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On the Occurrence of Naked Singularity in Spherically Symmetric Gravitational Collapse

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Abstract: Generalizing earlier results of [1], we analyze here the spherically symmetric gravitational collapse of a matter cloud with a general form of matter for the formation of a naked singularity. It is shown that this is related basically to the choice of initial data to the Einstein field equations, and would therefore occur in generic situations from regular initial data within the general context considered here, subject to the matter satisfying the weak energy condition. The condition on initial data which leads to the formation of black hole is also characterized.

1. Introduction

We considered recently the formation and structure of naked singularity in the self-similar gravitational collapse of a perfect fluid with an adiabatic equation of state, and also for a general form of matter subject only to the weak energy condition but with an arbitrary equation of state [1]. It was shown in those cases that strong curvature naked singularities form in the gravitational collapse from a regular initial data, from which non-zero measure families of non-spacelike trajectories could come out. The criterion for the existence of such singularities was characterized in terms of the existence of real positive roots of an algebraic equation constructed out of the field variables.

The considerations such as those in [1] and [2] provide many insights into the phenomena of gravitational collapse. For example, the Einstein equations, under the geometric assumption of self-similarity reduce to ordinary differential equations. This allows one to construct explicit collapse scenarios such as the Vaidya–Papapetrou radiation collapse with a linear mass function [3], or perfect fluid collapse [1, 4] which provide useful information on the phenomena of gravitational collapse. It is also known that such conclusions are not restricted to self-similar spacetimes only [2, 5].

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The scenarios such as above have, however, limitations. For example, while in [1] the equation of state is general, subject to the weak energy condition only, the geometric assumption of self-similarity is there. On the other hand, while [2] considers a more general collapse, the classes considered there have other limitations. For example, the later reference there is restricted to dust collapse (however, with a generic mass function with only C^1 differentiability) whereas the former, while considering a wide class of matter (with the condition that the mass function be expandable about the singularity), excludes useful collapse situations due to other assumptions made there (e.g. the metric coefficient $c(v\, r)$ is also expandable about the singularity and $c(v, 0) = 1$). The presently known collapse scenarios are restricted mainly to dust and perfect fluid. While the form of matter such as a perfect fluid has wide range of physical applications with the advantage of incorporating pressure which could be important in the later stages of collapse, it is certainly important to examine if similar conclusions will hold for other reasonable forms of matter. For example, as pointed out by Eardley [6], dust could be an approximation to a more fundamental form of matter, such as a massive scalar field. It is thus possible that the naked singularity is an artifact of the approximation used, and not a basic feature of collapse. It is therefore important to consider the collapse phenomena for a rather general form of matter, without limitations such as above. This should help us to understand gravitational collapse and the occurrence of naked singularity in a more clear manner, which should lead to a more precise formulation of the cosmic censorship hypothesis.

Our purpose here is to analyze the formation of naked singularities in spherically symmetric collapse from this perspective for a general form of matter, only subject to the weak energy condition with no restriction on the equation of state. All the presently known naked singular examples, such as radiation collapse, dust or perfect fluid models could apply only to a rather narrow class of equations of state. Our considerations here show, apart from other implications, that given any equation of state, for which there does exist a spherically symmetric, naked singularity example of present type, then for all sufficiently close equations of state there is also such an example. We reduce the spherically symmetric Einstein field equations to a single parabolic partial differential equation of second order. The class of naked singular spacetimes with such equations of state is defined in terms of solutions of this equation.

In Sect. 2 the basic equations for the collapse are set up and Sect. 3 discusses the initial value problem to consider the gravitational collapse of a spherically symmetric matter cloud which is initially non-singular. The existence of naked singularity is characterized in Sect. 4, also examining its curvature strength. It is pointed out that the occurrence of naked singularity or a black hole is more a problem of the choice of the initial data for the field equations rather than the form of matter or the equation of state. The concluding Sect. 5 briefly considers the implications.

2. Spherically Symmetric Collapse

We consider here the final fate of collapse of a matter cloud that evolves from a regular physical data defined on an initial spacelike surface. The energy-momentum tensor has a compact support on this initial surface where all the physical quantities such as density, etc. are regular and finite. For sufficiently high mass, there is no stability configuration possible for the system and the collapse results into a space-time singularity as implied by the singularity theorems in general relativity. This singularity is characterized by the existence of a future directed non-spacelike trajectory in the matter cloud which is future incomplete, having a finite affine length but no future end point.

