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On the oblique water-entry problem of a rigid sphere

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Abstract. The case of oblique water-entry of a rigid sphere into an ideal incompressible fluid is studied analytically in order to determine the hydrodynamical loads acting on the body. We consider the motion imparted to the fluid by an impulsively-started partially-submerged sphere under the large-impact approximation, in which the free surface is assumed flat and equipotential. Asymptotic small-time expressions are derived for both the vertical and horizontal time-dependent added masses and analytical expressions for the hydrodynamic forces are obtained by differentiating these added masses with respect to the instantaneous submergence depth. The resulting expressions are also compared with corresponding numerical solutions and with a known solution for a two-dimensional profile.

1. Introduction

The problem of the initial-stage impact between a solid body and an ideal incompressible fluid, initially at rest, has become recently a subject of great interest (see the comprehensive review by Korobkin and Pukhnachov [10]). Some related problems, for example, are: the impulsively-started two-dimensional wave maker (Roberts [23], Joo et al. [7]) and vertical circular cylinder (Wang and Chwang [33]), a two-dimensional cylinder in a current (Grosenbaugh and Yeung [4]), the oblique water entry of a 2-D profile (Korobkin [9]) and the vertical water-entry of a horizontal circular cylinder (Greenhow [3]) or a spherical projectile (Miloh [17]).

Most of the theoretical studies on the time-dependent impact problem have been limited to two-dimensional symmetrical shapes in normal entry where the powerful tool of complex variables may be employed. The case of an inclined water impact of a symmetric two-dimensional profile has been only recently discussed (Korobkin [9]). The few available three-dimensional examples are restricted only to vertical entries and to simple shapes such as cones, spheres or ellipsoids (Korobkin [8], Miloh [15]).

The purpose of this paper is to extend the analytical solution, of the vertical water-entry problem of a sphere (Miloh [15, 17]), to the case of an oblique entry. In this sense the paper may also be considered as a three-dimensional generalization of Korobkin's [9] inclined entry analysis of a two-dimensional profile. Here, because of the asymmetry of the pressure field on the wetted body surface, there exists a horizontal drag force in addition to the vertical (slamming) force. Such a force may give rise to a considerable pitching (whipping) moment about the center of gravity of an axisymmetric elongated projectile during early stages of water impact. It is well known that the hydrodynamical loads on a blunt rigid body penetrating a free surface reach a maximum value at rather small values of the dimensionless time, where the time scale is defined as the ratio between the radius of body curvature and its velocity. The initial whipping moment acting on the projectile, which results from the horizontal drag force, usually affects the trajectory of the body and may cause, in some circumstances, phenomena such as broaching or ricocheting off the free surface. On the other hand, most available numerical algorithms which have been developed for computing impact loads on water-entry bodies are inappropriate and tend to be inaccurate for very small body penetrations. Thus, it is useful to obtain, when possible, asymptotic small-time expressions for the initial impact loads, against which existing numerical codes can be checked. Such an expression is derived in this paper for the case of an oblique water-entry of a spherical-like shape.

The analytical treatment of the inclined entry case of a rigid sphere is different in some respects from the related axisymmetric vertical-entry problem, given in Miloh [15]. In both cases it was found convenient to formulate the problem in terms of toroidal harmonics and to employ the Kirchhoff–Lagrange energy method rather than using a momentum approach, in order to calculate the hydrodynamical loads. However, the advantage of employing the Stokes stream function in the vertical axisymmetric case, which fortunately renders a closed-form solution, does not prevail in the present oblique case and for this reason the final solution is obtained in terms of an integral equation for the velocity potential.

The general water-entry problem is formulated in Sec. 1, where we consider a partially submerged sphere in a quiescent fluid and examine the resulting flow due to an impulsively-started motion of the sphere. Of particular interest is the small-submergence case which may serve as an analytical framework model for treating the initial stage of water impact of arbitrary 3-D bluff shapes. The problem is postulated in terms of a small-time power-series expansion, in the manner of Peregrine [21], which provides a systematic procedure for obtaining consistent higher-order solutions. Only the first-order solution, in which the free surface is approximated by an equipotential horizontal surface (large-impact case) is treated here. A general method is proposed to calculate the hydrodynamical loads in terms of the time-dependent vertical and horizontal added-masses and their derivatives with respect to the instantaneous submergence depth (slamming coefficients). The resulting Fredholm integral equation of the first kind, whose solution determines the horizontal (sway) motion, is derived in Sec. 3 and some useful small-time approximations of this integral equation are presented in Sec. 4. The small-time analytic solution is compared with the full numerical solution of the corresponding problem in Sec. 5. Also discussed in this final section, is the analogy between the present 3-D asymptotic expressions and Korobkin's [9] corresponding solution for the inclined entry of two-dimensional profiles.

2. General formulation

We consider a rigid sphere of radius R entering obliquely a half-space filled with an ideal incompressible heavy fluid initially at rest. Let b denote the instantaneous penetration depth of the sphere below the undisturbed free surface and the constant vertical and horizontal velocities of the body centroid are denoted here by V and U, respectively. For large-impact velocities, viscous and surface-tension effcts may be ignored with respect to inertia (see discussion in Korobkin and Pukhnachov [10]) and the fluid may be assumed to be irrotational. The velocity field exterior to the body and the induced pressure distribution on its wetted surface may be described in terms of a velocity potential $\Phi(x, y, z, t)$, where (x, y, z) is a Cartesian coordinate system with origin at the undisturbed free surface, with z pointing upward in the direction opposite to gravity. Using R and $W = (U^2 + V^2)^{1/2}$ as the reference length and velocity, the non-dimensional formulation of the nonlinear boundary-value problem is

$$
\begin{aligned}
&\nabla^2\phi = 0 \quad \text{in } \mathcal{V}\,, && \\
&\phi_t + \tfrac{1}{2}\nabla\phi\cdot\nabla\phi + F^{-2}\eta = 0 && \text{on } z=\eta(x,y,t)\,, \\
&\eta_t = \phi_z + \phi_x\eta_x + \phi_y\eta_y && \text{on } z=\eta(x,y,t)\,, \\
&\frac{\partial\phi}{\partial n} = \cos\delta\,\frac{\partial x}{\partial n} + \sin\delta\,\frac{\partial z}{\partial n} && \text{on } S\,, \\
&|\nabla\phi| \to 0 && \text{as } x^2+y^2+z^2 \to \infty\,,
\end{aligned}
\tag{1}
$$

where $z = \eta(x, y, t)$ denotes the free-surface elevation, $\delta = \tan^{-1}(U/V)$ is the incidence angle between the total velocity vector and the undisturbed free surface, $F = W/(gR)^{1/2}$ is the Froude number, $\mathcal{V}$ denotes the fluid domain and S represents the submerged spherical surface.

