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Research Article

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On the numerical solutions of some fractional ordinary differential equations by fractional Adams-Bashforth-Moulton method

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Abstract: In this paper, we apply the Fractional Adams-Bashforth-Moulton Method for obtaining the numerical solutions of some linear and nonlinear fractional ordinary differential equations. Then, we construct a table including numerical results for both fractional differential equations. Then, we draw two dimensional surfaces of numerical solutions and analytical solutions by considering the suitable values of parameters. Finally, we use the L_2 *nodal norm* and L_∞ *maximum nodal norm* to evaluate the accuracy of method used in this paper.

Keywords: Fractional Adams-Bashforth-Moulton method, Fractional calculus, Fractional nonlinear ordinary differential equation

MSC: 65ZXX, 90CXX, 90-08, 90-00, 68WXX

1 Introduction

Globally, a physical phenomenon can be expressed by the help of theory of derivatives and integrals with fractional order. Therefore, fractional concepts have been seen as a tool in the fields such as physics, chemistry and engineering in terms of representing physical phenomena. Thus, a lot of powerful methods, such as fractional linear multistep methods, variational iteration, galerkin finite element, Sumudu transform, trial equation, Adomian's decomposition, extended trial equation, homotopy analysis, iteration, homotopy perturbation, modified homotopy perturbation, generalized trigonometry functions, homotopy perturbation, Sumudu transform or modified trial equation method, have been presented in literature [2–17, 26–28, 30]. Besides these methods some authors have investigated various properties of fractional concepts [18, 19, 29, 32–34].

The organization of this paper is as follows: we give some definitions and properties of the fractional calculus in Section 2. In Section 3, we introduce the general construction of Fractional Adams-Bashforth-Moulton Method (FABMM) for time-fractional linear and nonlinear ordinary differential equations. In Section 4, we apply FABMM to the linear time fractional ordinary differential equation (FODE) defined by [1],

$$D_t^\alpha u(t) + u(t) = t^{3+\alpha} + \frac{\Gamma[4+\alpha]}{6} t^3, 0 < \alpha \le 1, \tag{1}$$

where α is an arbitrary constant and a parameter describing the order of the fractional time-derivative. (1) have the following exact solution in the closed manner [1];

$$u(t) = t^{3+\alpha}. \tag{2}$$

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Then, we consider the nonlinear time fractional ordinary differential equation described as following;

$$D_t^{\alpha}u(t)+\eta u^2(t)=\frac{1}{24T^{4+\alpha}}\Gamma[5+\alpha]t^4+\eta(\frac{t}{T})^{8+2\alpha},1<\alpha\leq 2, \tag{3}$$

having the exact solution as:

$$u(t)=(\frac{t}{T})^{4+\alpha}, \tag{4}$$

where η and T are arbitrary constants and not zero [1].

2 Preliminaries

In this part of the paper it would be useful to introduce some definitions and properties of the fractional calculus theory. The Caputo fractional derivative of $f(t)$ function is defined by [24, 25]:

$${}_a^C D_t^{\alpha}[f(t)]=\frac{1}{\Gamma(\alpha-n)}\int_a^t[(t-\tau)^{n-1-\alpha}f^{(n)}(\tau)]d\tau,n-1<\alpha\leq n,n\in Z,\alpha\in R^{+}. \tag{5}$$

In addition to this expression, some of the useful formulas such as $f(x)=(x-a)^{\beta}$ and $g(x)=(b-x)^{\beta}$ are given by [1]

$${}_a^C D_x^{\alpha}[f(x)]=\frac{\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)}(x-a)^{\beta-\alpha},$$

$${}_x^C D_b^{\alpha}[g(x)]=\frac{\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)}(b-x)^{\beta-\alpha}, \tag{6}$$

in which $n-1<\alpha\leq n,n\in Z,\alpha\in R^{+}$.

