w_0, u \in D(A^\sigma)$, where $D(A^\sigma)$ denotes the domain of the operator A^σ.

Let γ be an arbitrary positive number and

$$T_\gamma(\tilde{A}) = (\gamma I - \tilde{A})(\gamma I + \tilde{A})^{-1}$$

be the Cayley transform of the operator $\tilde{A}$. The following result was proved in [**1, 8, 9**].

Theorem 2.1. *Let $\tilde{A}$ be a self-adjoint positive definite operator. Then*

$$u(x,t) = e^{-\gamma t} \sum_{p=0}^{\infty} (-1)^p L_p^{(0)}(2\gamma t) U_{\gamma,p}(x), \tag{2.6}$$

where $L_p^{(0)}(t)$ are the Laguerre polynomials and

$$U_{\gamma,p}(x) = T_\gamma^p(\tilde{A})(I + T_\gamma(\tilde{A}))w_0(x), \quad p = 0, 1, \dots .$$

It is easy to see that $U_{\gamma,p}$ is the solution of the following (stationary) operator equations

$$(\gamma I + \tilde{A})U_{\gamma,0} = 2\gamma w_0, \tag{2.7}$$

$$(\gamma I + \tilde{A})U_{\gamma,p} = (\gamma I - \tilde{A})U_{\gamma,p-1}, \quad p = 1, 2, \dots , \tag{2.8}$$

with the same operator $\gamma I + \tilde{A}$.

It is natural to take as an approximation for the exact solution (2.6) the following truncated sum

$$u_M(x,t) = e^{-\gamma t} \sum_{p=0}^{M} (-1)^p L_p(2\gamma t) U_{\gamma,p}(x). \tag{2.9}$$

The next theorem from [**1, 8**] shows that the quality of this approximation automatically depends on the smoothness of the initial data $w_0(x)$. Here and in what follows, c denotes various constants independent of N and M.

Theorem 2.2. *Let the assumptions of Theorem* 1 *hold and $w_0 \in D(\tilde{A}^\sigma)$. Then*

$$\|u(\cdot,t) - u_M(\cdot,t)\|_{L^2(\Gamma)} \le cM^{-\sigma} \|\tilde{A}^\sigma w_0\|_{L^2(\Gamma)}, \quad t > 0.$$

We call (2.9) together with (2.7) and (2.8) the semi-discrete approximation for the problem (2.4), (2.5), with the temporal discretization parameter M. In order to get a fully discrete approximation, one has to discretize equations (2.7) and (2.8). To this end, one has to study the structure and properties of the boundary operator $\tilde{A}$. We begin with the case when Γ is a circle.

3. A fully discrete approximation in the case of a circle. If the boundary Γ is a circle with radius R, then one can represent the solution of problem (2.1)–(2.2) by the Poisson integral [**11**]

$$w(x) = \frac{1}{2\pi} \int_\Gamma v(y) \frac{R - |x|^2}{|x-y|^2}\, ds(y),$$

which in polar coordinates (r, φ) takes the form

$$w(r,\varphi) = \frac{1}{2\pi} \int_0^{2\pi} v(\psi) \frac{R - r^2}{R^2 - 2Rr\cos(\psi - \varphi) + r^2}\, d\psi. \tag{3.1}$$

Hence, we get the following representation for the operator A,

$$(Au)(\varphi) = -\frac{1}{2\pi R} \int_0^{2\pi} u(\psi) \frac{1}{2\sin^2((\psi - \varphi)/2)}\, d\psi. \tag{3.2}$$

Taking into account the 2π-periodicity of the function $u(\varphi)$ in φ, we receive after integration by parts the following equivalent representation

$$(Au)(\varphi) = -\frac{1}{2\pi R} \int_0^{2\pi} u'(\psi) \cot \frac{\psi - \varphi}{2}\, d\psi.$$

Now the system of operator equations (2.7)–(2.8) takes the form

$$\gamma^+ U_{\gamma,0}(\varphi) - \frac{1}{2\pi R} \int_0^{2\pi} U_{\gamma,0}(\psi) \frac{1}{2\sin^2((\psi - \varphi)/2)}\, d\psi = 2\gamma w_0(\varphi), \tag{3.3}$$

$$\begin{aligned} &\gamma^+ U_{\gamma,p}(\varphi) - \frac{1}{2\pi R} \int_0^{2\pi} U_{\gamma,p}(\psi) \frac{1}{2\sin^2((\psi - \varphi)/2)}\, d\psi \\ &\quad = \gamma^- U_{\gamma,p-1}(\varphi) + \frac{1}{2\pi R} \int_0^{2\pi} U_{\gamma,p-1}(\psi) \frac{1}{2\sin^2((\psi - \varphi)/2)}\, d\psi, \end{aligned} \tag{3.4}$$

where $\gamma^+ = \gamma + \beta$, $\gamma^- = \gamma - \beta$, $p = 1, 2, \dots, \varphi \in [0, 2\pi]$.