We now seek a solution of the nonlinear problem given in (1) which is valid at early stages of impact. Towards this goal we formulate an initial boundary-value problem by postulating an asymptotic small-time power-series expansion for both the potential and the free-surface elevation (Peregrine [21], Chwang [1], Roberts [23]), i.e.

$$
\begin{aligned}
&\phi = \phi_0(x, y, z; b) + t\phi_1(x, y, z; b) + t^2\phi_2(x, y, z; b) + \cdots\,, \\
&\eta = t\eta_1(x, y; b) + t^2\eta_2(x, y; b) + \cdots\,.
\end{aligned}
\tag{2}
$$

Here ϕ_0 denotes the velocity potential induced by instantaneously introducing a moving rigid sphere (submerged to depth b) into a quiescent fluid.

The leading-order solution is then found by substituting (2) into (1) and letting $t \to 0$, which gives

$$
\begin{aligned}
&\phi_0 = \cos\delta\,\phi_0^{(1)} + \sin\delta\,\phi_0^{(2)}\,, \\
&\nabla^2\phi_0^{(i)} = 0 \quad \text{in } \mathcal{V}\,, \quad i=1,2\,, \\
&\phi_0^{(i)} = 0 \quad \text{on } z=0\,, \quad i=1,2\,, \\
&\frac{\partial\phi_0^{(1)}}{\partial n} = \frac{\partial x}{\partial n}\,, \qquad \frac{\partial\phi_0^{(2)}}{\partial n} = \frac{\partial z}{\partial n} \quad \text{on } S \\
&|\nabla\phi_0^{(i)}| \to 0 \quad \text{as } x^2+y^2+z^2\to\infty\,, \quad i=1,2\,.
\end{aligned}
\tag{3}
$$

The next-order solution, valid for large Froude numbers, is given by

$$
\begin{aligned}
&\nabla^2\phi_1 = 0 && \text{in } \mathcal{V}\,, \\
&\phi_1 = -\frac{1}{2}\left(\frac{\partial\phi_0}{\partial y}\right)^2 && \text{on } z=0\,, \\
&\eta_1 = \frac{\partial\phi_0}{\partial y} && \text{on } z=0\,, \\
&\frac{\partial\phi_1}{\partial n} = 0 && \text{on } S\,, \\
&|\nabla\phi_1| \to 0 && \text{as } x^2+y^2+z^2\to\infty\,,
\end{aligned}
\tag{4}
$$

and higher-order solutions of (1) may be obtained in a similar way. Following such an asymptotic solution, the leading gravity-dependent term will enter into the calculation only in the expression for the third-order velocity potential term. For a comprehensive discussion on the range of validity of the small-time expansion and its rate of convergence, the interested reader is referred to Tyvand [29].

The boundary-value problem defined in (3) is identical with the classical formulations of the high-impact water-entry problem (Von Karman [31], Wagner [32], Sedov [25], Trilling [27]). In deriving (3), it is argued in these works that the impact occurs during a very short time interval, and may therfore be considered as an impluse. It is also known that in cases of impulsive motion, the dynamic pressure is proportional to the velocity potential and for this reason the free surface may be considered as equipotential surface. The so-called splash contour, or the wetting correction, here denoted by η_1, may then be determined from the second-order solution (4) in terms of ϕ_0 (see for example the axisymmetric solution of Miloh [15]).

It should be noted that applying the more rigorous Lagrangian approach for treating the initial-stage water-entry problem (Korobkin and Pukhnachov [10]), leads to the same first-order problem as formulated in (3), from which analytic expressions for the small-time slamming coefficient may actually be derived. One way of deriving the hydrodynamic loads is by carrying out a pressure integration over the submerged part of the body (Pukhnachov and Korobkin [22]). However, a more direct approach for calculating the hydrodynamic forces in the present case is by using the generalized Kirchhoff–Lagrange equations (Miloh and Landweber [18]), which express these forces in terms of the fluid kinetic energy.

Thus, within the realm of the first-order problem, where the free surface may be replaced by a flat equipotential surface (3), we note that the corresponding dynamical system, comprising of body and fluid, is uniquely determined by specifying the initial submergence depth b and the velocities U, V (which may be also considered as generalized single coordinate and two generalized velocities). For the constant-velocity oblique water-entry case, we may therefore express the vertical and horizontal forces as (see also Lamb [11], pp. 187–192, and Miloh [16])

$$F_x = -\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial U}\right), \qquad F_z = -\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial V}\right) + \frac{\partial T}{\partial b}, \tag{5}$$

which may also be written in dimensionless form as

$$\begin{aligned} f_x &= \frac{F_x}{\pi\rho R^2 W^2} = -\frac{1}{2}\,\frac{\mathrm{d}C_1}{\mathrm{d}(b/R)}\sin 2\delta\,, \\ f_z &= \frac{F_z}{\pi\rho R^2 W^2} = \frac{1}{2}\,\frac{\mathrm{d}C_1}{\mathrm{d}(b/R)}\cos^2\delta - \frac{1}{2}\,\frac{\mathrm{d}C_2}{\mathrm{d}(b/R)}\sin^2\delta\,. \end{aligned} \tag{6}$$

Here δ denotes the velocity angle of incidence ($\delta = \pi/2$ for the vertical entry), and T is the kinetic energy

$$T = \tfrac{1}{2}\pi\rho R^3(C_1(b)U^2 + C_2(b)V^2)\,, \qquad W^2 = U^2 + V^2\,, \tag{7}$$

given in terms of the horizontal and vertical added masses,

$$C_1 = \frac{1}{\pi R^3} \int_S \phi_0^{(1)} \frac{\partial x}{\partial n} \, \mathrm{d}S \,, \qquad C_2 = \frac{1}{\pi R^3} \int_S \phi_0^{(2)} \frac{\partial z}{\partial n} \, \mathrm{d}S \,. \tag{8}$$

Thus, in order to determine the hydrodynamical loads acting on the penetrating sphere, it is necessary to compute the derivatives of both the vertical and horizontal added masses with respect to submergence depth b in the limit as $b \to 0$. Employing numerical techniques for the evaluation of this limit is not an easy task and for this reason asymptotic analytical expressions may be found to be very useful. An asymptotic small-time expansion for the vertical slamming coefficient $\mathrm{d}C_2/\mathrm{d}(b/R)$ has been recently derived in Miloh [17], and here we complete the solution for the oblique water-entry problem by providing the necessary small-time asymptotic expansion for the horizontal slamming coefficient $\mathrm{d}C_1/\mathrm{d}(b/R)$.

