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ON THE O'NAN-SCOTT THEOREM FOR FINITE PRIMITIVE PERMUTATION GROUPS

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Abstract

We give a self-contained proof of the O'Nan-Scott Theorem for finite primitive permutation groups.

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Introduction

The classification of finite simple groups has led to the solution of many problems in the theory of finite permutation groups. An important starting point in such applications is the reduction theorem for primitive permutation groups first stated by O'Nan and Scott (see [9]). The version particularly useful in this context is that given in Theorem 4.1 of [2]. Unfortunately a case was omitted in the statements in [2, 9] (namely, the case leading to our groups of type III(c) in Section 1 below). A corrected and expanded version of the theorem appears in the long papers [1] and [3]. Our aim here is to update and develop further the material in Section 4 of [2]. We give a self-contained proof of the theorem (stated in Section 2), which extends [2, Theorem 4.1], and which we have found to be in the form most useful for application to permutation groups (see [6] and [8] for example). Most of the ideas of the proof we owe to [1] and [2].

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NOTATION. For groups A, B we denote by $A.B$ an extension of A by B (not necessarily split). If G, H are permutation groups on Ω, Δ respectively, we say that G is permutation equivalent to H if there is a bijection $\varphi: \Omega \to \Delta$ and an isomorphism ψ: $G \to H$ such that $(\omega g)\varphi = (\omega\varphi)(g\psi)$ for all $g \in G$, $\omega \in \Omega$. Notice that if Ω and Δ are identified via the bijection φ, then G and H consist of the same set of permutations on Ω.

1. Classes of finite primitive permutation groups

Before stating our theorem we describe various classes of primitive permutation groups. For more details of these, see [5].

In what follows, X will be a primitive permutation group on a finite set Ω of size n, and α a point in Ω. Let B be the socle of X, that is, the product of all minimal normal subgroups of X. Then $B \cong T^k$ with $k \geq 1$, where T is a simple group.

I. *Affine groups.* Here $T = Z_p$ for some prime p, and B is the unique minimal normal subgroup of X and is regular on Ω of degree $n = p^k$. The set Ω can be identified with $B = Z_p^k$ so that X is a subgroup of the affine group $AGL(k,p)$ with B the translation group and $X_\alpha = X \cap GL(k,p)$ irreducible on B.

II. *Almost simple groups.* Here $k = 1$, T is a nonabelian simple group and $T \leq X \leq \operatorname{Aut} T$. Also $T_\alpha \neq 1$.

III. In this case $B \cong T^k$ with $k \geq 2$ and T a nonabelian simple group. We distinguish three types:

III(a). *Simple diagonal action.* Define

$$W = \{(a_1, \dots, a_k).\pi \mid a_i \in \operatorname{Aut} T,\ \pi \in S_k,\ a_i \equiv a_j \bmod \operatorname{Inn} T \text{ for all } i,j\},$$

where $\pi \in S_k$ just permutes the components a_i naturally. With the obvious multiplication, W is a group with socle $B \cong T^k$, and $W = B.(\operatorname{Out} T \times S_k)$, a (not necessarily split) extension of B by $\operatorname{Out} T \times S_k$. We define an action of W on Ω by setting

$$W_\alpha = \{(a, \dots, a).\pi \mid a \in \operatorname{Aut} T,\ \pi \in S_k\}.$$

Thus $W_\alpha \cong \operatorname{Aut} T \times S_k$, $B_\alpha \cong T$ and $n = |T|^{k-1}$.

For $1 \leq i \leq k$ let T_i be the subgroup of B consisting of the k-tuples with 1 in all but the ith component, so that $T_i \cong T$ and $B = T_1 \times \cdots \times T_k$. Put $\mathcal{T} = \{T_1, \dots, T_k\}$, so that W acts on $\mathcal{T}$. We say that the subgroup X of W is of type III(a) if $B \leq X$ and, letting P be the permutation group $X^{\mathcal{T}}$, one of the following holds:

(i) P is primitive on $\mathcal{T}$,

(ii) $k = 2$ and $P = 1$.