Let us consider the solvability of these integral equations in the Sobolev space H^s of 2π periodic functions.

Theorem 3.1. *For $w_0 \in H^s$, $s \in \mathbf{R}$ and $p = 0, 1, \dots,$ each of the integral equations* (3.3)–(3.4) *possesses the unique solution $U_{\gamma,p} \in H^{s+1}$.*

Proof. Let us determine the scalar product for the functions $f = \sum_{m=-\infty}^{\infty} f_m e^{im\varphi}$, $g = \sum_{m=-\infty}^{\infty} g_m e^{im\varphi} \in H^s$ through the Fourier coefficients f_n, g_n

$$(f, g)_s = \sum_{m=-\infty}^{\infty} f_m \bar{g}_m (1 + m^2)^s.$$

Obviously, it is sufficient to show that the operator $(\gamma I + \tilde{A}) : H^{s+1} \to H^s$ is an isomorphism. Since the integral operator in equation (3.3) corresponds to the Poincare boundary value problem for the Laplace equation, the uniqueness of its solution implies that the operator $\gamma I + \tilde{A}$ is an injection. Using the formula [**11**, p. 131],

$$\frac{1}{2\pi} \int_0^{2\pi} \cot \frac{t}{2} e^{\pm imt}\, dt = \begin{cases} \pm i, & m = 1, 2, \dots, \\ 0 & m = 0, \end{cases}$$

it is easy to check that, for an arbitrary function $f \in H^{s+1}$, the following representation holds true

$$(Af)(\varphi) = R^{-1} \sum_{m=-\infty}^{\infty} f_m |m| e^{im\varphi},$$

i.e., the operator A is a pseudodifferential operator of order 1, and the inverse operator $(\gamma I + \tilde{A})^{-1}$ is given by

$$((\gamma I + \tilde{A})^{-1} g)(\varphi) = \sum_{m=-\infty}^{\infty} \frac{g_m}{\gamma^+ + R^{-1}|m|} e^{im\varphi}$$

is bounded from H^s in H^{s+1}. Hence, the operator $\gamma I + \tilde{A}$ is an isomorphism. The assertion of the theorem for equations (3.4) follows by induction. $\square$

Next we consider a discretization method for the numerical solution of integral equations (3.3)–(3.4). To this end, we use the collocation method based on a special quadrature formula from [**4, 12**]. Since we deal with 2π-periodic functions, it is preferable to use trigonometric interpolation in order to get a suitable quadrature rule. We denote by $\mathbf{T}_N$ the set of all trigonometric polynomials of the form

$$u(t) = \sum_{j=0}^{N} d_j \cos jt + \sum_{j=1}^{N-1} \beta_j \sin jt,$$

which are the real parts of the complex polynomials $\sum_{j=-N+1}^{N} c_j e^{ijt}$ with $c_j = d_j,\ j = 1, 2, \ldots, N,\ c_{-j} = i\beta_j,\ j = 1, 2, \ldots, N-1$. Let $\varphi_i = i\pi/N,\ i = 0, 1, \ldots, 2N-1$ be an equidistant partition. Substituting the trigonometric interpolation polynomial related to this partition and to a function f, one gets the following quadrature formula [**4, 12**]

$$\begin{aligned}\frac{1}{2\pi}\int_0^{2\pi} f'(\psi)\cot\frac{\psi-\varphi}{2}\,d\psi &\approx \sum_{i=0}^{2N-1} f(\varphi_i)F_i(\varphi) \\ &\equiv (Q_N f)(\varphi),\end{aligned} \tag{3.5}$$

where

$$F_i(\varphi) = \frac{1}{2\pi}\int_0^{2\pi} l_i'(\psi)\cot\frac{\psi-\varphi}{2}\,d\psi \tag{3.6}$$

and

$$\begin{aligned} l_n(\varphi) &= \frac{1}{2N}\left\{1 + 2\sum_{m=1}^{N-1}\cos m(\varphi-\varphi_n) + \cos N(\varphi-\varphi_n)\right\} \\ &= \frac{1}{2N}\sin N(\varphi-\varphi_n)\cot\frac{\varphi-\varphi_n}{2}, \qquad \varphi \neq \varphi_n \end{aligned}$$

are the fundamental trigonometric Lagrange polynomials.

