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Journal of King Saud University – Science (2015) xxx, xxx–xxx

King Saud University
Journal of King Saud University – Science
www.ksu.edu.sa
www.sciencedirect.com

ORIGINAL ARTICLE

On the numerical solution of space fractional order diffusion equation via shifted Chebyshev polynomials of the third kind

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Received 16 April 2015; accepted 11 May 2015

KEYWORDS
Space fractional order diffusion equation;
Caputo derivative;
Chebyshev collocation method;
Finite difference method;
Chebyshev polynomials of the third kind

Abstract In this paper, we propose a numerical scheme to solve space fractional order diffusion equation. Our scheme uses shifted Chebyshev polynomials of the third kind. The fractional differential derivatives are expressed in terms of the Caputo sense. Moreover, Chebyshev collocation method together with the finite difference method are used to reduce these types of differential equations to a system of algebraic equations which can be solved numerically. Numerical approximations performed by the proposed method are presented and compared with the results obtained by other numerical methods. The results reveal that our method is a simple and effective numerical method.

1. Introduction

The subject of fractional calculus (that is, calculus of integrals and derivatives of any arbitrary real or complex order) has gained considerable popularity and importance during the past four decades or so, due mainly to its demonstrated applications in numerous seemingly diverse and widespread fields of science and engineering, chemistry and other sciences (Dalir and Bashour, 2010; Kilbas et al., 2006). It does indeed provide several potentially useful tools for solving differential and integral equations, and various other problems involving special functions of mathematical physics as well as their extensions and generalizations in one and more variables (see for instance, Boyd (2001), Bhrawy et al. (2013, 2014a,b), Kilbas et al. (2006), Miller and Ross (1993), Oldham and Spanier (1974), Podlubny (1999), Rossikhin and Shitikova (1997)).

In recent decades, the Chebyshev polynomials are one of the most useful polynomials which are suitable in numerical analysis including polynomial approximation, integral and differential equations and spectral methods for partial differential equations and fractional order differential equations (see,

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Peer review under responsibility of King Saud University.

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http://dx.doi.org/10.1016/j.jksus.2015.05.002
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Canuto et al., 2006; Dalir and Bashour, 2010; Mason and Handscomb, 2003; Scalas et al., 2003; Su et al., 2010; Sousa, 2011; Tadjeran et al., 2006).

In recent years, one of the attractive concepts in the initial and boundary value problems is the fractional order diffusion equation, it has found its extensive applications in many fields such as, physics, chemistry, engineering, mathematics, it included in a wide variety of practical situations and has emerged as an important area of investigation. For the general theory and applications of fractional diffusion equations see (Canuto et al., 2006; Dalir and Bashour, 2010; Oldham and Spanier, 1974; Podlubny, 1999; Scalas et al., 2003; Su et al., 2010; Sousa, 2011; Tadjeran et al., 2006). The fractional order differential equations have been much studied and many aspects of these equations are explored. For some recent work on fractional diffusion equations, we can refer to different publications (see for instance, Azizi and Loghmani (2013, 2014), Meerschaert and Tadjeran (2004, 2006), Saadatmandi and Dehghan (2007, 2006), Sweilam and Khader (2010), Sweilam et al. (2011, 2012, 2015)) and the references therein.

The fractional order (time–space) diffusion equation makes a great role in the mathematical modeling of several phenomena. It is well known that most of fractional differential equations cannot be solved exactly. Therefore, numerical methods would be proposed and investigated to get approximate solutions of these equations. The Chebyshev finite difference method and a semi-discrete scheme with Chebyshev collocation method have been introduced by Azizi and Loghmani (2013, 2014), for approximating the solution of the space fractional diffusion equations (FDEs). Also, Khader (2011) investigated the Chebyshev collocation method together with the finite difference method for solving FDEs. Moreover, Bhrawy et al. (2014b) introduced efficient generalized Laguerre-spectral methods for solving multi-term fractional differential equations on the half line. The authors in Saadatmandi and Dehghan (2006, 2011) have constructed new operational matrix and tau approach for solution of the space fractional diffusion equation. On the other side many researchers used the finite difference method (FDM) for solving FDEs (see, Elbarbary, 2003; Dehghan and Saadatmandi, 2008; Meerschaert and Tadjeran, 2004, 2006). Second kind shifted Chebyshev polynomials and Crank-Nicolson FDM are applied for solving fractional order diffusion equation in Sweilam et al. (2012, 2015).