We have $X_\alpha \lesssim \operatorname{Aut} T \times P$, and $X \leq B.(\operatorname{Out} T \times P)$. Moreover, in case (i) B is the unique minimal normal subgroup of X, and in case (ii) X has two minimal normal subgroups T_1 and T_2, both regular on Ω.

III(b). *Product action.* Let H be a primitive permutation group on a set Γ, of type II or III(a). For $l > 1$, let $W = H \operatorname{wr} S_l$, and take W to act on $\Omega = \Gamma^l$ in its natural product action. Then for $\gamma \in \Gamma$ and $\alpha = (\gamma, \ldots, \gamma) \in \Omega$ we have $W_\alpha = H_\gamma \operatorname{wr} S_l$, and $n = |\Gamma|^l$. If K is the socle of H then the socle B of W is K^l, and $B_\alpha = (K_\gamma)^l \neq 1$.

Now W acts naturally on the l factors in K^l, and we say that the subgroup X of W is of type III(b) if $B \leq X$ and X acts transitively on these l factors.

Finally, one of the following holds:

(i) H is of type II, $K \cong T$, $k = l$ and B is the unique minimal normal subgroup of X,

(ii) H is of type III(a), $K \cong T^{k/l}$ and X and H both have m minimal normal subgroups, where $m \leq 2$; if $m = 2$ then each of the two minimal normal subgroups of X is regular on Ω.

III(c). *Twisted wreath action.* Here X is a twisted wreath product $T \operatorname{twr}_\varphi P$, defined as follows. (The original construction is due to B. H. Neumann [7]; here we follow [10, page 269].) Let P be a transitive permutation group on $\{1, \ldots, k\}$ and let Q be the stabilizer P_1. We suppose that there is a homomorphism $\varphi: Q \to \operatorname{Aut} T$ such that $\operatorname{Im} \varphi$ contains $\operatorname{Inn} T$. Define

$$B = \{f: P \to T \mid f(pq) = f(p)^{\varphi(q)} \text{ for all } p \in P,\ q \in Q\}.$$

Then B is a group under pointwise multiplication, and $B \cong T^k$. Let P act on B by

$$f^p(x) = f(px) \quad \text{for } p, x \in P.$$

We define $X = T \operatorname{twr}_\varphi P$ to be the semidirect product of B by P with this action, and define an action of X on Ω by setting $X_\alpha = P$. We then have $n = |T|^k$, and B is the unique minimal normal subgroup of X and acts regularly on Ω.

We say that the group X is of type III(c) if it is primitive on Ω. (Note that the primitivity of X in the above construction depends on some quite complicated conditions on P which we do not investigate here.)

REMARKS. 1. The classes I,..., III(c) are pairwise disjoint; this is clear from the differing structures and actions of the socles B on Ω.

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2. Although III(b) is the only case where X is described as a subgroup of a group with a product action, some of the groups of other types are subgroups of a wreath product $S_a \operatorname{wr} S_b$ with product action on Ω (and $n = a^b$, $a > 1$, $b > 1$); these are

(i) groups of type I where X_α is imprimitive as a linear group, b divides k and $a = p^{k/b}$,

(ii) groups of type III(c), with $a = |T|$ and $b = k$; in this case X is contained in the wreath product $H \operatorname{wr} S_k$, where $H = T \times T$ is of type III(a); note that here the socle of $H \operatorname{wr} S_k$ is isomorphic to $B \times B$.

3. Although groups of type III(c) are the only groups described as twisted wreath products, various primitive groups of types I, III(a) or III(b) may also be nontrivial twisted wreath products as abstract groups. The distinguishing feature of III(c) is the existence of a unique nonabelian regular normal subgroup.