The direct calculation of the integrals in (3.6) leads to the following representation for $F_i(\varphi)$ [**12**]

$$F_i(\varphi) = -\frac{1}{N}\sum_{m=1}^{N-1} m\cos m(\varphi-\varphi_i) - \frac{1}{2}\cos N(\varphi-\varphi_i).$$

Let us introduce the approximate operators

$$(A_N U)(\varphi) = -\frac{1}{R}\sum_{i=0}^{2N-1} U(\varphi_i)F_i(\varphi), \quad \tilde{A}_N = A_N + \beta I.$$

It is evident that $A_N U = AP_N U$, where P_N is the trigonometric interpolation operator

$$(P_N U)(\varphi) = \sum_{i=0}^{2N-1} U(\varphi_i) l_i(\varphi).$$

We replace the equations (2.7) and (2.8) by the approximate equations

$$(\gamma I + \tilde{A}_N)U^N_{\gamma,0} = 2\gamma P_N w_0, \tag{3.7}$$

$$(\gamma I + \tilde{A}_N)U^N_{\gamma,p} = (\gamma I - \tilde{A}_N)U^N_{\gamma,p-1}, \quad p = 1, 2, \dots, \tag{3.8}$$

with an unknown $U^N_{\gamma,p} \equiv U^N_{\gamma,p}(\varphi) \in \mathbf{T}_N$. These operator equations are obviously equivalent to the following sequence of systems of linear algebraic equations

$$\gamma^+ U^N_{\gamma,0}(\varphi_i) - \frac{1}{R}\sum_{j=0}^{2N-1} U^N_{\gamma,0}(\varphi_j)F_{|j-i|} = 2\gamma w_0(\varphi_i), \tag{3.9}$$

$$\begin{aligned}\gamma^+ U^N_{\gamma,p}(\varphi_i) &- \frac{1}{R}\sum_{j=0}^{2N-1} U^N_{\gamma,p}(\varphi_j)F_{|j-i|} \\ &= \gamma^- U^N_{\gamma,p-1}(\varphi_i) + \frac{1}{R}\sum_{j=0}^{2N-1} U^N_{\gamma,p-1}(\varphi_j)F_{|j-i|},\end{aligned} \tag{3.10}$$

where

$$F_i = F_i(0) = \begin{cases} 1/(2N\sin^2(\varphi_i/2)) & i \text{ odd}, \\ 0 & i \text{ even}, \ i \neq 0, \\ -N/2 & i = 0. \end{cases}$$

One can write (3.9) and (3.10) in matrix form

$$\begin{aligned}&(\gamma^+ I - R^{-1}T_{2N})v_p = (\gamma^- I + R^{-1}T_{2N})v_{p-1}, \quad p = 1, 2, \dots, \\ &(\gamma^+ I - R^{-1}T_{2N})v_0 = 2\gamma w_0,\end{aligned} \tag{3.11}$$

where $v_p = \{U^N_{\gamma,p}(\varphi_j)\}_{j=0,\dots,2N-1}$, $w_0 = \{w_0(\varphi_i)\}_{i=0,\dots,2N-1} \in \mathbf{R}^{2N}$,

$$T_{2N} = \begin{pmatrix} F_0 & F_1 & F_2 & \cdots & F_{2N-1} \\ F_1 & F_0 & F_1 & \cdots & F_{2N-2} \\ \cdots & \cdots & \cdots & \cdots & \\ F_{2N-1} & & F_{2N-2} & F_{2N-3} & \cdots & F_0 \end{pmatrix}$$

is a circulant Toeplitz matrix. Let

$$\tilde{F}_{2N} = \left\{ \exp\left[(k-1)(l-1)\frac{\pi i}{N}\right]\right\}_{k,l=1,\dots,2N}$$

be the matrix of the discrete Fourier transform and $t = [F_0, \dots, F_{2N-1}]^T \in \mathbf{R}^{2N}$ be the first column of the matrix T_{2N}. Since T_{2N} is a circulant matrix, we have [**21, 22**],

$$T_{2N} = (2N)^{-1}\tilde{F}^*_{2N}\operatorname{diag}(d)\tilde{F}_{2N}, \tag{3.12}$$

where $\operatorname{diag}(d) = \operatorname{diag}(d_0, d_1, \dots, d_{2N-1}) = \operatorname{diag}(\tilde{F}_{2N}t)$.