To consider a general matter field, we note that the stress-energy tensor $T^a{}_b$ describing the matter distribution of a space-time can be classified as being one of the following four types [7]. These are the possibilities when it has either in a timelike invariant 2-plane (i) two real orthogonal eigenvectors, (ii) one double null real eigenvector, (iii) no real eigenvector, or has in null invariant 2-plane, (iv) one triple null real eigenvector. The matter distributions of type (iii) and (iv) necessarily violate the energy conditions ensuring the positivity of the mass-energy density. Furthermore, such fields have not been observed so far. Thus, we would not attribute any physical interpretation to the same presently. The only observed occurrence of type (ii) matter distribution corresponds to zero rest mass fields representing directed radiation. In a spherically symmetric space-time these could be effectively described by the Vaidya metric. The radiation collapse, as described by the Vaidya space-times has already been analyzed in detail and strong curvature naked singularities do form in such a collapse in generic situations, either with or without the geometric condition of self-similarity [3, 5].

Hence, we need to examine only the gravitational collapse of type (i) matter fields for spherically symmetric space-times considered here. This is the form of matter and stress-energy for all the observed fields so far with either non-zero rest mass and also for zero rest mass fields, except the special cases described by type (ii). We take the matter fields to satisfy the weak energy condition, i.e. the energy density as measured by any observer is non-negative and for any timelike vector V^a,

$$T_{ab}V^aV^b \geqq 0 \,. \tag{1}$$

For the stress-energy tensor T^a_b of type (i), we can write

$$T^{ab} = \lambda_1 E^a_1 E^b_1 + \lambda_2 E^a_2 E^b_2 + \lambda_3 E^a_3 E^b_3 + \lambda_4 E^a_4 E^b_4 \,, \tag{2}$$

where (E_1, E_2, E_3, E_4) is an orthonormal basis with E_4 being timelike eigenvector and $\lambda_i (i = 1, 2, 3, 4)$ are the eigenvalues. For such a spherically symmetric matter distribution we can choose coordinates $(x^i = t, r, \theta, \phi)$ to write the metric as,

$$ds^2 = -e^{2\nu}dt^2 + e^{2\psi}dr^2 + R^2 d\Omega^2 \,, \tag{3}$$

where $d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2$ is the line element on two-sphere. Here ν, ψ and R are functions of t and r and the stress energy tensor T^a_b given by Eq. (1) has only diagonal components in this coordinate system (i.e. we are using a comoving coordinate system)

$$T^t_t = -\rho, \quad T^r_r = p_1, \quad T^\theta_\theta = p_2 = T^\phi_\phi = p_3, \quad T^t_r = T^r_t = 0 \,. \tag{4}$$

The quantities ρ, p_1, p_2, and p_3 are the eigenvalues of T^a_b and are interpreted as the density and principle pressures. Then, the weak energy condition holds for type (i) matter fields provided,

$$\rho \geqq 0, \quad \rho + p_\alpha \geqq 0, \quad \alpha = 1, 2, 3 \,. \tag{5}$$

We note that $R(t,r) \geqq 0$ here is the area coordinate in the sense that the quantity $4\pi R^2(t,r)$ gives the proper area of the mass shells and the area of such a shell at $r=$ const. goes to zero when $R(t,r)=0$. In this sense, the curve $R(t,r)=0$ describes the singularity in the space-time where the mass shells are collapsing to a vanishing volume. Such a singularity is often called a "shell-focusing." It thus follows that the range of the coordinates in metric (3) is given by

$$0 \leqq r < \infty, \quad -\infty < t < t_0(r), \tag{6}$$

with θ and ϕ having the usual range of values. The time $t = t_0(r)$ corresponds to the value of area coordinate $R = R(t,r) = R(t_0(r), r) = 0$, where the area of the shell of matter at a constant value of r vanishes. This corresponds to the time when the matter shells meet the physical singularity.

The Einstein field equations and the Bianchi identities $T^a_{b;a} = 0$ are written as below:

$$G^t_t = k_0 T^t_t \Rightarrow \frac{\partial}{\partial r}[R(-1 + e^{-2\psi}R'^2 - e^{-2\nu}\dot{R}^2)] = R^2 R' k_0 T^t_t\,, \tag{7a}$$

$$G^r_r = k_0 T^r_r \Rightarrow \frac{\partial}{\partial t}[R(-1 + e^{-2\psi}R'^2 - e^{-2\nu}\dot{R}^2)] = R^2 \dot{R} k_0 T^r_r\,, \tag{7b}$$

$$G^t_r = k_0 T^t_r = 0 \Rightarrow \dot{R}' - \dot{\psi}R' - \nu'\dot{R} = 0\,, \tag{7c}$$