3. Solutions in terms of toroidal harmonics

In order to solve the boundary-value problem formulated in (3), it was found convenient to adopt a triply-orthogonal toroidal coordinate system (η, θ, ψ), which is related to the Cartesian system (x, y, z) by

$$z = \frac{a \sin \theta}{\cosh \eta - \cos \theta} \,, \qquad x + \mathrm{i}y = \frac{a \sinh \eta}{\cosh \eta - \cos \theta} \, \mathrm{e}^{\mathrm{i}\psi} \,, \tag{9}$$

where a is a characteristic parameter which is equal to the radius of the contour of intersection of the sphere and the free surface. We also note, following Moon and Spencer [20], that $\eta = \text{const}\ (0 \leq \eta \leq \infty)$ denote toroidal surfaces, $\theta = \text{const}\ (-\pi \leq \theta \leq \pi)$ are spherical bowls and $\psi = \text{const}\ (0 < \psi < 2\pi)$ represent half-planes. Thus, the submerged portion of the sphere is given by $\theta = \theta_0 = \text{const}$, where

$$a = R \sin \theta_0 \,, \qquad b = 2R \cos^2 \frac{\theta_0}{2} \,, \tag{10}$$

with b again denoting the instantaneous vertical submergence-depth below the undisturbed free surface.

It is known that a general solution of the Laplace equation may be expressed in terms of exterior toroidal harmonics in a form which is only partially separable. Hence, following Sneddon [26], an arbitrary exterior potential function which decays at infinity, may be represented as

$$\phi(\eta, \theta, \psi) = (\cosh \eta - \cos \theta)^{1/2} \, \mathrm{e}^{\mathrm{i}m\psi} \int_0^\infty [\alpha_m(p) \cosh p\theta + \beta_m(p) \sinh p\theta] K_p^m(\cosh \eta) \, \mathrm{d}p \,, \tag{11}$$

where α_m and β_m are some complex-valued unknown coefficients and K_p^m denotes the Legendre polynomial of the first kind of order $-\frac{1}{2} + \mathrm{i}p$ and degree m, i.e.

$$K_p^m(\cosh \eta) = P^m_{-1/2+\mathrm{i}p}(\cosh \eta) \,. \tag{12}$$

Thus, since the surface $z = 0$ is being represented now by $\theta = 0$, equations (3) and (11) suggest that

$$\phi_0^{(1)}(\eta, \theta, \psi; \theta_0) = a \cos \psi (\cosh \eta - \cos \theta)^{1/2} \int_0^\infty A(p; \theta_0) K_p^1(\cosh \eta) \frac{\sinh p\theta}{\cosh p\theta_0} \, \mathrm{d}p \tag{13}$$

represents the horizontal velocity potential with $K_p^1(\cosh \eta) = (\mathrm{d}/\mathrm{d}\eta) K_p(\cosh \eta)$, and that the vertical potential is similarly given by

$$\phi_0^{(2)}(\eta, \theta, \psi; \theta_0) = a(\cosh \eta - \cos \theta)^{1/2} \int_0^\infty B(p; \theta_0) K_p(\cosh \eta) \frac{\sinh p\theta}{\cosh p\theta_0} \, \mathrm{d}p \,, \tag{14}$$

where $A(p)$ and $B(p)$ are some unknown real coefficients. The particular representation given in (13) and (14) automatically satisfies the field equation, the linearized free-surface condition and renders the proper decay at infinity. The only boundary condition not yet satisfied is the no-flow condition across the wetted part of the body, which, when applied, provides a closure integral equation for determining these coefficients.

Substituting (9) and (13) into (3) and recalling that the normal derivative on S is proportional to a partial derivative with respect to θ evaluated at $\theta = \theta_0$, yields the following integral relationship

$$\int_0^\infty pA(p; \theta_0) K_p^1(\cosh \eta) \, \mathrm{d}p + \frac{1}{2} \frac{\sin \theta_0}{\cosh \eta - \cos \theta_0} \int_0^\infty A(p; \theta_0) K_p^1(\cosh \eta) \tanh p\theta_0 \, \mathrm{d}p$$
$$= -\frac{\sinh \eta \sin \theta_0}{(\cosh \eta - \cos \theta_0)^{5/2}} \,. \tag{15}$$

Next, we employ a particular form of the Fourier–Mehler integral transform (Robin [24], Sneddon [26])

$$f(\cosh \eta) = \int_0^\infty p \tanh(p\pi) K_p(\cosh \eta) \, \mathrm{d}p \int_0^\infty K_p(\cosh \eta') f(\cosh \eta') \sinh \eta' \, \mathrm{d}\eta' \tag{16}$$

together with the relationship (Miloh [15])

$$(\cosh \eta - \cos \theta)^{-1/2} = \sqrt{2} \int_0^\infty \operatorname{sech}(p\pi) \cosh[p(\pi - \theta)] K_p(\cosh \eta) \, \mathrm{d}p \,, \tag{17}$$

and substitute both (16) and (17) into (15). Making use of the orthogonality properties of the toroidal function K_p, finally leads to the following Fredholm integral equation of the second kind for $A(p)$:

$$A(p; \theta_0) + \frac{1}{2} \sin \theta_0 \tanh p\pi \int_0^\infty A(q; \theta_0) I(p, q; \theta_0) \tanh q\theta_0 \, \mathrm{d}q = \frac{4\sqrt{2}}{3} \frac{\sinh p(\pi - \theta_0)}{\cosh p\pi} \,, \tag{18}$$

where

$$I(p, q; \theta_0) = \int_0^\infty \frac{K_p(\cosh \eta') K_q(\cosh \eta')}{\cosh \eta' - \cos \theta_0} \sinh \eta' \, \mathrm{d}\eta' \,. \tag{19}$$

A similar integral equation may also be obtained for $B(p)$, representing the axisymmetric motion (14), by substituting (16), (17) and (9) into (3), which eventually gives

$$B(p;\theta_0)+\frac{1}{2}\sin\theta_0\tanh p\pi\int_0^\infty B(q;\theta_0)I(p,q;\theta_0)\tanh q\theta_0\,\mathrm{d}q$$
$$=\frac{2\sqrt{2}}{3\cosh p\pi}\left[\cot\theta_0\sinh p(\pi-\theta_0)-3p\cosh p(\pi-\theta_0)\right]. \tag{20}$$

Once $A(p)$ and $B(p)$ are determined by solving (18) and (20), the added-mass coefficients for both the horizontal and vertical motions are evaluated by substituting (13) and (14) into (8), which yields (Appendix A)

$$C_1(\theta_0)=\frac{4\sqrt{2}}{3}\sin^3\theta_0\int_0^\infty\left(p^2+\frac{1}{4}\right)A(p;\theta_0)\,\frac{\sinh p(\pi-\theta_0)}{\sinh p\pi}\tanh p\theta_0\,\mathrm{d}p\,, \tag{21}$$

and

$$C_2(\theta_0)=\frac{4\sqrt{2}}{3}\sin^3\theta_0\int_0^\infty[2p\coth p(\pi-\theta_0)-\cot\theta_0]B(p;\theta_0)\,\frac{\sinh p(\pi-\theta_0)}{\sinh p\pi}\tanh p\theta_0\,\mathrm{d}p\,. \tag{22}$$