The matrix $(2N)^{-1/2}\tilde{F}_{2N}$ is a symmetric unitary matrix, hence

$$\gamma^+ I - R^{-1}T_{2N} = (2N)^{-1}\tilde{F}^*_{2N}(\gamma^+ I - \operatorname{diag}(d))\tilde{F}_{2N}. \tag{3.13}$$

Since T_{2N} is the collocation discretization of the operator $-RA$ and $AU = A_N U$ for all $U \in \mathbf{T}_N$, we get that the operator $-RA_N$ and the matrix T_{2N} have the same eigenvalues $d_k = -k$, $k = 0, 1, \dots, N$, $d_{2N-k} = d_k$, $k = 1, 2, \dots, N-1$. The eigenfunctions of $-RA_N$ are $e^{ik\psi}$, $k = -N+1, -N+2, \dots, N$, or their linear combinations. Analogously, the eigenvector of T_{2N} are the columns of the matrix $\tilde{F}_{2N}$ or their linear combinations.

The eigenvalues of the matrix $\gamma^+ I - R^{-1}T_{2N}$ are $\tilde{d}_k = \gamma^+ - R^{-1}d_k$, $k = 0, 1, \dots, 2N-1$. Since the eigenvalues are multiple eigenvalues, the eigenvectors are not uniquely determined.

Thus, the solution of each of the linear algebraic equation (3.11) exists, is unique and can be found by $2N(1 + 2\log_2 2N)$ complex multiplications and $4N \log_2 2N$ complex additions using the formula

$$(\gamma^+ I - R^{-1}T_{2N})^{-1} = (2N)^{-1}\tilde{F}^*_{2N}(\gamma^+ I - \operatorname{diag}(d))^{-1}\tilde{F}_{2N}$$

and the Fast Fourier Transform (FFT). In order to find all vectors $\{U^N_{\gamma,p}(\varphi_j)\}_{j=0}^{2N-1}$, $p=0,1,\dots,M$, one has to perform $O(NM\log N)$ arithmetical operations contrary to $O(N^2M)$ operations using, for example, the LU-decomposition of the matrix $\gamma^+ I - R^{-1}T_{2N}$.

The fully discrete approximation for the solution of the problem (2.4), (2.5), is defined by

$$u^N_M(t,\varphi) = e^{-\gamma t}\sum_{p=0}^{M}(-1)^p L_p^{(0)}(2\gamma t)U^N_{\gamma,p}(\varphi), \tag{3.14}$$

where

$$U^N_{\gamma,0}(\varphi) = \frac{2\gamma}{\gamma^+}P_N w_0(\varphi) + \frac{1}{\gamma^+ R}\sum_{j=0}^{2N-1}U^N_{\gamma,0}(\varphi_j)F_j(\varphi),$$

$$U^N_{\gamma,p}(\varphi) = \frac{\gamma^-}{\gamma^+}U^N_{\gamma,p-1}(\varphi) + \frac{1}{\gamma^+ R}\sum_{j=0}^{2N-1}[U^N_{\gamma,p}(\varphi_j) + U^N_{\gamma,p-1}(\varphi_j)]F_j(\varphi)$$

and $U^N_{\gamma,p}(\varphi_j)$ are the solutions of the systems of linear algebraic equations (3.9), (3.10).

The last formulas represent an interpolation step which is analogous to the interpolation step of Nyström type, since it represents the trigonometric interpolation connected with a corresponding interpolatory quadrature rule which takes into account the hypersingularity of the kernel. This quadrature formula does not possess the "saturation effect," i.e., its accuracy depends automatically on the smoothness of the integrand. This is the reason why, despite the fact that Nyström's interpolation is usually applied to smoothing operators, one can expect good numerical results also for hypersingular kernels which is confirmed by numerical experiences. An analogous approach is used in Nyström's method for weakly singular kernels, see [**11**, Chapter 12.3] and references therein.

Concerning the convergence of this approximation as $N,M\to\infty$, the next theorem shows that its rate automatically depends on the smoothness of the initial data.