Our fundamental goal of this work is to develop a suitable way to approximate the space fractional order diffusion equation using the shifted Chebyshev polynomials of the third kind with finite difference method together with Chebyshev collocation method. In what follows, we give some necessary definitions and mathematical relations which are used in this paper.

Definition 1. The Caputo fractional derivative operator D^μ of order μ is defined as the following form:

$$D^\mu f(x) = \frac{1}{\Gamma(m-\mu)} \int_0^x \frac{f^{(m)}(t)}{(x-t)^{\mu-m+1}}\, dt, \;\; \mu > 0, \tag{1}$$

where $m-1 < \mu \leq m, \; m \in N, \; x > 0$. The linear property of the Caputo fractional derivative exists similar to the integer order differentiation:

$$D^\mu(\lambda f(x) + \gamma g(x)) = \lambda D^\mu f(x) + \gamma D^\mu g(x), \tag{2}$$

where λ and γ are constants.

For the Caputo derivative we can obtain the following result:

$$D^\mu k = 0, \quad k \text{ is a constant}, \tag{3}$$

$$D^\mu x^n = \begin{cases} 0, & \text{for } n \in \mathbb{N}_0 \text{ and } n < \lceil \mu \rceil, \\ \frac{\Gamma(n+1)}{\Gamma(n+1-\mu)} x^{(n-\mu)}, & \text{for } n \in \mathbb{N}_0 \text{ and } n \geq \lceil \mu \rceil. \end{cases} \tag{4}$$

The function $\lceil \mu \rceil$ is used to denote the smallest integer greater than or equal to μ. Also $N_0 = \{0, 1, 2, \ldots\}$. Recall that, for $\mu \in N$ the Caputo differential operator coincides with the usual differential operator of integer order. For more details on fractional derivatives definitions, theorems and its properties see Podlubny (1999).

The main aim of this work is to find approximate solution of space fractional order diffusion equation using the shifted Chebyshev polynomials of the third kind. Consider the one-dimensional space fractional order diffusion equation of the form:

$$\frac{\partial u(x,t)}{\partial t} = p(x)\frac{\partial^\mu u(x,t)}{\partial x^\mu} + q(x,t), \tag{5}$$

on a finite domain $0 < x < L, \; 0 < t \leq T$ and the parameter μ refers to the fractional order of spatial derivative with $1 < \mu \leq 2$. The function $q(x,t)$ is the source term. We also assume an initial condition:

$$u(x,0) = f(x), \quad 0 < x < L, \tag{6}$$

and the boundary conditions:

$$u(0,t) = v_0(t), \quad 0 < t \leq T \tag{7}$$

$$u(L,t) = v_1(t), \quad 0 < t \leq T. \tag{8}$$

In case of $\mu = 2$, Eq. (5) is the classical second order diffusion equation:

$$\frac{\partial u(x,t)}{\partial t} = p(x)\frac{\partial^2 u(x,t)}{\partial x^2} + q(x,t). \tag{9}$$

In this paper, we use shifted Chebyshev polynomials of third kind and recall some important properties and its analytical form. Next we use these polynomials to approximate the numerical solution of (FDE) with the aid of the Chebyshev collocation method together with the finite difference method to convert the system of equations in algebraic equations that can be solved numerically.

For this purpose, organization of paper is expressed as follows. In Section 2, we give some properties of Chebyshev polynomials of the third kind which are of fundamental importance in what follows. In Section 3, we introduce main theorem of our technique for solving space fractional order diffusion equation subject to homogeneous and nonhomogeneous boundary conditions using a shifted Chebyshev polynomials of the third kind. Numerical scheme is given in Section 4. In Section 5, we present numerical examples to exhibit the accuracy and the efficiency of our proposed method where our numerical results are computed by Matlab program. Conclusions are presented in Section 6.