4. A full discussion of the permutation isomorphism classes of type III(a) groups is contained in [4].

2. The theorem and its proof

THEOREM. *Any finite primitive permutation group is permutation equivalent to one of the types* I, II, III(a), III(b) *and* III(c) *described in Section* 1.

PROOF. Let G be a primitive permutation group on a finite set Ω of size n, let $\alpha \in \Omega$, and let $M = \operatorname{soc} G$, the socle of G.

Let J be a minimal normal subgroup of G. Then J is transitive on Ω. The centralizer $C_G(J)$ is also a normal subgroup of G. If $C_G(J) \neq 1$ then $C_G(J)$ is transitive on Ω, whence J and $C_G(J)$ are both regular on Ω; and J and $C_G(J)$ are equal if and only if J is abelian. Here J and $C_G(J)$ are minimal normal subgroups of G and there are no further minimal normal subgroups as such subgroups would centralize J. Moreover J and $C_G(J)$ are isomorphic as they are right and left regular representations of the same group. If on the other hand $C_G(J) = 1$ then J is the unique minimal normal subgroup of G. Thus in either case $M = JC_G(J) = T_1 \times \cdots \times T_k$ with $k \geq 1$ and $T_i \simeq T$ for each i, where T is a simple group.

If M is abelian then G is of type I, so assume that M is nonabelian. If $k = 1$ then G is of type II; the fact that $M_\alpha \neq 1$ here will be shown at the end of the proof. Assume then that $k \geq 2$. In this case G permutes the set $\{T_1, \ldots, T_k\}$ and since G_α is maximal in G,

(1) $\qquad M_\alpha$ is a maximal proper G_α-invariant subgroup of M.

For $1 \leq i \leq k$ let p_i be the projection of M onto T_i.

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Case 1. First suppose that $p_i(M_\alpha) = T_i$ for some i. Then it follows from (1) that $p_j(M_\alpha) = T_j$ for all $j = 1, \dots, k$ and so M_α is a direct product $D_1 \times \cdots \times D_l$ of full diagonal subgroups D_i of subproducts $\prod_{j \in I_i} T_j$ where the I_i partition $\{1, \dots, k\}$.

Choose notation so that $I_1 = \{1, \dots, m\}$ (so $m \geq 2$). By (1) G_α is transitive on $\{D_1, \dots, D_l\}$ and hence each D_i involves precisely m of the factors T_i, so $k = lm$. Let P be the permutation group induced by G on the set $\mathcal{T} = \{T_1, \dots, T_k\}$.

Assume first that $l = 1$. If P preserved a nontrivial partition of $\mathcal{T}$ then the subgroup Y of all elements of M constant on each block of the partition would be G_α-invariant with $M_\alpha < Y < M$, contradicting (1). Thus P leaves invariant no nontrivial partition of $\mathcal{T}$, and so either P is primitive on $\mathcal{T}$, or $P = 1$, $k = 2$ and G has two minimal normal subgroups. In either case we show that G is of type III(a) as follows. First we claim that up to permutation equivalence we can identify M with $(\operatorname{Inn} T)^k$ so that

$$M_\alpha = D_1 = \{(i, \dots, i) | i \in \operatorname{Inn} T\}.$$

For let $M = (\operatorname{Inn} T)^k$ and let $D = \{(i, \dots, i) | i \in \operatorname{Inn} T\} \leq M$. Let E be another diagonal subgroup of M; thus

$$E = \{(i^{\varphi_1}, \dots, i^{\varphi_{k-1}}, i) | i \in \operatorname{Inn} T\}$$

for some $\varphi_1, \dots, \varphi_{k-1} \in \operatorname{Aut} T$. Define $\varphi\colon (M : D) \to (M : E)$ (where $(M : L)$ denotes the set of right cosets of a subgroup L in M), and $\psi \in \operatorname{Aut} M$ by