Theorem 3.2. *For N, M large enough, the following error estimate holds*

$$\sup_{t\in(0,\infty)} \|u(\cdot,t) - u_M^N(\cdot,t)\|_s \leq c(M^{s-r} + N^{s-r})\|w_0\|_r.$$

Proof. Let E_λ be the spectral family corresponding to $\tilde{A}$ and λ_0 the minimal eigenvalue of $\tilde{A}$. Using the operator calculus for self-adjoint operators, we can represent

$$[(\gamma I - \tilde{A})(\gamma I + \tilde{A})^{-1}]^n = \int_{\lambda_0}^{\infty} \left(\frac{\gamma-\lambda}{\gamma+\lambda}\right)^n dE_\lambda.$$

Since $\tilde{A}_N U_{\gamma,p}^N = \tilde{A} U_{\gamma,p}^N$, we get from (2.7), (2.8), (2.9) and from (3.7), (3.8)

$$u_M(\varphi,t) = \int_{\lambda_0}^{\infty} f^{(M)}(t,\lambda)\, dE_\lambda w_0 \tag{3.15}$$

and

$$u_M^N(\varphi,t) = \int_{\lambda_0}^{\infty} f^{(M)}(t,\lambda)\, dE_\lambda P_N w_0, \tag{3.16}$$

where

$$f^{(M)}(t,\lambda) = \frac{2\gamma e^{-\gamma t}}{\gamma+\lambda} \sum_{p=0}^{M} (-1)^p L_p^{(0)}(2\gamma t) \left(\frac{\gamma-\lambda}{\gamma+\lambda}\right)^p.$$

Due to the inequality $|e^{-t/2} L_p^{(0)}(t)| \leq 1$ for all t, p, we get

$$|f^{(M)}(t,\lambda)| \leq \frac{2\gamma}{\gamma+\lambda} \sum_{k=0}^{\infty} \left|\frac{\lambda-\gamma}{\lambda+\gamma}\right|^p \leq \max\{1, \gamma/\lambda_0\}. \tag{3.17}$$

We have, for the global error,

$$\|u - u_M^N\| \leq \|z_1\| + \|z_2\|,$$

where $z_1 = u - u_M$, $z_2 = u_M - u_M^N$. It follows from [**1, 4**] that

$$\|z_1\| \le cM^{-\sigma}\|\tilde{A}^\sigma w_0\|$$

or, due to Theorem 3.1,

$$\|z_1(\cdot,t)\|_s \le cM^{s-r}\|w_0\|_r.$$

Taking into account (3.20)–(3.22), we get, for z_2,

$$\|z_2\| = \left\| \int_{\lambda_0}^{\infty} f^{(M)}(t,\lambda)\, dE_\lambda(w_0 - P_N w_0) \right\| \le c\|w_0 - P_N w_0\|,$$

in particular,

$$\|z_2(\cdot,t)\|_s \le c\|w_0(\cdot) - P_N w_0(\cdot)\|_s$$

and further, due to [**13**],

$$\|z_2(\cdot,t)\|_s \le cN^{s-r}\|w_0\|_r.$$

The proof is complete. $\square$

4. The case of an arbitrary curve in a parametric representation. Let the curve Γ have a parametric representation

$$\Gamma = \{x(\sigma) = (x_1(\sigma), x_2(\sigma)) : 0 \le \sigma \le 2\pi\},$$

where $x : \mathbf{R} \to \mathbf{R}^2$ is a 2π-periodic function of class C^3 with $|x'(\sigma)| > 0$ for all σ.

In order to get an integral representation for the operator A in (2.3), we use the theory of potentials. Let us consider the following auxiliary problem

$$\Delta\Phi = 0 \quad \text{in} \quad D, \tag{4.1}$$

$$\Phi(x) = U_{\gamma,p}(x), \quad x \in \Gamma, \quad p = 0, 1, \dots . \tag{4.2}$$

The logarithmic double layer potential

$$W_{\gamma,p}(x) = \frac{1}{2\pi}\int_\Gamma \mu_{\gamma,p}(y)\frac{\partial}{\partial\nu(y)} \ln\frac{1}{|x-y|}\, ds(y),$$

where $\mu_{\gamma,p}(x)$ is an arbitrary function (density) satisfying equation (4.1).

Taking into account the behavior of the potential $W_{\gamma,p}$ and that of its derivative when passing through the boundary Γ, see [**11**], and the definition of the operator A, we obtain

$$U_{\gamma,p}(x) = -\frac{1}{2}\mu_{\gamma,p}(x) + \frac{1}{2\pi}\int_\Gamma \mu_{\gamma,p}(y)\frac{\partial}{\partial\nu(y)}\ln\frac{1}{|x-y|}\,ds(y) \tag{4.3}$$

and

$$(AU_{\gamma,p})(x) = \frac{1}{2\pi}\frac{\partial}{\partial\nu(x)}\int_\Gamma \mu_{\gamma,p}(y)\frac{\partial}{\partial\nu(y)}\ln\frac{1}{|x-y|}\,ds(y), \quad x\in\Gamma. \tag{4.4}$$