It is important to note that there exists an alternative analytic expression for the 'heave' added mass, (22) which does not require the solution of an integral equation. Such an expression is obtained by formulating the axisymmetric vertical motion in terms of a Stokes stream function rather than a velocity potential. The resulting Dirichlet-type boundary-value problem (instead of a Neumann type), may then be solved exactly (Miloh [15]) yielding

$$C_2(\theta_0)=\frac{8}{3}\sin^3\theta_0\int_0^\infty\left(p^2+\frac{1}{4}\right)[3\tanh p\theta_0-\tanh p(\pi-\theta_0)]\,\frac{\cosh^2 p(\pi-\theta_0)}{\sinh(2p\pi)}\,\mathrm{d}p\,. \tag{23}$$

However, it is rather unfortunate that such an approach cannot be used when solving for the horizontal (sway) motion and that the computation of the corresponding added mass, using (21), involves first the solution of a Fredholm integral equation (18). A general procedure for solving such an integral equation has been recently outlined by McIver [13] in his study of the sloshing frequencies of a partially-filled spherical container. The proposed numerical scheme is rather involved and is based on the representation of the toroidal functions in terms of the Gauss hypergeometric functions (Gradshteyn and Ryzhik [2], p. 1039). These functions were evaluated by approximating them in terms of the Chebyshev polynomials, using Luke's [12] algorithm. Problems of very slow convergence were encountered for small submergence depth, even when using Shank's transformation in order to accelerate the convergence. Since our interest lies mainly in the case of small submergence, it is next demonstrated how asymptotic analytic solutions of the integral equation (18) may be obtained without actually having to solve it numerically.

4. The small-depth approximation

We consider here the initial-stage of an oblique water impact with constant velocity W and angle of incidence δ. The instantaneous vertical submergence below the undisturbed free surface, varies linearly with time t according to

$$b(t) = Wt \sin \delta \ . \tag{24}$$

Let us next define a dimensionless time τ by

$$\tau = b(t)/R = 2\cos^2 \frac{\theta_0}{2} = \frac{1}{2}\,\varepsilon^2 + O(\varepsilon^3)\ , \tag{25}$$

where ε is a small parameter defined in terms of $\theta_0(\tau)$ as

$$\varepsilon(\tau) = \pi - \theta_0(\tau)\ , \tag{26}$$

which follows from the geometrical relationship (10).

We seek an asymptotic solution of the integral equation (18), valid for small τ, in the form

$$A(p;\varepsilon) = \varepsilon A_1(p) + \varepsilon^2 A_2(p) + \cdots\ , \tag{27}$$

since just prior to first contact $A(p;0)\rightarrow 0$. Substituting (27) into (18) and collecting terms of $O(\varepsilon)$, yields

$$A_1(p) = \frac{4\sqrt{2}}{3}\,\frac{p}{\cosh p\pi}\ . \tag{28}$$

The next-order term in (27) is then obtained from (18) as

$$A_2(p) = -\frac{2\sqrt{2}}{3}\tanh p\pi \int_0^\infty \frac{q \sinh q\pi}{\cosh^2 q\pi}\, I(p,q;\pi)\, dq\ , \tag{29}$$

which, after some elaborate transformations (see Appendix B), renders

$$A_2(p) = -\frac{\sqrt{2}}{3}\,(p^2 + 1/4)\,\frac{\sinh p\pi}{\cosh^2 p\pi}\ . \tag{30}$$

In order to determine the 'sway' added mass, we substitute (28) and (30) into (27) and (21), which finally leads to (Appendix C)

$$C_1(\varepsilon) = \frac{\varepsilon^5}{\pi}\left(\frac{8}{45} - \frac{23}{216\pi}\,\varepsilon\right). \tag{31}$$

Using the relationship between ε and the dimensionless submergence τ (25), we obtain from (31) the following two-term asymptotic expansion for the sway slamming coefficient

$$\frac{dC_1(\tau)}{d\tau} = \frac{1}{\varepsilon}\,\frac{dC_1(\varepsilon)}{d\varepsilon} = \frac{1}{\pi}\left(\frac{8}{9}\,\varepsilon^3 - \frac{23}{36\pi}\,\varepsilon^4\right) = \frac{16\sqrt{2}}{9\pi}\,\tau^{3/2} - \frac{23}{9\pi^2}\,\tau^2 + O(\tau^{5/2})\ . \tag{32}$$

A similar small-time expansion may be also found for the heave slamming factor by letting $\varepsilon = \pi - \theta_0 \ll 1$ in the added-mass expression (23), and by differentiating the resulting expansion with respect to the submergence depth b, thus following Miloh [17] it is shown that

$$\frac{\mathrm{d}C_2(\tau)}{\mathrm{d}\tau} = \frac{4\sqrt{2}}{\pi}\tau^{1/2} - 1.19\tau - 1.05\tau^{3/2} + \mathrm{O}(\tau^2)\,, \tag{33}$$

implying that the vertical slamming coefficient varies for small τ like $\tau^{1/2}$, reaching a maximum of 0.47 at $\tau \sim 0.21$, in agreement with the analytical solution of Miloh [15]. The first-order asymptotic term in (33) has been also obtained by Pukhnachov and Korobkin [22] by using a Lagrangian approach and employing a pressure integration. Experimental results for the early stages of vertical water impact of a rigid sphere (Moghisi and Squire [19]) were also found to correlate very well (when introducing a wetting correction) with the theoretical asymptotic prediction (33) (Miloh [17]).The corresponding asymptotic expression for the horizontal drag (32) has not been considered before and it is given here for the first time.

5. Discussion and conclusions

The horizontal drag force experienced by a sphere during an oblique water entry, varies initially as $\tau^{3/2}$ (32), which exhibits a somewhat stronger dependence on the dimensionless depth τ than the corresponding two-dimensional case, where the horizontal force was found to vary linearly with τ (Korobkin [9]). It follows also from (6) that the presence of a horizontal component of the velocity always tends to reduce the maximum vertical slamming loads; a conclusion which has been also demonstrated numerically by Trilling [27], and experimentally by Troesch and Kang [28] for the sphere inclined-entry problem. For example, for an entry angle of 45°, the maximum vertical slamming load, which in the case of normal impact occurs at $\tau \sim 0.2$, is only 85 percent of the corresponding value for a pure vertical entry with the same vertical velocity (see Fig. 7 in Trilling [27]). We may also conclude that to $\mathrm{O}(\tau)$ and for moderate angles of incidence, the vertical slamming force is almost unaffected by the horizontal velocity and the angle of attack and may be thus obtained by solving only for the vertical entry problem. A similar conclusion has been also obtained for a two-dimensional entry but there the range of validity of the small-time expansion was found to be smaller (when compared against the present 3-D case) and is correct only to $\mathrm{O}(\tau^2)$ (Korobkin [9]). Nonetheless, the horizontal drag force acting on the sphere solely depends on the solution for the horizontal motion, and is given in terms of the derivative of the horizontal added-mass coefficient C_1 with respect to the instantaneous submergence depth.