3 Construction of FABMM

In this section of our study, an approach to the FODE will be given. The general form of FABMM used to obtain numerical solutions of fractional differential equations (FODEs) given along with initial conditions can be considered as the following form [20–24];

$$D_*^{\alpha}[y(t)]=f(t,y(t)),\alpha>0, \tag{7}$$

with initial conditions:

$$y^{(k)}(0)=y_0^{(k)}, \tag{8}$$

where $k=0,1,2,3,\cdots,\lceil\alpha\rceil-1$, and $D_*^{\alpha}[.]$ an operator in the sense of Caputo defined by:

$$D_*^{\alpha}[z(t)]=J^{n-\alpha}[D^n[z(t)]], \tag{9}$$

where n is bigger than α and smallest integer number, D is integer order derivative operator and J is an integral operator defined in the following way in the sense of Riemann-Liouville integral operator:

$$J^{\mu}[z(t)]=\frac{1}{\Gamma(\mu)}\int_0^t[(t-u)^{\mu-1}z(u)]du,\mu>0. \tag{10}$$

If we take the integral of (7) according to (10), it gives us a second order Volterra integral equation well known [6, 22, 23, 31]:

$$y(t)=\Sigma_{v=0}^{n-1}y^{(v)}(0)\frac{t^v}{v!}+\frac{1}{\Gamma[\alpha]}\int_0^t[(t-u)^{\alpha-1}f(u,y(u))]du. \tag{11}$$

Before submitting the general structure of FABMM, we can set off by remembering the general form of Adams-Bashforth-Moulton method for the integer order differential equations. Here, the general form of the integer order differential equations is defined by:

$$D[y(t)] = f(t, y(t)), \tag{12}$$

and initial condition is

$$y(0) = y_0. \tag{13}$$

The grid points of (12) for every single point by dividing N steps to the interval of $[0, T]$ can be written in the following way [23, 24]:

$$h = \frac{T}{N}, t_j = jh, j = 0, 1, 2, 3, \cdots, N, \tag{14}$$

and

$$y_j = y(t_j), j = 0, 1, 2, 3, \cdots, k, \tag{15}$$

where $y(t_{k+1})$ approaches are defined as [22, 23]:

$$y(t_{k+1}) = y(t_k) + \int_{t_k}^{t_{k+1}} [f(z, y(z))]dz. \tag{16}$$

The integral of (16) can be rearranged as:

$$\int_a^b [g(z)]dz \approx \frac{b-a}{2}[g(a) + g(b)]. \tag{17}$$

By applying trapezoidal rule, we get (18) for (16) as follows:

$$y(t_{k+1}) = y(t_k) + \frac{h}{2}[f(t_k, y(t_k) + f(t_{k+1}, y(t_{k+1})]. \tag{18}$$

If in (18) we represent equations of (15) as $y(t_k) = y_k$ and $y(t_{k+1}) = y_{k+1}$, we receive:

$$y_{k+1} = y_k + \frac{h}{2}[f(t_k, y_k + f(t_{k+1}, y_{k+1})]. \tag{19}$$

As (19) has y_{k+1} on both sides, it may be very hard to directly obtain solution of this equation. Therefore, we have to take predictor y_{k+1}^p for the first approach of y_{k+1} in the following way [22, 23]:

$$\int_a^b [g(z)]dz \approx (b-a)g(a). \tag{20}$$

Then, by means of applying trapezoidal rule, we can obtain the construction known as Euler formula or one-step Adams-Bashforth method getting [22, 23]:

$$y_{k+1}^p = y_k + hf(t_k, y_k). \tag{21}$$

When we rewrite (19) under the terms of (21), we can get the following equation:

$$y_{k+1} = y_k + \frac{h}{2}[f(t_k, y_k + f(t_{k+1}, y_{k+1}^p)]. \tag{22}$$

When we take into account both (21) and (22), we obtain the general form of one-step Adams-Bashforth-Moulton method of finding numerical solution of (12), being the integer order differential equations, for which we will investigate FABMM for solving FODEs numerically. The fundamental difference between (11) and (16) is the fact of starting from zero at the lower bound of integral. Therefore, we have to set out by taking t_k instead of zero as the lower boundary of integral to solve FODEs by using FABMM. When we apply trapezoidal rule according to weight function $(t_{k+1}-)^{\alpha-1}$ for (11), it gives us following equation:

$$\int_0^{t_{k+1}} [(t_{k+1}-z)^{\alpha-1} g(z)]dz \approx \int_0^{t_{k+1}} [(t_{k+1}-z)^{\alpha-1} \tilde{g}_{k+1}(z)]dz, \tag{23}$$