Hence, equations (2.7), (2.8) take the following form

$$\begin{aligned} -\frac{\gamma^+}{2}\mu_{\gamma,p}(x) &+ \frac{\gamma^+}{2\pi}\int_\Gamma \mu_{\gamma,p}(y)\frac{\partial}{\partial\nu(y)}\ln\frac{1}{|x-y|}\,ds(y) \\ &+ \frac{1}{2\pi}\frac{\partial}{\partial\nu(x)}\int_\Gamma \mu_{\gamma,p}(y)\frac{\partial}{\partial\nu(y)}\ln\frac{1}{|x-y|}\,ds(y) = g_{\gamma,p}(x), \\ & x\in\Gamma \end{aligned} \tag{4.5}$$

where

$$\begin{aligned} g_{\gamma,p}(x) = &-\frac{\gamma^-}{2}\mu_{\gamma,p-1}(x) + \frac{\gamma^-}{2\pi}\int_\Gamma \mu_{\gamma,p-1}(y)\frac{\partial}{\partial\nu(y)}\ln\frac{1}{|x-y|}\,ds(y) \\ &- \frac{1}{2\pi}\frac{\partial}{\partial\nu(x)}\int_\Gamma \mu_{\gamma,p}(y)\frac{\partial}{\partial\nu(y)}\ln\frac{1}{|x-y|}\,ds(y) \end{aligned} \tag{4.6}$$

for $p = 1, 2, \dots$ and $g_{\gamma,0}(x) = 2\gamma w_0(x)$, $x \in \Gamma$.

Using the connection of the normal derivative of the double layer potential with the single layer potential [**11**],

$$\begin{aligned} \frac{1}{2\pi}\frac{\partial}{\partial\nu(x)}\int_\Gamma \varphi(y)\frac{\partial}{\partial\nu(y)}\ln\frac{1}{|x-y|}\,ds(y) \\ = \frac{1}{2\pi}\frac{\partial}{\partial t(x)}\int_\Gamma \frac{\partial}{\partial t(y)}\varphi(y)\ln\frac{1}{|x-y|}\,ds(y), \end{aligned} \tag{4.7}$$

where $t(x)$ is the tangential vector at the point x, we transform (4.5) in the following parametric form

$$(4.8)\quad -\gamma^+ q_{\gamma,p}(\sigma) + \frac{1}{2\pi}\int_0^{2\pi} (\gamma^+ q_{\gamma,p}(\tau)\tilde{K}_1(\sigma,\tau) + q'_{\gamma,p}(\tau)K_1(\sigma,\tau))\, d\tau = G_{\gamma,p}(\sigma), \quad 0 \le \sigma \le 2\pi$$

with the righthand side

$$(4.9)\quad G_{\gamma,p}(\sigma) = -\gamma^- q_{\gamma,p-1}(\sigma) + \frac{1}{2\pi}\int_0^{2\pi} (\gamma^- q_{\gamma,p-1}(\tau)\tilde{K}_1(\sigma,t) - q'_{\gamma,p-1}(\tau)K_1(\sigma,\tau))\, d\tau$$

for $p = 1, 2, \dots,$ and $G_{\gamma,0}(\sigma) = g_{\gamma,0}(x(\sigma))$. Here $q_{\gamma,n}(\sigma) = \mu_{\gamma,n}(x(\sigma))$ and the kernels $\tilde{K}_1(\sigma,\tau)$ and $K_1(\sigma,\tau)$ have the form

$$(4.10)\quad \tilde{K}_1(\sigma,\tau) = 2|x'(\tau)|\frac{(x(\sigma)-x(\tau))\nu(x(\tau))}{|x(\sigma)-x(\tau)|^2},$$

$$(4.11)\quad K_1(\sigma,\tau) = 2\frac{(x(\tau)-x(\sigma))t(x(\sigma))}{|x(\sigma)-x(\tau)|^2}$$

and

$$\tilde{K}_1(\sigma,\sigma) = \frac{x''_1(\sigma)x'_2(\sigma) - x''_2(\sigma)x'_1(\sigma)}{|x'(\sigma)|^2}.$$