In order to numerically determine the variation of the sway added-mass with submergence, one has to solve the integral equation (18) and substitute the results in (21). This integral equation has been solved numerically by using McIver's [13] procedure for evaluating the kernel function $I(q, p;\, \theta_0)$ for $\pi > \theta_0 > 0$, which covers the whole range of depths starting from first contact ($\tau = 0$) to full submergence ($\tau = 2$). Once $C_1(\theta_0)$ has been calculated, the horizontal slamming coefficient

$$\frac{\mathrm{d}C_1(\tau)}{\mathrm{d}\tau} = -\sin\theta_0\,\frac{\mathrm{d}C_1(\theta_0)}{\mathrm{d}\theta_0} \tag{34}$$

is determined from (25) by numerical differentiation. The analytical results thus obtained were also compared with the full numerical solution of a boundary-value problem defined in (3), obtained by distributing singularities (sources) on the wetted surface of the spherical bowl and by determining their strength by solving a Fredholm integral equation of the second kind. These two methods were found to render almost identical results (depending on the mesh size selected), but the first method of solving directly the integral equation (18) was much more efficient in the sense that it does not involve any surface integration over the body wetted part, which becomes rather problematic for small submergences.

The numerical results thus obtained for the sway added mass and for the horizontal drag coefficient, are plotted in Figs 1 and 2, respectively. Two limiting cases for which an exact solution exists may be readily confirmed. The first is the sway added-mass coefficient of a

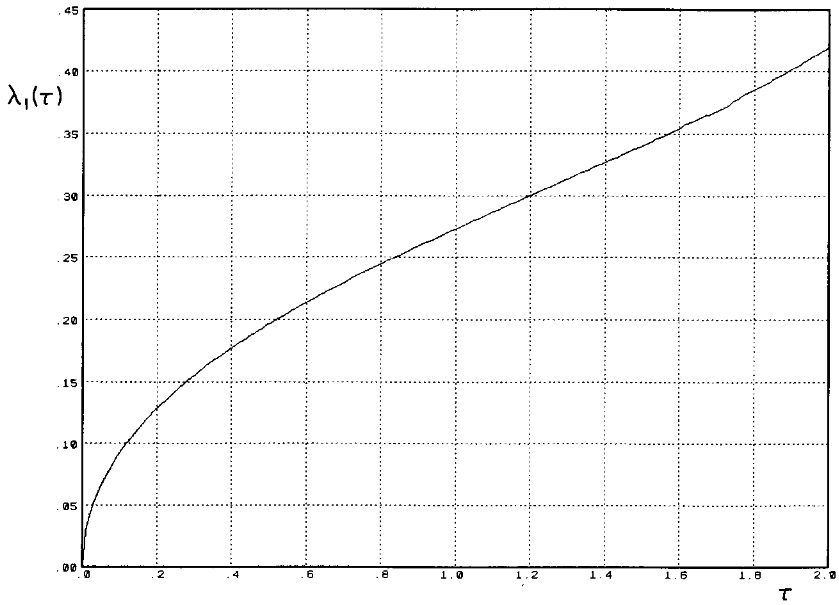


Fig. 1. The horizontal added-mass coefficient $\lambda_1(\tau)$ of a spherical bowl versus dimensionless depth τ.

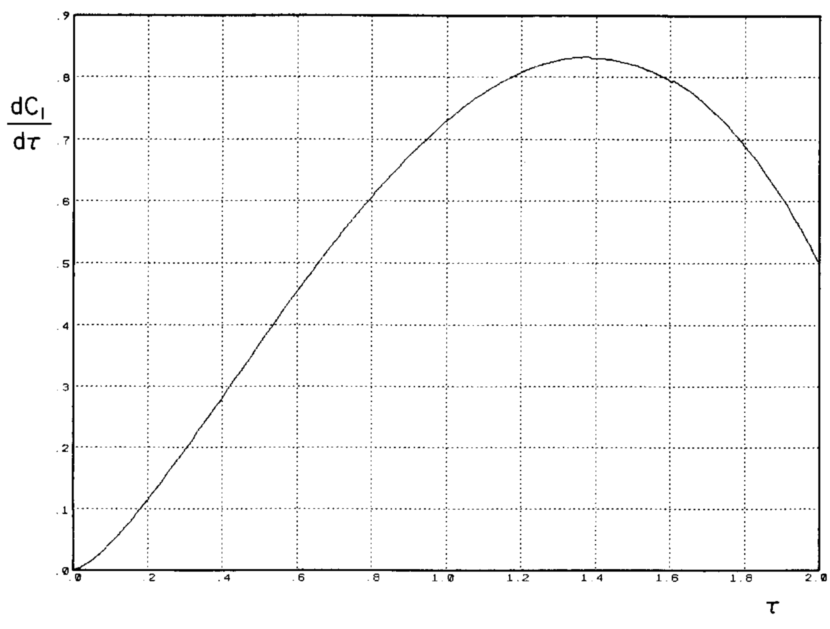
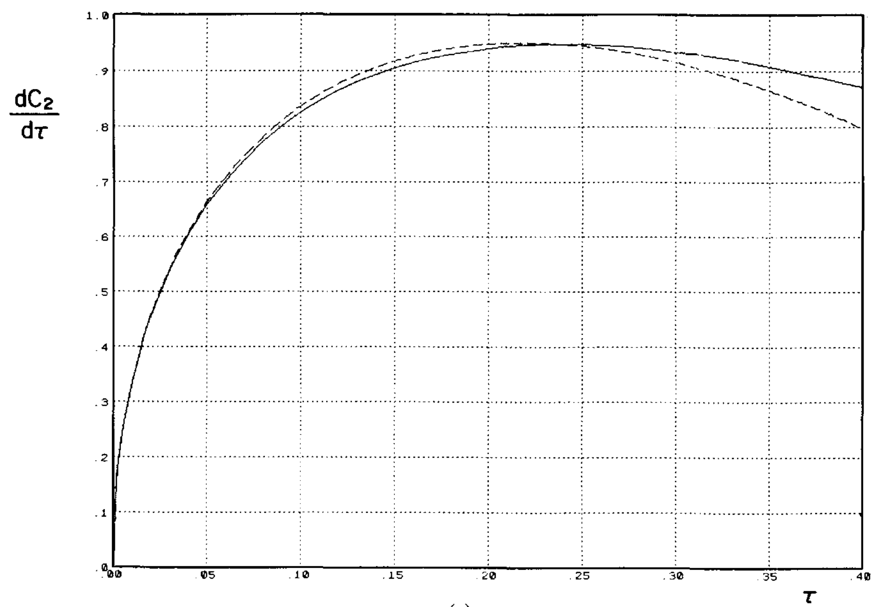
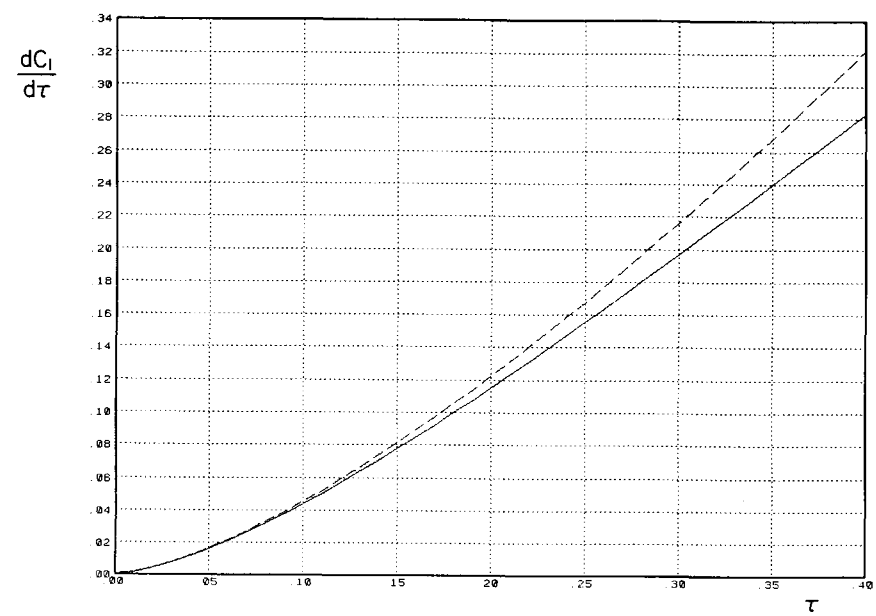