where $g(z)$ is the piecewise linear interpolant and it has nodes $\tilde{g}_{k+1}, t_j = jh$. Under the rules of standard quadrature technique [22, 23], right side integral of (23) can be rewritten in the following way:

$$\int_0^{t_{k+1}} [(t_{k+1}-z)^{\alpha-1} \tilde{g}_{k+1}(z)]dz = \sum_{j=0}^{k+1} a_{j,k+1} g(t_j), \tag{24}$$

where $a_{j,k+1}$ is defined by:

$$a_{j,k+1} = \frac{h^\alpha}{\alpha(\alpha+1)} \begin{cases} k^{\alpha+1} - (k-\alpha)(k+1)^\alpha, j = 0, \\ (k-j+2)^{\alpha+1} + (k-j)^{\alpha+1} - 2(k-j+1)^{\alpha+1}, 1 \le j \le k, \\ 1, j = k+1, \end{cases} \tag{25}$$

By substituting (23), (24) and (25) in (11), we form the corrector formula for one-step FABMM as follows [22, 23];

$$y_{k+1} = \Sigma_{j=0}^{n-1} y^{(j)}(0) \frac{t_{k+1}^j}{j!} + \frac{1}{\Gamma[\alpha]}[\sum_{j=0}^{k} a_{j,k+1} f(t_j, y_j) + a_{k+1,k+1} f(t_{k+1}, y_{k+1}^p)]. \tag{26}$$

Then, y_{k+1}^p predictor under the constructions of Adams-Moulton method can be rewritten in the following form [22, 23]:

$$\int_0^{t_{k+1}} [(t_{k+1}-z)^{\alpha-1} g(z)]dz = \sum_{j=0}^{k} b_{j,k+1} g(t_j), \tag{27}$$

where $b_{j,k+1}$ is defined by:

$$b_{j,k+1} = \frac{h^\alpha}{\alpha}[(k+1-j)^\alpha - (k-j)^\alpha]. \tag{28}$$

Then, the predictor y_{k+1}^p is the same as the one defined above for the Adams-Moulton method [22, 23]:

$$y_{k+1}^p = \Sigma_{j=0}^{n-1} y_0^{(j)} \frac{t_{k+1}^j}{j!} + \frac{1}{\Gamma[\alpha]}[\sum_{j=0}^{k} b_{j,k+1} f(t_j, y_j)]. \tag{29}$$

When we consider both (26) and (29), we obtain the general form of FABMM to solve numerically (7).

4 Applications

In this section, we applied FABMM to the linear and nonlinear time fractional differential equations as follows.

Example 1. *Firstly, we consider (1) linear fractional differential equation along with exact solution (2). We can rearrange (1), in a way similar to (7);*

$$D_t^\alpha u(t) = f(t, u(t)) = -u(t) + t^{\alpha+3} + \frac{\Gamma[4+\alpha]}{6} t^3. \tag{30}$$

If we apply FABMM to (30) by taking $0 < t \le 1$, *step size* $n = 300$ *and initial condition* $u(0) = 0$, *we can obtain a numerical solution for (30) for the first ten term and error accounts. Next, to determine the accuracy of the technique, we use* L_2 nodal norm *defined by:*

$$L_2 = \|U^{Analytical} - U_M\|_2 \simeq \sqrt{h \sum_{j=0}^{M} |U_j^{Analytical} - (U_M)_j|^2}, \tag{31}$$

and L_∞ maximum nodal norm *defined by:*

$$L_\infty = \|U^{Analytical} - U_M\|_\infty \simeq max_j |U_j^{Analytical} - (U_M)_j|. \quad (32)$$