2. Some properties of Chebyshev polynomials of the thirdkind

2.1. Chebyshev polynomials of the thirdkind

Definition 2. The Chebyshev polynomials $V_n(x)$ of the third kind are orthogonal polynomials of degree n in x defined on the $[-1,1]$ (see, Mason and Handscomb (2003))

$$V_n(x) = \frac{\cos\left(n+\frac{1}{2}\right)\theta}{\cos\frac{\theta}{2}},$$

where $x = \cos\theta$ and $\theta \in [0,\pi]$. They can be obtained explicitly using the Jacobi polynomials $P_k^{(\alpha,\beta)}(x)$, for the special case $\beta = -\alpha = \frac{1}{2}$. These are given by:

$$V_k(x) = \frac{2^{2k}}{\binom{2k}{k}} P_k^{\left(-\frac{1}{2},\frac{1}{2}\right)}(x). \tag{10}$$

Also, these polynomials $V_n(x)$ are orthogonal on $[-1,1]$ with respect to the inner product:

$$\langle V_n(x), V_m(x)\rangle = \int_{-1}^{1}\sqrt{\frac{1+x}{1-x}}\,V_n(x)\,V_m(x)\,dx = \begin{cases}0, & n \neq m,\\ \pi, & n = m,\end{cases} \tag{11}$$

where $\sqrt{\frac{1+x}{1-x}}$ is weight function corresponding to $V_n(x)$. The polynomials $V_n(x)$ may be generated by using the recurrence relations

$$V_{n+1}(x) = 2x\,V_n(x) - V_{n-1}(x), \quad n = 1,2,\ldots,$$

with

$$V_0(x) = 1, \quad V_1(x) = 2x - 1.$$

Using Eq. (10) and properties of Jacobi polynomials to obtain the analytical form of the Chebyshev polynomials of the third kind $V_n(x)$ of degree n, they are given as:

$$V_n(x) = \sum_{k=0}^{\left\lceil\frac{2n+1}{2}\right\rceil}(-1)^k(2)^{n-k}\frac{(2n+1)\Gamma(2n-k+1)}{\Gamma(k+1)\Gamma(2n-2k+2)}(x+1)^{n-k} \quad n \in Z^+, \tag{12}$$

where $\left\lceil\frac{2n+1}{2}\right\rceil$ denotes the integral part of $(2n+1)/2$.

2.2. The shifted Chebyshev polynomials of the third kind

Since the range $[0,1]$ is quite often more convenient to use than the range $[-1,1]$, we sometimes map the independent variable $x \in [0,1]$ to the variable s in $[-1,1]$ by the transformation $s = 2x - 1$ or $x = \frac{(s+1)}{2}$, and this leads to a shifted Chebyshev polynomials of the third kind $V_n^*(x)$ of degree n in x on $[0,1]$ given by

$$V_n^*(x) = V_n(2x-1).$$

These polynomials are orthogonal on the support interval $[0,1]$ as the following inner product:

$$\langle V_n^*(x), V_m^*(x)\rangle = \int_0^1\sqrt{\frac{x}{1-x}}\,V_n^*(x)\,V_m^*(x)\,dx = \begin{cases}0, & n \neq m,\\ \frac{\pi}{2}, & n = m,\end{cases} \tag{13}$$

where $\sqrt{\frac{x}{1-x}}$ is weight function corresponding to $V_n^*(x)$ and normalized by the requirement that $V_n^*(1) = 1$. Also, $V_n^*(x)$ may be generated by using the recurrence relations

$$V_{n+1}^*(x) = 2(2x-1)\,V_n^*(x) - V_{n-1}^*(x), \quad n = 1,2,\ldots,$$

with starting values

$$V_0^*(x) = 1, \quad V_1^*(x) = 4x - 3.$$

The analytical form of the shifted Chebyshev polynomials of the third kind $V_n^*(x)$ of degree n in x is given by

$$V_n^*(x) = \sum_{k=0}^{n}(-1)^k(2)^{2n-2k}\frac{(2n+1)\Gamma(2n-k+1)}{\Gamma(k+1)\Gamma(2n-2k+2)}x^{n-k}, \quad n \in Z^+. \tag{14}$$

In a spectral method, in contrast, the function $g(x)$, square integrable in $[0,1]$, is represented by an infinite expansion of the shifted Chebyshev polynomials of the third kind as follows:

$$g(x) = \sum_{i=0}^{\infty} b_i V_i^*(x), \tag{15}$$

where b_i is a chosen sequence of prescribed basis functions. One then proceeds somehow to estimate as many as possible of the coefficients b_i, thus approximating $g(x)$ by a finite sum of $(m+1)$-terms such as:

$$g_m(x) = \sum_{i=0}^{m} b_i V_i^*(x), \tag{16}$$

where the coefficients $b_i, (i = 0,1,\ldots)$ are given by

$$b_i = \frac{1}{\pi}\int_{-1}^{1} g\left(\frac{x+1}{2}\right)\sqrt{\frac{1+x}{1-x}}V_i(x)dx \tag{17}$$

or

$$b_i = \frac{2}{\pi}\int_0^1 g(x)\sqrt{\frac{x}{1-x}}V_i^*(x)dx. \tag{18}$$

3. Main results

The main approximate formula for the function $g_m(x)$ given in (16) is presented in the following theorem.