Fig. 2. The horizontal slamming coefficient $dC_1/d\tau$ versus τ.

totally submerged sphere, tangent to a horizontal equipotential surface, given by $\lambda_1(2) = 0.4198$ (Miloh [14]). The second is the infinite-frequency limit of a half-submerged swaying sphere, which following Hulme [6] is given by $\lambda_1(1) = 0.2732$. Note that the added-mass coefficients λ_i $(i = 1, 2)$ are here defined with respect to the submerged volume of the sphere, given by $\frac{1}{3}\pi R^3(2 + 3\cos\theta_0 - \cos^3\theta_0)$. Thus the relationship between λ_i and the coefficients C_i, defined in (7) and (8), is



(a)



(b)

Fig. 3. Comparison between the exact (full line) and the asymptotic (broken line) small-depth values for the slamming coefficients: a) vertical case; b) horizontal case.

$$\lambda_i(\tau) = 3(3\tau^2 - \tau^3)^{-1} C_i(\tau) \,. \tag{35}$$

The asymptotic small-time expression for the horizontal drag coefficient (32) is plotted against the exact solution in Fig. 3, and it is shown that it may serve as a reasonable approximation in the range of $\tau \leq 0.3$, with a maximum error of less than 8 percent. Also shown in the same figure is a comparison between the asymptotic expansion for the vertical

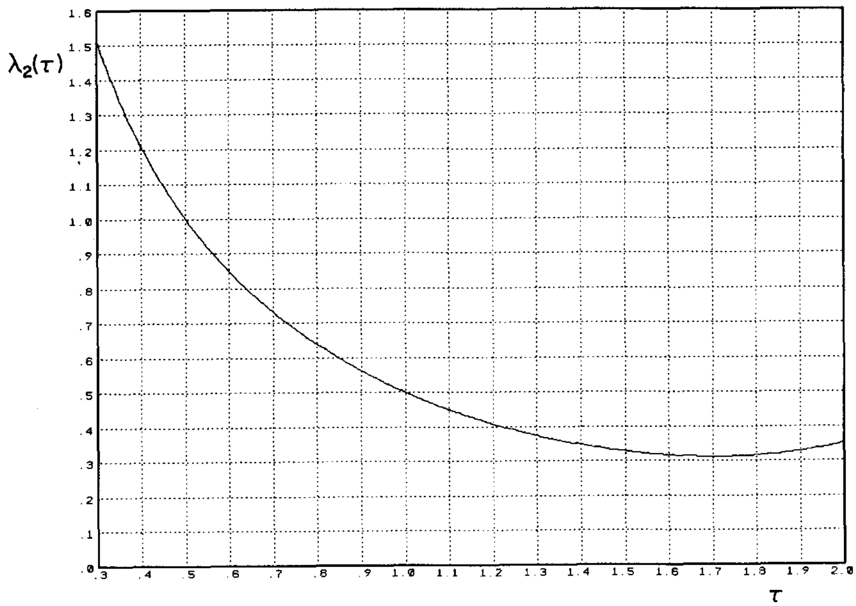


Fig. 4. The vertical added-mass coefficient $\lambda_2(\tau)$ versus τ.

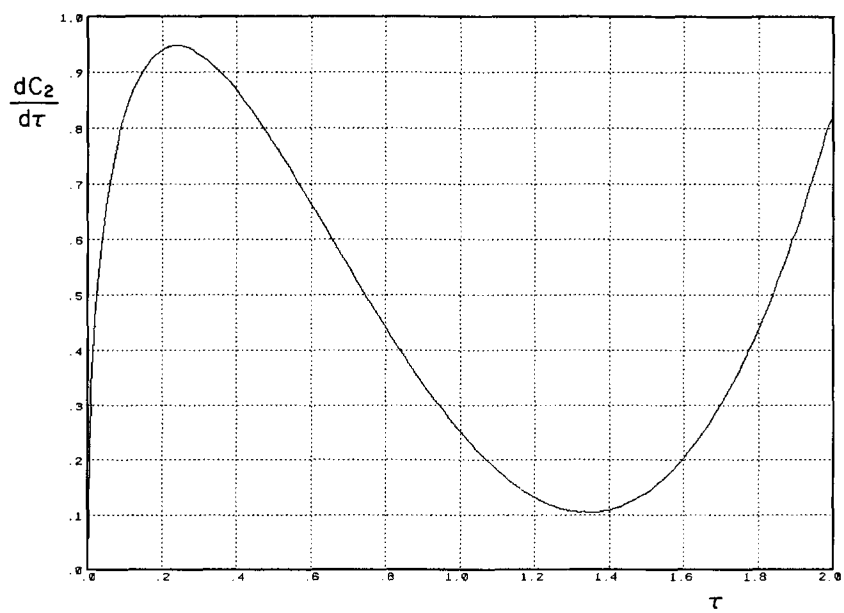


Fig. 5. The vertical slamming coefficient $\mathrm{d}C_2/\mathrm{d}\tau$ versus τ.

slamming force (33), and the exact solution found from (23). The agreement between the two is almost perfect for the same range of depths. For reasons of completeness we also present here the plots of the added-mass and slamming coefficients, which were obtained analytically by Miloh [15] for the vertical motion (Figs 4 and 5). Using these figures in conjuction with (6) enables us to evaluate the first-order hydrodynamical loads acting on a spherical projectile during early stages of an inclined water-entry at arbitrary angles of attack. The analytic solution for the sphere oblique-entry problem thus obtained is employed in a subsequent paper to determine the critical conditions for the ricocheting of a spherical projectile off a free surface.