Table 1. The exact solution, numerical solution of (1) and the numerical errors for $\alpha = 0.25$ obtained by using FABMM

n	t_n	u_Exact	u_Num.	u_Exact−u_Num.
1	0.0033333	8.8930153E-09	1.2733339E-08	-3.8340377E-09
2	0.0066667	8.4664901E-08	1.0586060E-07	-2.1195501E-08
3	0.0100000	3.1622776E-07	3.7064679E-07	-5.4419030E-08
4	0.0133333	8.0547292E-07	9.0951855E-07	-1.0404572E-07
5	0.0166667	1.6634449E-06	1.8339278E-06	-1.7048289E-07
6	0.0200000	3.0084824E-06	3.2655390E-06	-2.5407143E-07
7	0.0233333	4.9650605E-06	5.3201650E-06	-3.5510443E-07
8	0.0266667	7.6629921E-06	8.1368311E-06	-4.7383897E-07
9	0.0300000	1.1236836E-05	1.1847341E-05	-6.1050410E-07
10	0.0333333	1.5825444E-05	1.6590751E-05	-7.6530645E-07

Using L_2 nodal norm *algorithms for measuring the accuracy of the technique used for solving (1) by taking* $n = 300$ *and* $\alpha = 0.25$*, we obtain the following* L_2 *sum of numerical error:*

0.000440091.

Similarly, when we use L_∞ maximum nodal norm *algorithms for measuring the accuracy of the technique used for solving (1) by taking* $n = 300$ *and* $\alpha = 0.25$*, we obtain the following* L_∞ *maximum numerical errors:*

0.000998594.

Fig. 1. The two-dimensional surfaces of the numerical solution, analytical solution and absolute errors of (1) obtained by using FABMM for $0 \le t \le 1, \alpha = 0.25$

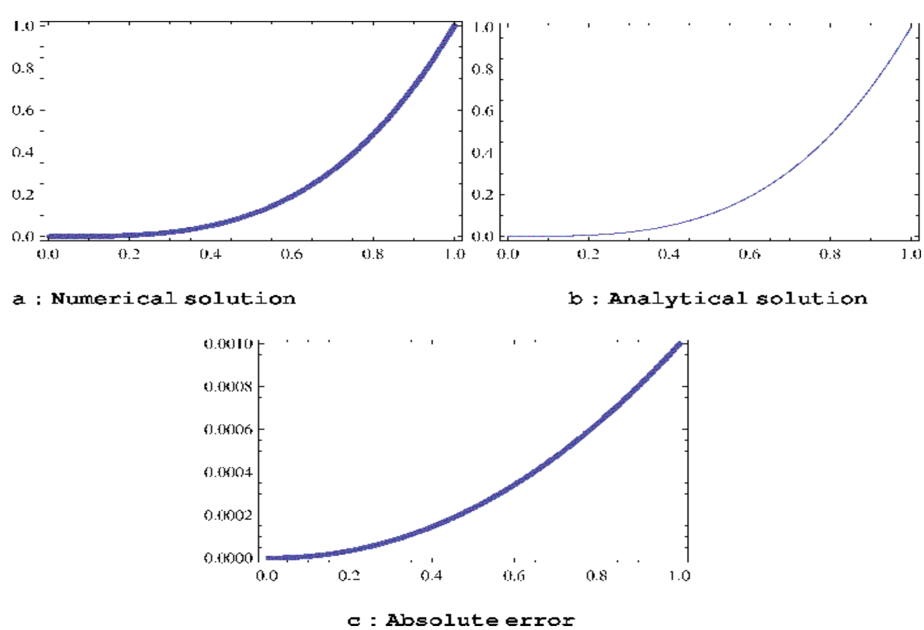


Table 2. The exact solution, numerical solution of (1) and the numerical errors for $\alpha = 0.75$ obtained by using FABMM

n	t_n	u_Exact	u_Num.	u_Exact−u_Num.
1	0.0033333	5.1380141E-10	8.8752788E-10	-3.7372647E-10
2	0.0066667	6.9128602E-09	8.3108958E-09	-1.3980355E-09
3	0.0100000	3.1622776E-08	3.4595664E-08	-2.9728881E-09
4	0.0133333	9.3007990E-08	9.8076290E-08	-5.0682993E-09
5	0.0166667	2.1474982E-07	2.2242107E-07	-7.6712350E-09
6	0.0200000	4.2546367E-07	4.3624009E-07	-1.0776420E-08
7	0.0233333	7.5842553E-07	7.7280824E-07	-1.4382715E-08
8	0.0266667	1.2513613E-06	1.2698530E-06	-1.8491667E-08
9	0.0300000	1.9462772E-06	1.9693837E-06	-2.3106527E-08
10	0.0333333	2.8893176E-06	2.9175494E-06	-2.8231731E-08