$$\varphi\colon D(i_1, \dots, i_{k-1}, 1) \mapsto E(i_1^{\varphi_1}, \dots, i_{k-1}^{\varphi_{k-1}}, 1),$$
$$\psi\colon (i_1, \dots, i_k) \mapsto (i_1^{\varphi_1}, \dots, i_{k-1}^{\varphi_{k-1}}, i_k),$$

where $i_j \in \operatorname{Inn} T$ for $1 \leq j \leq k$. Then for $\omega \in (M : D)$ and $m \in M$, we have $(\omega m)\varphi = (\omega\varphi)(m\psi)$. Thus the actions of M on $(M : D)$ and on $(M : E)$ are permutation equivalent, as claimed. Now the full normalizer of M in $\operatorname{Sym}(\Omega)$ is $M.(\operatorname{Out} T \times S_k)$, and hence G^Ω is permutation equivalent to a subgroup of W^Ω, where W is as described in III(a).

Now let $l > 1$ and set $K = T_1 \times \cdots \times T_m$ and $N = N_G(K)$. It follows from (1) that D_1 is a maximal N_α-invariant subgroup of K. For $L \leq N$ denote by L^* the group of automorphisms of K induced by L by conjugation, so that $L^* = LC_G(K)/C_G(K)$. Since N contains M, we see that N is transitive on Ω and so $N = MN_\alpha$. Hence $N^* = K^* N_\alpha^*$. Let Y be a maximal subgroup of N containing $N_\alpha C_G(K)$. Then $Y \cap K$ is an N_α-invariant subgroup of K containing D_1, and by maximality of D_1 we have $Y \cap K = D_1$. Thus $Y \cap M = D_1 \times T_{m+1} \times \cdots \times T_k$. Also $Y = (Y \cap M)N_\alpha$ so that $Y^* = D_1^* N_\alpha^* = N_\alpha^*$, and hence $Y = N_\alpha C_G(K)$, that is, $N_\alpha C_G(K)$ is a maximal subgroup of N. Set $H = N^*$ and let Γ be the coset space $(H : N_\alpha^*)$. Then H has socle $K^* \simeq K$ and H is a primitive permutation group on Γ of type III(a). Also $|\Omega| = |\Gamma|^l$.

We now claim that G^{Ω} is permutation equivalent to a subgroup of $H \operatorname{wr} S_l$ in its natural product action on Γ^l, hence to a group of type III(b)(ii) (the transitivity of G on the l factors in K^l follows from primitivity). To see this, let $\mathcal{R} = \{g_1, \ldots, g_l\}$ be a right transversal for N_α in G_α and for N in G, such that $D_1^{g_i} = D_i$ for $1 \le i \le l$. Write $K_i = K^{g_i}$ $(1 \le i \le l)$, so that G permutes the set $\{K_1, \ldots, K_l\}$. For $g \in G$, write $g = n_g \overline{g}$ with $\overline{g} \in \mathcal{R}$ and $n_g \in N$. Writing elements of $H \operatorname{wr} S_l$ in the form $(h_1, \ldots, h_l)\pi$ with $h_i \in H$, $\pi \in S_l$ and π permuting the components h_i naturally, we define a map $\rho: G \to H \operatorname{wr} S_l$ by

$$\rho: g \mapsto (a_1^*, \ldots, a_l^*)\pi \qquad (g \in G),$$

where π is the permutation induced by g on $\{K_1, \ldots, K_l\}$, for $1 \le i \le l$ we have $a_i = g_i g(\overline{g_i g})^{-1}$, and a_i^* denotes the automorphism of K induced by conjugation by a_i. Then ρ is a monomorphism; moreover, since $g_i \in G_\alpha$ for all i, we have $G_\alpha \rho \le N_\alpha^* \operatorname{wr} S_l$, the point stabilizer in the natural action of $H \operatorname{wr} S_l$ on Γ^l. Since $|\Omega| = |\Gamma|^l$, identification of G with its image $G\rho$ gives the required embedding of G in $H \operatorname{wr} S_l$, acting naturally on Γ^l. This proves our claim.