Appendix A

Substitution of (13) in (8) yields

$$C_1 = -\frac{1}{\pi R^3}\int_0^\infty \int_0^{2\pi} \phi_0^{(1)} \frac{\partial x}{\partial \theta}\frac{h_\eta h_\psi}{h_\theta}\, \mathrm{d}\psi\, \mathrm{d}\eta \quad \text{at } \theta = \theta_0 , \tag{A.1}$$

where h_η, h_θ and h_ψ are the three linearizing coefficients of the orthogonal transformation (9), which are given by (Moon and Spencer [20])

$$h_\eta = h_\theta = h_\psi/\sinh\eta = a(\cosh\eta - \cos\theta)^{-1} . \tag{A.2}$$

Thus, because of (9),

$$C_1 = -\sin^4\theta_0 \int_0^\infty \frac{\sinh^2\eta}{(\cosh\eta - \cos\theta_0)^{5/2}}\, \mathrm{d}\eta \int_0^\infty A(p)K_p^{\prime}(\cosh\eta)\tanh p\theta_0\, \mathrm{d}p . \tag{A.3}$$

Following Miloh [15], we also have

$$\sin\theta_0 \int_0^\infty \frac{\sinh^2\eta}{(\cosh\eta - \cos\theta_0)^{5/2}} \frac{\mathrm{d}}{\mathrm{d}\eta} K_p(\cosh\eta)\, \mathrm{d}\eta = -\frac{4\sqrt{2}}{3}\left(p^2 + \frac{1}{4}\right)\frac{\sinh p(\pi - \theta_0)}{\sinh p\pi} , \tag{A.4}$$

which, when substituted in (A.3), leads to (21).

The added mass for the vertical (heave) motion is given in a similar manner to (A.3) by

$$C_2 = -\frac{1}{R^3}\int_0^\infty \phi_0^{(2)} \frac{\partial z}{\partial\theta}\frac{h_\eta h_\psi}{h_\theta}\, \mathrm{d}\eta = -\sin^3\theta_0 \int_0^\infty \frac{\sinh\eta}{(\cosh\eta - \cos\theta)^{1/2}} \frac{\partial}{\partial\theta}\left(\frac{\sin\theta}{\cosh\eta - \cos\theta}\right) \mathrm{d}\eta$$
$$\times \int_0^\infty B(p)K_p(\cosh\eta)\tanh p\theta_0\, \mathrm{d}p , \quad \text{at } \theta = \theta_0 . \tag{A.5}$$

Using the following identity (Robin [24], p. 170)

$$\sin\theta_0 \int_0^\infty \frac{K_p(\cosh\eta)\sinh\eta}{(\cosh\eta - \cos\theta_0)^{3/2}}\, \mathrm{d}\eta = 2^{3/2}\frac{\sinh p(\pi - \theta_0)}{\sinh p\pi} \tag{A.6}$$

in (A.5), the latter reduces to (22).

Appendix B

Substitution of (19) in (29) yields

$$A_2(p) = -\frac{2\sqrt{2}}{3} \tanh p\pi \int_0^\infty \frac{q \sinh q\pi}{\cosh^2 q\pi} \, \mathrm{d}q \int_0^\infty \frac{K_p(\cosh\eta')K_q(\cosh\eta')}{\cosh\eta' + 1} \sinh\eta' \, \mathrm{d}\eta' . \quad \text{(B.1)}$$

In order to evaluate the above integral we employ the following Mehler formula for the toroidal function (Hobson [5], p. 451)

$$K_q(\cosh\eta) = \frac{\sqrt{2}}{\pi} \coth q\pi \int_\eta^\infty \frac{\sin qu \, \mathrm{d}u}{(\cosh u - \cosh\eta)^{1/2}} , \quad \text{(B.2)}$$

which, when substituted in (B.1) yields

$$A_2(p) = -\frac{4}{3\pi} \tanh p\pi \int_0^\infty \frac{q \sin qu}{\cosh q\pi} \, \mathrm{d}q \int_0^\infty \int_{\eta'}^\infty \frac{K_p(\cosh\eta') \sinh\eta'}{(\cosh\eta' + 1)(\cosh u - \cosh\eta')^{1/2}} \, \mathrm{d}u \, \mathrm{d}\eta'$$

$$= -\frac{\sqrt{2}}{6\pi} \tanh p\pi \int_0^\infty \frac{K_p(\cosh\eta') \sinh\eta'}{\cosh\eta' + 1} \, \mathrm{d}\eta' \int_{\eta'}^\infty \frac{\sinh\frac{u}{2} \, \mathrm{d}u}{\cosh^2\frac{u}{2}\left(\cosh^2\frac{u}{2} - \cosh^2\frac{\eta'}{2}\right)^{1/2}} , \quad \text{(B.3)}$$

since (Gradshteyn and Ryzhik [2], p. 525)

$$\int_0^\infty \frac{q \sin qu}{\cosh q\pi} \, \mathrm{d}q = \frac{1}{4} \frac{\sinh\frac{u}{2}}{\cosh^2\frac{u}{2}} . \quad \text{(B.4)}$$

Using in (B.3) the following transformations:

$$\cosh\lambda = \cosh u/2/\cosh\eta'/2 , \qquad \mu = \cosh\eta' \quad \text{(B.5)}$$

and the identity

$$\int_0^\infty \frac{\mathrm{d}\lambda}{\cosh^2\lambda} = 1 , \quad \text{(B.6)}$$

leads to

$$A_2(p) = -\frac{2\sqrt{2}}{3\pi} \tanh p\pi \int_1^\infty \frac{K_p(\mu) \, \mathrm{d}\mu}{(\mu + 1)^2} . \quad \text{(B.7)}$$

Following Robin [24] (p. 168), we have for arbitrary $|\nu| \leq 1$

$$\int_1^\infty \frac{K_p(\mu)}{\mu + \nu} \, \mathrm{d}\mu = \frac{\pi}{\cosh p\pi} K_p(\nu) , \quad \text{(B.8)}$$

or

$$\int_1^\infty \frac{K_p(\mu)}{(\mu+\nu)^2}\,\mathrm{d}\mu = -\frac{\pi}{\cosh p\pi}\frac{\mathrm{d}}{\mathrm{d}\nu}K_p(\nu)\,. \tag{B.9}$$

Recalling that (Robin [24], p. 150)

$$K_p(\nu) = 1 + \sum_{m=1}^{\infty} \frac{(4p^2+1)(4p^2+3)\cdots[4p^2+(2m-1)^2]}{2^{3m}(m!)^2}(1-\nu)^m\,, \tag{B.10}$$

one gets

$$\lim_{\nu\to 1}\frac{\mathrm{d}}{\mathrm{d}\nu}K_p(\nu) = -\frac{4p^2+1}{8}\,, \tag{B.11}$$

which, when substituted in (B.7), finally leads to (30).