Integrating by parts and separating the hypersingular terms of (4.8) and (4.9), we obtain

$$(4.12)\quad -\gamma^+ q_{\gamma,p}(\sigma) + \frac{1}{|x'(\sigma)|}\left\{\frac{1}{2\pi}\int_0^{2\pi} q'_{\gamma,p}(\tau)\cot\frac{\tau-\sigma}{2}\, d\tau + \frac{1}{2\pi}\int_0^{2\pi} q_{\gamma,p}(\tau)(\gamma^+|x'(\sigma)|\tilde{K}_1(\sigma,\tau) + K_2(\sigma,\tau))\, d\tau\right\} = G_{\gamma,p}(\sigma), \quad 0 \le \sigma \le 2\pi$$

and

$$(4.13)\quad G_{\gamma,p}(\sigma) = -\gamma^- q_{\gamma,p-1}(\sigma) + \frac{1}{|x'(\sigma)|}\left\{\frac{1}{2\pi}\int_0^{2\pi} q'_{\gamma,p-1}(\tau)\cot\frac{\tau-\sigma}{2}\, d\tau + \frac{1}{2\pi}\int_0^{2\pi} q_{\gamma,p-1}(\tau)(\gamma^-|x'(\sigma)|\tilde{K}_1(\sigma,\tau) + K_2(\sigma,\tau))\, d\tau\right\},$$

where the kernel $K_2(\sigma,\tau)$ has the following representation

$$(4.14)\qquad \begin{aligned} K_2(\sigma,\tau) = {} & 4\frac{(x'(\tau)(x(\tau)-x(\sigma)))(x'(\sigma)(x(\tau)-x(\sigma)))}{|x(\sigma)-x(\tau)|^4} \\ & - 2\frac{(x'(\sigma)x'(\tau))}{|x(\sigma)-x(\tau)|^2} - \frac{1}{2\sin^2((\sigma-\tau)/2)} \end{aligned}$$

with the diagonal term

$$K_2(\sigma,\sigma) = \frac{1}{3}\frac{(x'(\sigma)x'''(\sigma))}{|x'(\sigma)|^2} + \frac{1}{2}\frac{|x''(\sigma)|^2}{|x'(\sigma)|^2} - \frac{1}{6} - \frac{(x'(\sigma)x''(\sigma))^2}{|x'(\sigma)|^4}.$$

Using the quadrature formula (3.5) and the trigonometric Gauss type quadrature for 2π-periodic functions

$$\frac{1}{2\pi}\int_0^{2\pi} f(\sigma)\,d\sigma \approx \frac{1}{2N}\sum_{i=0}^{2N-1} f(\sigma_i), \quad \sigma_i = \frac{i\pi}{N}$$

together with the collocation method, we arrive at the following sequence of linear algebraic equations

$$(4.15)\qquad \begin{aligned} & -\gamma^+ q^N_{\gamma,p}(\sigma_k) + \frac{1}{|x'(\sigma_k)|}\sum_{i=0}^{2N-1} q^N_{\gamma,p}(\sigma_i)\biggl(F_{|i-k|} \\ & \qquad + \frac{1}{2N}[\gamma^+|x'(\sigma_k)|\tilde{K}_1(\sigma_k,\sigma_i) + K_2(\sigma_k,\sigma_i)]\biggr) \\ & \qquad\qquad = G^N_{\gamma,p}(\sigma_k), \end{aligned}$$

where
(4.16)

$$\begin{aligned} G^N_{\gamma,p}(\sigma_k) = {} & -\gamma^- q^N_{\gamma,p-1}(\sigma_k) - \frac{1}{|x'(\sigma_k)|}\sum_{i=0}^{2N-1} q^N_{\gamma,p-1}(\sigma_i) \\ & \cdot\biggl(F_{|i-k|} + \frac{1}{2N}[-\gamma^-|x'(\sigma_i)|\tilde{K}_1(\sigma_k,\sigma_i) + K_2(\sigma_k,\sigma_i)]\biggr) \end{aligned}$$

$k = 0, 1, \ldots, 2N-1$, $p = 1, 2, \ldots$ and $G^N_{\gamma,0}(\sigma_k) = G_{\gamma,0}(\sigma_k)$.

The approximate solution of the initial problem is given by the formula

$$u_M^N(x(\sigma),t) = e^{-\gamma t}\sum_{p=0}^{M}(-1)^p U_{\gamma,p}^N(x(\sigma))L_p(2\gamma t). \tag{4.18}$$

A justification of this approach and the error estimates using the results of Section 3 and of [**1, 4, 8, 9, 12, 13**] will be reported in a forthcoming paper of the authors.

5. Numerical experiments. 1. Let us consider the problem (2.3), (2.4), with the initial functions

$$w_{01}(\varphi) = \exp(\sin\varphi), \quad \varphi \in [0, 2\pi]$$

and

$$w_{02}(\varphi) = \ln(1+\sin^2\varphi), \quad \varphi \in [0, 2\pi].$$

Table 1 presents the results of the numerical solution of problem (2.4), (2.5), using the suggested method for the case of a circle with radius 1, $\gamma = 2$, $\beta = 1$.