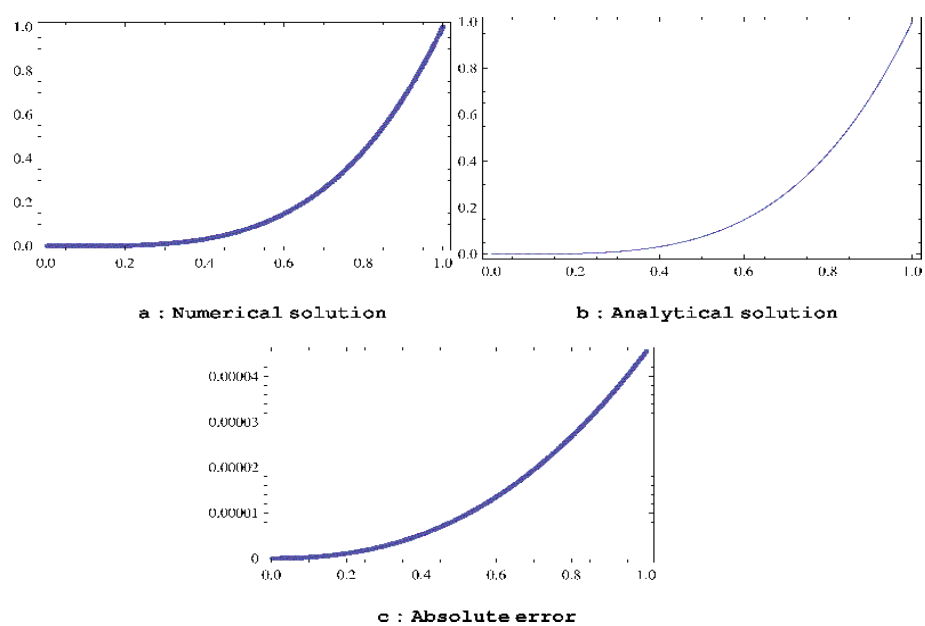
Using L_2 nodal norm *algorithms for measuring the accuracy of the technique used for solving (1) by taking* $n = 300$ *and* $\alpha = 0.75$*, we obtain the following* L_2 *sum of numerical errors:*

0.0000190654.

Similarly, when we use L_∞ maximum nodal norm *algorithms for measuring the accuracy of the technique used for solving (1) by taking* $n = 300$ *and* $\alpha = 0.75$*, we obtain the following* L_∞ *maximum numerical error:*

0.0000455084.

Fig. 2. The two-dimensional surfaces of the numerical solution, analytical solution and absolute errors of (1) obtained by using FABMM for $0 \le t \le 1, \alpha = 0.75$



Example 2. *Secondly, let's consider (3) nonlinear fractional ordinary differential equation along with exact solution of (4). We can rearrange (3);*

$$D_t^\alpha u(t) = -\eta u^2(t) + \frac{\Gamma[5+\alpha]}{24T^{4+\alpha}} t^4 + \eta(\frac{t}{T})^{8+2\alpha}, 1 < \alpha \le 2, \tag{33}$$

where α, η, T *are constants and not zero. When we rewrite (33) by substituting* $\eta = T = 1$*, it gives us the following differential equation [1]:*

$$D_t^\alpha u(t) = f(t, u(t)) = -u^2(t) + \frac{\Gamma[5+\alpha]}{24} t^4 + t^{8+2\alpha}, \tag{34}$$

with the exact solution [1];

$$u(t) = t^{4+\alpha}. \tag{35}$$

If we apply FABMM to (34) by getting $\alpha = 1.25, \alpha = 1.75, 0 < t \leq 1$, *step size* $n = 300$ *and initial condition* $u(0) = 0$, *one can obtain a numerical solution for (34) for the first ten terms and error accounts. Next, to measure to the accuracy of the technique, we will use* L_2 nodal norm *and* L_∞ maximum nodal norm.