Case 2. Now suppose that $R_i = p_i(M_\alpha)$ is a proper subgroup of T_i for each $i = 1, \ldots, k$. Since each R_i is an $N_G(T_i)$-invariant subgroup of T_i, it follows from (1) that G_α is transitive on $\{T_1, \ldots, T_k\}$ and hence for $i = 1, \cdots, k$, R_i is the image of R_1 under an isomorphism $T_1 \to T_i$. Since $R_1 \times \cdots \times R_k$ is G_α-invariant we have $M_\alpha = R_1 \times \cdots \times R_k$. Also R_1 must be a maximal $N_{G_\alpha}(T_1)$-invariant proper subgroup of T_1. Set $N = N_G(T_1)$ and for $L \le N$ denote by L^* the group of automorphisms of T_1 induced by L by conjugation, so that $L^* = LC_G(T_1)/C_G(T_1)$. Since N contains the transitive subgroup M, we have $N = MN_\alpha$. Hence $N^* = T_1^* N_\alpha^*$.

Case 2(a). Suppose that $T_1^* \le N_\alpha^*$. Thus $N^* = T_1^* N_\alpha^* = N_\alpha^*$. If $R_1 \ne 1$ then

$$T_1 = \langle R_1^{T_1} \rangle \le \langle R_1^{C_G(T_1)N_\alpha} \rangle = \langle R_1^{N_\alpha} \rangle \le G_\alpha$$

which is not so. Hence $R_1 = 1$ and so $M_\alpha = 1$. Define $\varphi: N_\alpha \to \operatorname{Aut} T_1$ to be the natural homomorphism (that is, for $n \in N_\alpha$ and $t \in T_1$, $\varphi(n): t \mapsto t^n$), so that $\ker \varphi = C_G(T_1) \cap G_\alpha$ and $\operatorname{Im} \varphi = N_\alpha^*$ contains $\operatorname{Inn} T_1 = T_1^*$. Write $Z = \varphi^{-1}(\operatorname{Inn} T_1)$. Also let Y be the kernel of the action of G on $\{T_1, \ldots, T_k\}$.

We show first that $Y = M$. Now $Y_\alpha \simeq Y_\alpha M/M$ is isomorphic to a subgroup of $(\operatorname{Out} T_1)^k$ and hence is soluble by the Schreier "Conjecture". Also $Z/C \simeq T_1$ is simple, where we write $C = \ker \varphi$. Since $Y_\alpha C$ and Z are both normal subgroups of N_α we therefore have $[Z/C, Y_\alpha C/C] = 1$ in N_α/C. Thus $Y_\alpha \le C$, that is, Y_α centralizes T_1. Similarly Y_α centralizes T_i for all i and hence $Y_\alpha = 1$. Thus $Y = M$.

Set $P = G_\alpha$ and $Q = N_\alpha$ so that P acts faithfully and transitively on $\{T_1, \ldots, T_k\}$ and $G = MP$. Abusing notation slightly, take P to act on $I =$

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$\{1,\ldots,k\}$ by $T_i^p = T_{ip}$ ($i \in I$, $p \in P$). We show finally that there is an isomorphism of G onto the twisted wreath product $T_1 \operatorname{twr}_\varphi(P^I)$ (defined in Section 1) which maps M onto the base group B and $G_\alpha = P$ onto the top group P^I, so that G is of type III(c). For $1 \le i \le k$ choose $c_i \in P$ such that $T_i^{c_i} = T_1$ (so that $\{c_1,\ldots,c_k\}$ is a transversal for Q in P). Now each $m \in M$ is of the form $m = \prod_{i=1}^k a_i$ with $a_i \in T_i$ and hence $a_i^{c_i} \in T_1$ for $1 \le i \le k$. We define a map ϑ from G into $T_1 \operatorname{twr}_\varphi P$ by