Theorem 1. *Let $g_m(x)$ be approximated function in terms of shifted Chebyshev polynomials of the third kind as given in (16), suppose $\mu > 0$ then, we obtain:*

$$D^\mu(g_m(x)) = \sum_{i=\lceil\mu\rceil}^{m}\sum_{k=0}^{i-\lceil\mu\rceil} b_i\, N_{i,k}^{(\mu)}\, x^{i-k-\mu}, \tag{19}$$

where

$$N_{i,k}^{(\mu)} = (-1)^k\, 2^{(2i-2k)}\,\frac{(2n+1)\Gamma(2i-k+1)\Gamma(i-k+1)}{\Gamma(k+1)\Gamma(2i-2k+2)\Gamma(i-k+1-\mu)}. \tag{20}$$

Proof. Using definition of approximated function $g_m(x)$ given in Eq. (16) and the Caputo fractional differentiation properties given in Eq. (2) we obtain:

Please cite this article in press as: Sweilam, N.H. et al., On the numerical solution of space fractional order diffusion equation via shifted Chebyshev polynomials of the third kind. Journal of King Saud University – Science (2015), http://dx.doi.org/10.1016/j.jksus.2015.05.002

$$D^{\mu}(g_m(x)) = \sum_{i=0}^{m} b_i \; D^{\mu}(V_i^*(x)). \tag{21}$$

The properties of linearity of the Caputo derivative together with Eqs. (3) and (4) are used to claim that:

$$D^{\mu}(V_i^*(x)) = 0, \quad i = 0, 1, \ldots, \lceil\mu\rceil - 1, \quad \mu > 0. \tag{22}$$

Also, we obtain:

$$D^{\mu}(V_i^*(x)) = \sum_{k=0}^{i} (-1)^k \; 2^{(2i-2k)} \; \frac{(2i+1)\Gamma(2i-k+1)}{\Gamma(k+1)\Gamma(2i-2k+2)} \; D^{\mu} x^{i-k}. \tag{23}$$

The above Eq. (23) can be rewritten with the aid of Eqs. (3) and (4) as follows:

$$D^{\mu}(V_i^*(x)) = \sum_{k=0}^{i-\lceil\mu\rceil} (-1)^k \; 2^{(2i-2k)} \times \frac{(2i+1)\Gamma(2i-k+1)\Gamma(i-k+1)}{\Gamma(k+1)\Gamma(2i-2k+2)\Gamma(i-k+1-\mu)} \; x^{i-k-\mu}. \tag{24}$$

By combinations Eqs. (21), (22) and (24) we obtain:

$$D^{\mu}(g_m(x)) = \sum_{i=\lceil\mu\rceil}^{m} \sum_{k=0}^{i-\lceil\mu\rceil} b_i \; (-1)^k 2^{(2i-2k)} \times \frac{(2i+1)\Gamma(2i-k+1)\Gamma(i-k+1)}{\Gamma(k+1)\Gamma(2i-2k+2)\Gamma(i-k+1-\mu)} \; x^{i-k-\mu}, \tag{25}$$

the above Eq. (25) can be rewritten as the following form:

$$D^{\mu}(g_m(x)) = \sum_{i=\lceil\mu\rceil}^{m} \sum_{k=0}^{i-\lceil\mu\rceil} b_i \; N_{i,k}^{(\mu)} \; x^{i-k-\mu}, \tag{26}$$

where

$$N_{i,k}^{(\mu)} = (-1)^k \; 2^{(2i-2k)} \; \frac{(2i+1)\Gamma(2i-k+1)\Gamma(i-k+1)}{\Gamma(k+1)\Gamma(2i-2k+2)\Gamma(i-k+1-\mu)}. \tag{27}$$