Appendix C

A two-term expansion of the sway added-mass is obtained by substituting (28) and (30) into (21) and (27), which yields

$$C_1(\varepsilon) = \frac{32}{9}\varepsilon^5\int_0^\infty \frac{p^2(p^2+1/4)}{\cosh^2 p\pi}\,\mathrm{d}p - \frac{8}{9}\varepsilon^6\int_0^\infty \frac{p(p^2+1/4)^2\sinh p\pi}{\cosh^3 p\pi}\,\mathrm{d}p\,. \tag{C.1}$$

Integration by parts of the last integral in (C.1) gives

$$C_1(\varepsilon) = \frac{32}{9}\varepsilon^5\int_0^\infty \frac{p^2(p^2+1/4)}{\cosh^2 p\pi}\,\mathrm{d}p - \frac{4}{9\pi}\varepsilon^6\int_0^\infty \frac{(p^2+1/4)(p^2+4p+1/4)}{\cosh^2 p\pi}\,\mathrm{d}p\,. \tag{C.2}$$

Following Gradshteyn and Ryzhik [2], we use in (C.2)

$$\int_0^\infty \frac{p^{\nu-1}}{\cosh^2 \pi p}\,\mathrm{d}p = \frac{4}{(2\pi)^\nu}(1-2^{2-\nu})\Gamma(\nu)\zeta(\nu-1)\,, \tag{C.3}$$

where ν is an arbitrary integer, Γ is the Gamma function and ζ denotes the Riemann zeta function. Substitution of (C.3) in (C.2) leads to (31). The above relationship may also be used to derive the asymptotic two-term expansion of (23) as given in (33).

References

1. A.T. Chwang, Nonlinear hydrodynamic pressure on an accelerating plate. *Phys. of Fluids* 25 (1983) 383–387.
2. I.S. Gradshteyn and I.M. Ryzhik, *Tables of Integrals, Series and Products*, Academic Press, New York (1965).
3. M.J. Greenhow, Water-entry and exit of a horizontal circular cylinder. *Appl. Ocean Res.* 10 (1988) 191–198.
4. M.A. Grosenbaugh and R.W. Yeung, Nonlinear bow flows – An experimental and theoretical investigation. *17th Symp. on Naval Hydrodynamics*, The Hague (1988) 195–214.
5. E.W. Hobson, *The Theory of Spherical and Ellipsoidal Harmonics*, Cambridge University Press, Cambridge (1931).
6. A. Hulme, A ring-source integral-equation for the calculation of hydrodynamical forces exerted on floating bodies of revolution. *J. Fluid Mech.* 128 (1983) 387–412.

7. S.W. Joo, W.W. Schultz and A.F. Messiter, Uniformly valid solution to the initial-value wavemaker problem. The Univ. of Michigan program in ship Hydrodynamics Rep. 89-01 (1989).
8. A.A. Korobkin, Initial asymptotic behavior of the solution of the three-dimensional problem of the entry of a blunt body into an ideal liquid. *Sov. Phys. Dokl.* 30 (1985) 656–658.
9. A.A. Korobkin, Inclined entry of a blunt profile into an ideal fluid. *Fluid Dynamics* 23 (1988) 443–447.
10. A.A. Korobkin and V.V. Pukhnachov, Initial stage of water impact. *Ann. Rev. Fluid Mech.* 20 (1988) 159–185.
11. H. Lamb, *Hydrodynamics*, Dover Publications, New York (1945).
12. Y.L. Luke, *Algorithm for the Computations of Mathematical Functions*, Academic Press, New York (1977).
13. P. McIver, Sloshing frequencies for cylindrical and spherical containers filled to an arbitrary depth. *J. Fluid Mech.* 201 (1989) 243–257.
14. T. Miloh, Hydrodynamics of deformable contiguous spherical shapes in an incompressible inviscid fluid. *J. Eng. Math.* 11 (1977) 349–372.
15. T. Miloh, Wave slam on a sphere penetrating a free-surface. *J. Eng. Math.* 15 (1981) 221–240.
16. T. Miloh, Hamilton's principle, Lagrange's method and ship motion theory. *J. Ship. Res.* 28 (1984) 229–237.
17. T. Miloh, On the initial stage slamming of a rigid sphere in a verical water entry, to appear in *Appl. Ocean Research* (1990).
18. T. Miloh and L. Landweber, Generalization of the Kelvin–Kirchhoff equations for the motion of a body through a fluid. *Phys. of Fluids* 24 (1981) 6–9.
19. M. Moghisi and P.T. Squire, An experimental investigation of the initial force of impact on a sphere striking a free-surface. *J. Fluid Mech.* 198 (1981) 133–146.
20. P. Moon and D.E. Spencer, *Field Handbook*, Springer, Berlin (1971).
21. D.H. Peregrine, Flow due to a vertical plate moving in a channel. Unpublished note (1972).
22. V.V. Pukhnachov and A.A. Korobkin, Initial asymptotic in problems of blunt body entrance into liquid. *Proc. of 3rd International Conference on Numerical Ship Hydrodynamics*, Paris (1981) 579–592.
23. A.J. Roberts, Transient free-surface flows generated by a moving vertical plate. *Quart. J. Mech. Appl. Math.* 40 (1987) 129–158.
24. L. Robin, *Fonctions Spheriques de Legendre et Fonctions Spheroidales*. Collection Technique et Scientifique du Centre National d'Etude des Telecommunications (1959).
25. L. Sedov, The impact of a solid body floating on the surface of an incompressible fluid. TSAGI Report No. 187, Moscow (1934).
26. I.H. Sneddon, *The Use of Integral Transforms*, McGraw-Hill, New York (1972).
27. L. Trilling, The impact of a body on a water surface at an arbitrary angle. *J. Appl. Phys.* 21 (1950) 161–170.
28. A.W. Troesch and C.G. Kang, Hydrodynamic impact loads on three-dimensional bodies. *Proc. 16th Symp. on Naval Hydrodynamics*, Berkeley (1987) 537–538.
29. P.A. Tyvand, On the interaction between a strong vortex pair and a free-surface, to appear in *Physics of Fluids* (1990).
30. T. Vinje, On small time expansion of non linear free surface problems. *Proc. 4th International Workshop on Water Waves and Floating Bodies*, Oystese, Norway (1989) 245–247.
31. Th. Von Karman, The impact on seaplane floats during landing. NACA TN 321 (1929).
32. H. Wagner, Uber Stoss und Gleitvorgange an der Oberflache von Flussigkeiten *ZAMM* 12 (1932) 193–215.
33. K.H. Wang and A. Chwang, Nonlinear free surface flow around an impulsively moving cylinder. *J. Ship Res.* 33 (1989) 194–202.