Table 3. The exact solution, numerical solution of (34) and the numerical errors for $\alpha = 1.25$ obtained by using FABMM

n	t_n	u_Exact	u_Num.	u_Exact−u_Num.
1	0.0033333	9.8881128E-14	2.9876778E-13	-1.9988665E-13
2	0.0066667	3.7628845E-12	5.6039360E-12	-1.8410515E-12
3	0.0100000	3.1622776E-11	3.8373813E-11	-6.7510365E-12
4	0.0133333	1.4319551E-10	1.6022741E-10	-1.7032249E-11
5	0.0166667	4.6206804E-10	4.9704561E-10	-3.4977573E-11
6	0.0200000	1.2033929E-09	1.2664240E-09	-6.3031050E-11
7	0.0233333	2.7031996E-09	2.8069634E-09	-1.0376375E-10
8	0.0266667	5.4492388E-09	5.6090960E-09	-1.5985717E-10
9	0.0300000	1.0113153E-08	1.0347244E-08	-2.3409097E-10
10	0.0333333	1.7583827E-08	1.7913160E-08	-3.2933348E-10

Using L_2 nodal norm *algorithms for measuring the accuracy of the technique used for solving (34) by taking* $n = 300$ *and* $\alpha = 1.25$, *we obtain the following* L_2 *sum of numerical errors:*

0.00000000809849.

Similarly, when we use L_∞ maximum nodal norm *algorithms for measuring the accuracy of the technique used for solving (34) by taking* $n = 300$ *and* $\alpha = 1.25$, *we obtain the following* L_∞ *maximum numerical errors:*

0.0000225821.

Fig. 3. The two-dimensional surfaces of the numerical solution, analytical solution and absolute errors of (34) obtained by using FABMM for $0 \leq t \leq 1, \alpha = 1.25$

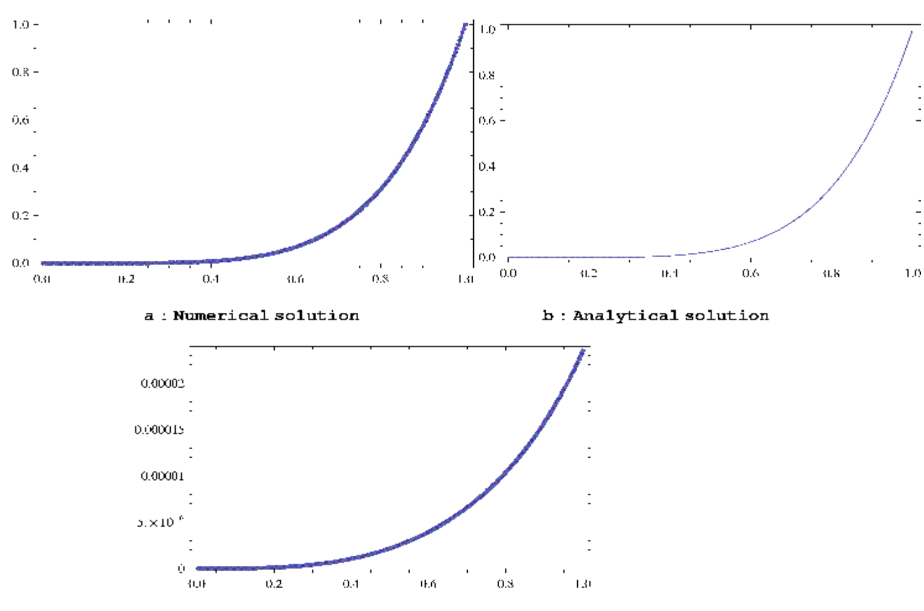


Table 4. The exact solution, numerical solution of (34) and the numerical errors for $\alpha = 1.75$ obtained by using FABMM

n	t_n	u_Exact	u_Num.	u_Exact−u_Num.
1	0.0033333	5.7089045E-15	2.4363196E-14	-1.8654291E-14
2	0.0066667	3.0723823E-13	5.0498014E-13	-1.9774191E-13
3	0.0100000	3.1622776E-12	4.0125200E-12	-8.5024240E-13
4	0.0133333	1.6534753E-11	1.8974819E-11	-2.4400657E-12
5	0.0166667	5.9652728E-11	6.5218503E-11	-5.5657757E-12
6	0.0200000	1.7018546E-10	1.8113626E-10	-1.0950080E-11
7	0.0233333	4.1292056E-10	4.3235707E-10	-1.9436511E-11
8	0.0266667	8.8985697E-10	9.2183378E-10	-3.1976810E-11
9	0.0300000	1.7516495E-09	1.8012834E-09	-4.9633889E-11
10	0.0333333	3.2103529E-09	3.2839275E-09	-7.3574614E-11