Simple test example 1. Consider $g(x) = x^2$ with $m = 3$ and $\mu = 1.5$. Using Eq. (4) we obtain:

$$D^{1.5} x^2 = \frac{\Gamma(2+1)}{\Gamma(2+1-1.5)} x^{(2-1.5)} = \frac{\Gamma(3)}{\Gamma(1.5)} x^{\frac{1}{2}} = \frac{2}{\Gamma(\frac{3}{2})} x^{\frac{1}{2}}.$$

Then, using the proposed method given in Theorem 1 together with Eqs. (17) we obtain:

$$\mathcal{D}^{1.5} x^2 = \sum_{i=2}^{3} \sum_{k=0}^{i-2} b_i \; N_{i,k}^{(1.5)} \; x^{(i-k-1.5)} = b_2 \; N_{2,0}^{\frac{3}{2}} \; x^{\frac{3}{2}} + b_3 \; N_{3,0}^{\frac{3}{2}} \; x^{\frac{3}{2}} + b_3 \; N_{3,1}^{\frac{3}{2}} \; x^{\frac{1}{2}}, \tag{28}$$

where,

$$N_{2,0}^{\frac{3}{2}} = \frac{32}{\Gamma(\frac{3}{2})}, \quad N_{3,0}^{\frac{3}{2}} = \frac{256}{\Gamma(\frac{3}{2})}, \quad N_{3,1}^{\frac{3}{2}} = \frac{-224}{\Gamma(\frac{3}{2})}.$$

The constants b_2 and b_3 are computed by using Eq. (17) or (18) and then substituting in Eq. (28) we get:

$$D^{1.5} x^2 = \frac{2}{\Gamma(\frac{3}{2})} x^{\frac{1}{2}}.$$

4. Numerical scheme

Consider the space fractional order diffusion equation of the type given in Eq. (5) with initial condition as in Eq. (6) and boundary conditions given in Eqs. (7) and (8) respectively. In order to use the Chebyshev collocation method, let us approximate $u(x,t)$ as follows:

$$u_m(x,t) = \sum_{i=0}^{m} u_i(t) V_i^*(x). \tag{29}$$

From Eqs. (5), (19) and Theorem 1 we can claim:

$$\sum_{i=0}^{m} \frac{du_i(t)}{dt} V_i^*(x) = p(x) \sum_{i=\lceil\mu\rceil}^{m} \sum_{k=0}^{i-\lceil\mu\rceil} u_i(t) N_{i,k}^{(\mu)} x^{i-k-\mu} + q(x,t). \tag{30}$$

Now, we collocate Eq. (30) at $(m+1-\lceil\mu\rceil)$ points x_p as follows:

$$\sum_{i=0}^{m} \frac{du_i(t)}{dt} V_i^*(x_p) = p(x_p) \sum_{i=\lceil\mu\rceil}^{m} \sum_{k=0}^{i-\lceil\mu\rceil} u_i(t) N_{i,k}^{(\mu)} x_p^{i-k-\mu} + q(x_p,t). \tag{31}$$

Using the roots of shifted Chebyshev polynomials of the third kinds $V_{m+1-\lceil\mu\rceil}^*(x)$ to suitable the collocation points. Using Eqs. (18) and (29) in the initial condition, we obtain the constants (u_i) in the initial case at $(t=0)$, moreover by substituting it in the boundary conditions, we obtain $\lceil\mu\rceil$ equations. For example by substituting Eq. (29) in Eqs. (7) and (8) respectively, in case of $0 < x < 1$ we obtain:

$$\sum_{i=0}^{m} (-1)^{(i)} (2i+1) u_i(t) = v_0(x), \quad \sum_{i=0}^{m} u_i(t) = v_1(x). \tag{32}$$

Notice that: in case the boundary conditions are zeros Dirichlet conditions Eq. (32) can be rewritten as:

$$\sum_{i=0}^{m} (-1)^{(i)} (2i+1) u_i(t) = 0, \quad \sum_{i=0}^{m} u_i(t) = 0. \tag{33}$$

Eq. (31), together with $\lceil\mu\rceil$ equations of the boundary conditions (32), give $(m+1)$ ordinary differential equations which can be solved numerically to get the unknown $u_i, i = 0, 1, \ldots, m$.