One can see that the doubling of M provides the doubling of correct digits in the results, which indicates the exponential convergence which we expect from our error analysis.

2. We choose the initial function $w_0(x(\sigma))$ as

$$w_0(x(\sigma)) = |x(\sigma)|,$$

and consider the following two boundary curves

$$\Gamma_1 = \{x(\sigma) = (0.2\cos\sigma, 0.4\sin\sigma - 0.3\sin^2\sigma) : 0 \le \sigma \le 2\pi\}$$

and

$$\Gamma_2 = \{x(\sigma) = (\cos\sigma + 0.65\cos 2\sigma - 0.65, 1.5\sin\sigma) : 0 \le \sigma \le 2\pi\}.$$

Table 2 contains the results of the approximate solution of the nonstationary problem (2.4), (2.5) for these data. The calculations were carried out for $\gamma = 2$ and $\beta = 1$ at the points $x(0)$ and $x(\pi/2)$ on the curves Γ_1 and Γ_2.

TABLE 1. Numerical example 1.

		Init. function w_{01}		Init. function w_{02}	
t	$N = M$	$\varphi = 0$	$\varphi = \pi/2$	$\varphi = 0$	$\varphi = \pi/2$
0.0	8	1.00006201	2.71833977	0.00005835	0.69317063
	16	1.00000000	2.71828183	0.00000055	0.69314713
	32	1.00000000	2.71828183	0.00000000	0.69314718
0.4	8	0.76765034	1.44808634	0.14479431	0.35191473
	16	0.76764364	1.44807131	0.14478900	0.35190710
	32	0.76764364	1.44807131	0.14478908	0.35190708
0.8	8	0.54433663	0.82361551	0.13746168	0.19974877
	16	0.54435067	0.82362905	0.13746970	0.19975360
	32	0.54435067	0.82362905	0.13746973	0.19975361
1.2	8	0.37394115	0.49168242	0.10394547	0.12269438
	16	0.37392701	0.49166900	0.10393570	0.12268921
	32	0.37392701	0.49166900	0.10393566	0.12268921
1.6	8	0.25338162	0.30399862	0.07316690	0.07881819
	16	0.25338179	0.30399843	0.07317057	0.07881868
	32	0.25338179	0.30399843	0.07317059	0.07881869
2.0	8	0.17065670	0.19272047	0.05008806	0.05179171
	16	0.17067066	0.19273398	0.05009541	0.05179660
	32	0.17067066	0.19273398	0.05009544	0.05179659

TABLE 2. Numerical example 2.

		Curve Γ_1		Curve Γ_2	
t	$N = M$	$\varphi = 0$	$\varphi = \pi/2$	$\varphi = 0$	$\varphi = \pi/2$
0.0	8	0.19948462	0.108223035	1.00006577	1.98533687
	16	0.20002136	0.10111592	1.00000019	1.98499310
	32	0.19999801	0.10007176	1.00000000	1.98494036
	64	0.19999950	0.10000604	1.00000000	1.98494334
0.4	8	0.17250707	0.16891444	0.67287245	1.17187137
	16	0.17199524	0.16813362	0.67346702	1.19854445
	32	0.17199025	0.16821564	0.67346871	1.19840510
	64	0.17199022	0.16821887	0.67346872	1.19840487
0.8	8	0.13345000	0.13295398	0.46913798	0.71602519
	16	0.13325004	0.13166291	0.46958856	0.74728496
	32	0.13324906	0.13173696	0.46958723	0.74709801
	64	0.13324909	0.13173828	0.46958724	0.74709791
1.2	8	0.09779201	0.09772105	0.32833388	0.44843766
	16	0.09616120	0.09564923	0.32892609	0.47506104
	32	0.09615882	0.09557894	0.32892251	0.47489031
	64	0.09615855	0.09557631	0.32892251	0.47489040
1.6	8	0.06881942	0.06808395	0.22851905	0.28553955
	16	0.06702790	0.06677271	0.22938722	0.30611231
	32	0.06702524	0.06680489	0.22938172	0.30598317
	64	0.06702543	0.06680586	0.22938172	0.30598317
2.0	8	0.04750654	0.04723133	0.15788191	0.18402470
	16	0.04589389	0.04579482	0.15893679	0.19918533
	32	0.04589006	0.04580835	0.15893014	0.19908893
	64	0.04588996	0.04580770	0.15893014	0.19908893

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