$$\vartheta\colon mu \mapsto \vartheta_m u$$

for $m = \prod a_i \in M$ and $u \in P$, where $\vartheta_m\colon P \to T_1$ is the map given by $\vartheta_m(c_i q) = a_i^{c_i q}$ for $1 \le i \le k$ and $q \in Q$. Clearly ϑ_m belongs to the base group B of $T_1 \operatorname{twr}_\varphi P^I$, and ϑ is 1-1 and hence bijective. To see that ϑ is a homomorphism we need to show that

$$\vartheta_{m^u} = (\vartheta_m)^u \quad \text{for } m = \prod a_i \in M,\ u \in P.$$

Write $b = \vartheta_m$. By the definition of b^u in Section 1, $b^u(c_i q) = b(uc_i q)$, which equals $b(c_{iu^{-1}} y_i q)$ where $y_i = c_{iu^{-1}}^{-1} u c_i \in Q$. Since $b \in B$, this equals $b(c_{iu^{-1}})^{y_i q}$, and therefore

$$b^u(c_i q) = a_{iu^{-1}}^{uc_i q}.$$

But this is clearly the same as $\vartheta_{m^u}(c_i q)$, and hence $\vartheta_{m^u} = (\vartheta_m)^u$, as required. Thus ϑ is an isomorphism, and since $\vartheta(M) = B$ and $\vartheta(G_\alpha) = P$ it follows that G is of type III(c).

Case 2(b). Thus we may assume that $T_1^* \not\le N_\alpha^*$. If Y is a maximal subgroup of N containing $N_\alpha C_G(T_1)$ then $Y \cap T_1$ is an N_α-invariant subgroup of T_1 containing R_1. By the maximality of R_1 and since $Y \ne N$ we have $Y \cap T_1 = R_1$. Thus $Y \cap M = R_1 \times T_2 \times \cdots \times T_k$. Also $Y = (Y \cap M)N_\alpha$ so that $Y^* = R_1^* N_\alpha^* = N_\alpha^*$, and hence $Y = N_\alpha C_G(T_1)$, that is, $N_\alpha C_G(T_1)$ is a maximal subgroup of N. Let $H = N^*$ and let Γ be the coset space $(H : N_\alpha^*)$. Then H has socle $T_1^* \simeq T_1$ and H is primitive on Γ. Also $|\Gamma| = |T_1 : R_1|$ and so $|\Omega| = |\Gamma|^k$. A calculation along the lines of the case $l > 1$ of Case 1 shows that G^Ω is permutation equivalent to a subgroup of $H \operatorname{wr} S_k$ in its product action on Γ^k. Then G is of type III(b)(i); for this it remains to show that $M_\alpha \ne 1$. This will follow from the corresponding assertion in the simple socle case II.

Thus suppose that $T \lhd G \le \operatorname{Aut} T$ and $T_\alpha = 1$. Then G_α is soluble by the Schreier "Conjecture". Let Q be a minimal normal subgroup of G_α. Then Q is an elementary abelian q-group for some prime q. Now $C_T(Q) = 1$, since both T and $C_G(Q)$ are normalized by G_α, and G_α is maximal in G. It follows that q does not divide $|T|$. Hence Q normalizes a Sylow 2-subgroup S of T. We assert that S is the unique such Sylow 2-subgroup. For suppose that Q normalizes S_1, where $S_1^x = S$ and $x \in T$. Then Q and Q^x are Sylow q-subgroups of $N_{TQ}(S)$, so

$Q = Q^{xy}$ for some $y \in N_T(S)$. We have $[Q, xy] \leq Q \cap T = 1$, so $xy \in C_T(Q) = 1$. Hence $x \in N_T(S)$ and so $S_1 = S$, as asserted. Thus $G_\alpha = N_G(Q) \leq N_G(S)$ and so $G_\alpha < G_\alpha S < G$, contradicting the maximality of G_α.

This completes the proof of the theorem.

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