5. Numerical experiments and comparison

Example 1. Consider Eq. (5) with $\mu = 1.8$ it is given in the following form:

$$\frac{\partial u(x,t)}{\partial t} = p(x) \frac{\partial^{1.8} u(x,t)}{\partial x^{1.8}} + q(x,t), \quad 0 < x < 1, \quad t > 0,$$

with the diffusion coefficient

$$p(x) = \Gamma(1.2) \; x^{1.8},$$

the source function

$$q(x,t) = 3x^2 \; (2x-1) \; e^{-t},$$

with the initial condition

$$u(x,0) = x^2(1-x),$$

Table 1 The absolute error of the methods given in Khader (2011), Saadatmandi and Dehghan (2011) and our method for Example 1, at $T = 2$ with different values of Δt.

x	In Khader (2011) ($m = 5$)	In Saadatmandi and Dehghan (2011) ($m = 5$)	Presented method ($m = 3$)	Presented method ($m = 3$)
0	2.74×10^{-5}	0	0	0
0.1	4.20×10^{-5}	4.47×10^{-6}	3.77×10^{-7}	3.79×10^{-10}
0.2	3.76×10^{-5}	2.78×10^{-7}	6.25×10^{-7}	6.26×10^{-10}
0.3	8.44×10^{-5}	5.81×10^{-6}	7.59×10^{-7}	7.61×10^{-10}
0.4	3.27×10^{-5}	1.02×10^{-5}	7.97×10^{-7}	7.99×10^{-10}
0.5	3.61×10^{-5}	1.17×10^{-5}	7.58×10^{-7}	7.60×10^{-10}
0.6	1.94×10^{-5}	1.08×10^{-5}	6.58×10^{-7}	6.59×10^{-10}
0.7	2.95×10^{-5}	8.54×10^{-6}	5.14×10^{-7}	5.16×10^{-10}
0.8	4.92×10^{-5}	6.06×10^{-6}	3.45×10^{-7}	3.46×10^{-10}
0.9	2.83×10^{-5}	3.76×10^{-6}	1.68×10^{-7}	1.68×10^{-10}
1	7.73×10^{-5}	0	0	0

and the boundary conditions

$$u(0,t) = u(1,t) = 0, \quad t > 0.$$

The exact solution of this problem is given by:

$$u(x,t) = x^2(1-x)e^{-t}.$$

Let us consider $m = 3$ then, we have:

$$u_3(x,t) = \sum_{i=0}^{3} u_i(t)\ V_i^*(x). \tag{34}$$

Using Eq. (31), we claim:

$$\sum_{i=0}^{3} \frac{du_i(t)}{dt} V_i^*(x_p) = p(x_p)\sum_{i=2}^{3}\sum_{k=0}^{i-2} u_i(t) N_{i,k}^{(1.8)} x_p^{i-k-1.8} + q(x_p,t), \quad p = 0,1, \tag{35}$$

where x_p are the roots of the shifted Chebyshev polynomial of the third kind $V_2^*(x)$. Using Eqs. (33) and (35) we obtain the following system of ordinary differential equations:

$$\dot{u}_0(t) + G_1\ \dot{u}_1(t) + G_2\ \dot{u}_3(t) = H_1\ u_2(t) + H_2\ u_3(t) + q(x_0,t), \tag{36}$$

$$\dot{u}_0(t) + G_{11}\ \dot{u}_1(t) + G_{22}\ \dot{u}_3(t) = H_{11}\ u_2(t) + H_{22}\ u_3(t) + q(x_1,t), \tag{37}$$

$$u_0(t) - 3u_1(t) + 5u_2(t) - 7u_3(t) = 0, \tag{38}$$

$$u_0(t) + u_1(t) + u_2(t) + u_3(t) = 0, \tag{39}$$

where

$$G_1 = V_1^*(x_0), \quad G_2 = V_3^*(x_0), \quad G_{11} = V_1^*(x_1), \quad G_{22} = V_3^*(x_1),$$

$$H_1 = p(x_0)N_{2,0}^{(1.8)}x_0^{2-1.8}, \quad H_2 = p(x_0)[N_{3,0}^{(1.8)}x_0^{3-1.8} + N_{3,1}^{(1.8)}x_0^{2-1.8}],$$

$$H_{11} = p(x_1)N_{2,0}^{(1.8)}x_1^{2-1.8}, \quad H_{22} = p(x_1)[N_{3,0}^{(1.8)}x_1^{3-1.8} + N_{3,1}^{(1.8)}x_1^{2-1.8}].$$