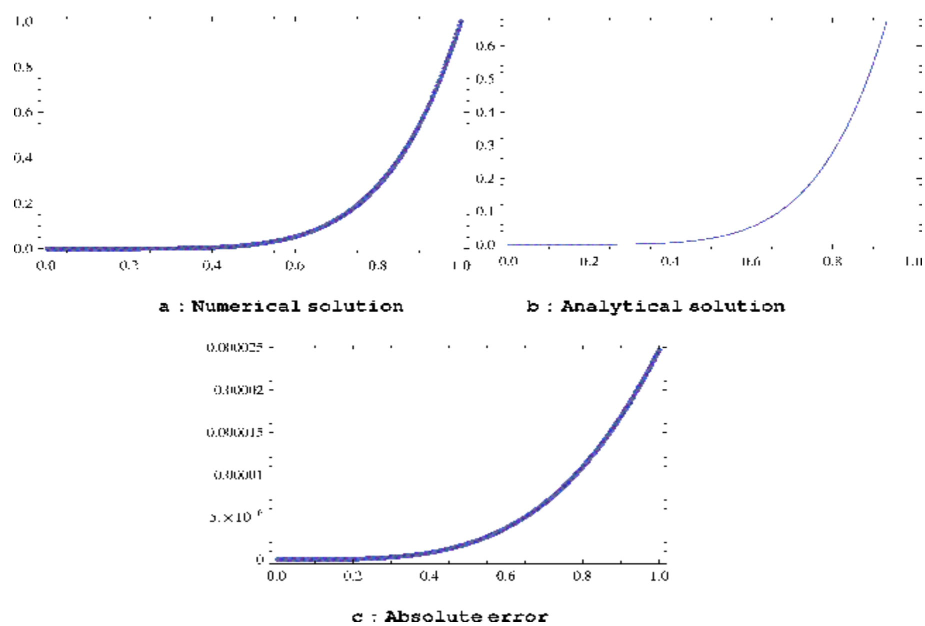
Using L_2 nodal norm *algorithms for measuring the accuracy of the technique used for solving (34) by taking* $n = 300$ *and* $\alpha = 1.75$*, we obtain the following* L_2 *sum of numerical errors:*

0.00000000863107.

Similarly, when we use L_∞ maximum nodal norm *algorithms for measuring the accuracy of the technique used for solving (34) by taking* $n = 300$ *and* $\alpha = 1.75$*, we obtain the following* L_∞ *maximum numerical error:*

0.0000247177.

Fig. 4. The two-dimensional surfaces of the numerical solution, analytical solution and absolute errors of (34) obtained by using FABMM for $0 \leq t \leq 1, \alpha = 1.75$



5 Remark

The numerical results for Example 1 and Example 2 have been obtained by using the programming language Wolfram Mathematica 9. To the best of our knowledge, these numerical solutions have not been published previously, and these results are new numerical solutions for (1) and (34).

6 Conclusions

In this paper, we have successfully applied FABMM for obtaining the numerical solutions of some linear and nonlinear FODEs. We have constructed a table including numerical results for both fractional differential equations. Next, we have drawn two dimensional surfaces of numerical solutions and analytical solutions by using Wolfram Mathematica 9. Applying suitable values of parameters before we use L_2 *nodal norm* and L_∞ *maximum nodal norm* to evaluate the validity of the method used in this paper. It can be seen that this method is a powerful tool for obtaining the numerical solutions of such FODEs, taking into account the numerical errors obtained by using L_2 *nodal norm* and L_∞ *maximum nodal norm* for the numerical errors of (1) and (34). We think that the proposed method can also be applied to other fractional differential equations.

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