Now, we use the finite difference method to solve the system (36)–(39) with the following notations: $T = T_{final}, 0 < t_j \le T$ and suppose $\Delta t = T/N, t_j = j\Delta t$, for $j = 0, 1, \ldots, N$. Also, we define

$$u_i(t_n) = u_i^n, \quad q_i(t_n) = q_i^n.$$

Then, the system in Eqs. (36)–(39), is discretized in the time and has the following form:

$$\frac{u_0^n - u_0^{n-1}}{\Delta t} + G_1\ \frac{u_1^n - u_1^{n-1}}{\Delta t} + G_2\ \frac{u_3^n - u_3^{n-1}}{\Delta t} = H_1\ u_2^n + H_2\ u_3^n + q_0^n, \tag{40}$$

$$\frac{u_0^n - u_0^{n-1}}{\Delta t} + G_{11}\ \frac{u_1^n - u_1^{n-1}}{\Delta t} + G_{22}\ \frac{u_3^n - u_3^{n-1}}{\Delta t} = H_{11}\ u_2^n + H_{22}\ u_3^n + q_1^n, \tag{41}$$

$$u_0^n - 3u_1^n + 5u_2^n - 7u_3^n = 0, \tag{42}$$

$$u_0^n + u_1^n + u_2^n + u_3^n = 0. \tag{43}$$

The above system of Eqs. (40)–(43) can be rewritten as the following matrix form:

$$AU^n = BU^{n-1} + \Delta t q^n, \quad or \quad U^n = A^{-1}BU^{n-1} + \Delta t A^{-1}q^n, \tag{44}$$

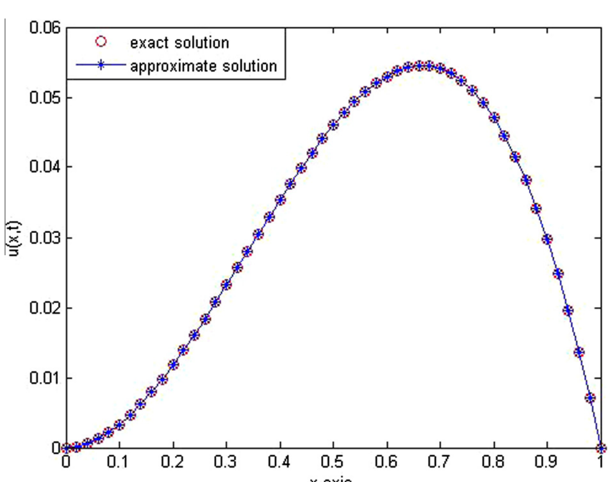

Fig. 1 The behavior of exact solution and approximation solution at $m = 5$ and $T = 1$.

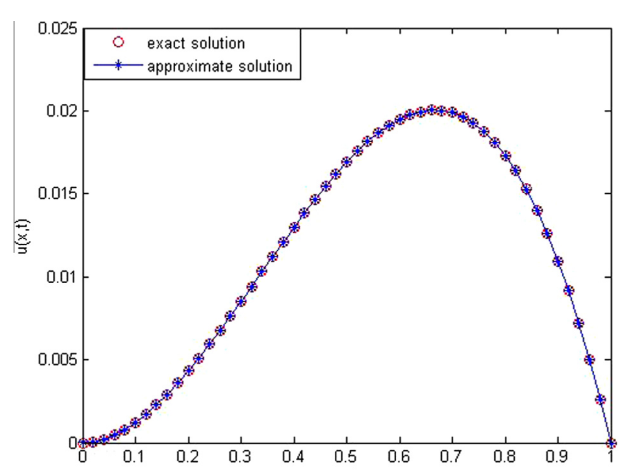

Fig. 2 The behavior of exact solution and approximation solution at $m = 7$ and $T = 2$.

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Table 2 Comparison of maximum error of the method in Sweilam et al. (2015) and our presented method for Example 2, at $m = 3$ and $T = 1$.

Max error (Sweilam et al., 2015)	Max error present method
8.3830×10^{-10}	1.008×10^{-10}

where

$$A = \begin{pmatrix} 1 & G_1 & -\Delta t H_1 & G_2 - \Delta t H_2 \\ 1 & G_{11} & -\Delta t H_{11} & G_{22} - \Delta t H_{22} \\ 1 & -2 & 3 & -4 \\ 1 & 2 & 3 & 4 \end{pmatrix},$$

$$B = \begin{pmatrix} 1 & G_1 & 0 & G_2 \\ 1 & G_{11} & 0 & G_{22} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$U^n = (u_0^n, u_1^n, u_2^n, u_3^n)^T$ and $q^n = (q_0^n, q_1^n, 0, 0)^T$.

In order to obtain the initial solution U^0 of Eq. (44) we use the initial condition of the problem, $u(x,0)$ combining with Eq. (17) *or* (18). Moreover, the approximation solution in Eq. (34) is obtained by substituting analytical form series of the shifted Chebyshev polynomials of the third kind $V_i^*(x)$, $i = 0, 1, 2, 3$ as well as the coefficients (u_i , $i = 0, 1, 2, 3$) which are computed in (44).

Table 1, presents the numerical results of Example 1 at $T = 2$. The results obtained by our method with $\Delta t = 0.25e - 3$ and $\Delta t = 0.25e - 6$ are given in the fourth and the fifth columns respectively. In this table, we have computed the absolute error between the exact and the approximate solutions. In order to validate the accuracy of the proposed method, we have compared the results of the presented method with the previously published data given in Khader (2011) and Saadatmandi and Dehghan (2011). From the results in Table 1, we can derive that the accuracy of the presented method is better than other methods. Also, the computational cost (CPU time) seems for the proposed method faster than other methods because we just need few terms (i.e., $m = 3$) to obtain a solution with high accuracy. In Figs. 1 and 2 we have plotted the exact and the numerical solutions at $m = 5$, $T = 1$ and $m = 7$, $T = 2$ respectively.

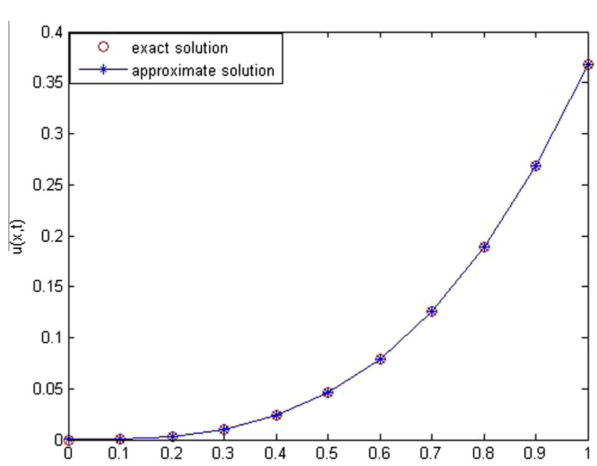

Fig. 3 The behavior of exact solution and approximation solution for Example 2 at $m = 3$ and $T = 1$.

Example 2. Consider Eq. (5) with $\mu = 1.8$ as follows:

$$\frac{\partial u(x,t)}{\partial t} = p(x)\frac{\partial^{1.8} u(x,t)}{\partial x^{1.8}} + q(x,t), \quad 0 < x < 1, \quad t > 0,$$

with the diffusion coefficient

$$p(x) = \frac{\Gamma(2.2)}{6}\ x^{2.8},$$

the source function

$$q(x,t) = -(1 + x)\ x^3\ e^{-t},$$

with the initial condition

$$u(x,0) = x^3,$$

and the boundary conditions

$$u(0,t) = 0, \quad u(1,t) = e^{-t}, \quad t > 0.$$

The exact solution of this problem is given by:

$$u(x,t) = x^3 e^{-t}.$$

From the results of Table 2 and Fig. 3, it is obvious that the presented method gives more accurate results than the results discussed in Sweilam et al. (2015).

6. Conclusions

In this paper, the shifted Chebyshev polynomials of the third kind and its properties together with the Chebyshev collocation method are used to reduce the space fractional order diffusion equation to a system of ordinary differential equations. The fractional derivative is considered in the Caputo sense. The validity and applicability of our presented method are illustrated through some numerical results. These results are compared with other published results in some papers. From the numerical results, it is obvious that our proposed method exhibits good accuracy and efficiency than the other methods. All computed results are obtained using Matlab program.

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