$$T^a_{t;a} = 0 \Rightarrow \dot{T}^t_t + T^t_t\left(\dot{\psi} + \frac{2\dot{R}}{R}\right) - T^r_r\dot{\psi} = 2p_2\frac{\dot{R}}{R}\,, \tag{7d}$$

$$T^a_{r;a} = 0 \Rightarrow (T^r_r)' + T^r_r\left(\nu' + \frac{2R'}{R}\right) - T^t_t\nu' = 2p_2\frac{R'}{R}\,, \tag{7e}$$

where $k_0 = 8\pi G/c^2$ is the gravitational constant, and in Eqs. (7a) and (7b), Eq. (7c) has been used. The ($'$) and ($^{\cdot}$) denote partial derivatives with respect to r and t. As we consider the collapse problem, we take $\dot{R} < 0$. Eliminating p_2 from Eqs. (7d), (7e) and using (7c) imply

$$\frac{\partial}{\partial r}[T^r_r R^2 \dot{R}] - \frac{\partial}{\partial t}[T^t_t R^2 R'] = 0\,,$$

and we conclude that

$$T^t_t = -\rho = -\frac{F'}{k_0 R^2 R'}\,, \quad T^r_r = p_1 = -\frac{\dot{F}}{k_0 R^2 \dot{R}}\,, \tag{8}$$

where $F = F(t,r)$ is an arbitrary function of t and r. For the Tolman–Bondi dust collapse [8] or perfect fluid space-times [9], F is physically interpreted as mass function. For dust collapse, $F = F(r)$ represents the total mass within a coordinate radius r. Thus the function F is treated as mass function for the cloud with $F \geqq 0$. In general, R' may not be positive, however, in the cases when $R' \geqq 0$, the weak energy condition implies from (8) that $F' \geqq 0$. In the case of gravitational collapse ($\dot{R} < 0$), it will be seen later from the null geodesic equations, that $R' < 0$ at the singularity implies that no rays will be outgoing and the singularity will be censored.

Using (8), Eqs. (7d) and (7c) become

$$4p_2 R\dot{R}R' = -2\dot{F}' + F'\frac{\dot{G}}{G} + \dot{F}\frac{H'}{H}\,, \tag{9a}$$

$$-2\dot{R}' + R'\frac{\dot{G}}{G} + \dot{R}\frac{H'}{H} = 0\,, \tag{9b}$$

where we have put

$$G = G(t, r) = e^{-2\psi}(R')^2\,, \quad H = H(t, r) = e^{-2\nu}\dot{R}^2\,. \tag{9c}$$

Integration of the remaining field equations $G^t_t = k_0 T^t_t$, $G^r_r = k_0 T^r_r$ is straightforward and some simplification gives

$$\rho = \frac{1}{k_0 R^2}\left(F_{,R} + \frac{F_{,r}}{T}\right), \quad p_1 = -\frac{F_{,R}}{k_0 R^2}\,, \tag{10a}$$

$$(F_{,RR} + 2k_0 p_2 R)T \equiv \frac{T}{p} = F_{,r}\frac{G_{,R}}{2G} - F_{,rR}\,, \tag{10b}$$

$$-2T_{,R} + T\frac{G_{,R}}{G} + T\frac{H_{,R}}{H} + \frac{H_{,r}}{H} = 0$$

$$\Rightarrow\; -2\left[p\left(F_{,r}\frac{G_{,R}}{2G} - F_{,rR}\right)\right]_{,R} + p\left(F_{,r}\frac{G_{,R}}{2G} - F_{,rR}\right)\left(\frac{G_{,R}}{G} + \frac{(G-1+\frac{F}{R})_{,R}}{(G-1+\frac{F}{R})}\right)$$
$$= \frac{G_{,r} + \frac{F_{,r}}{R}}{G - 1 + \frac{F}{R}}\,, \tag{10c}$$

$$H = G - 1 + \frac{F}{R}\,. \tag{10d}$$

We have used R instead of t as variable in the above equations. The function $p = p(R, r)$ is defined in (10b), $F(t, r) = F(R, r)$, $T(R, r) = R'$ and likewise $G(R, r)$ and $H(R, r)$ are to be treated as functions of R and r. Here $(_{,R})$ and $(_{,r})$ denote partial derivatives with respect to R and r and are defined by

$$\left[\frac{\partial}{\partial r}\right]_{t=\text{const.}} = R'\left[\frac{\partial}{\partial R}\right]_{r=\text{const.}} + \left[\frac{\partial}{\partial r}\right]_{R=\text{const.}}\,,$$
$$\left[\frac{\partial}{\partial t}\right]_{r=\text{const.}} = \dot{R}\left[\frac{\partial}{\partial R}\right]_{r=\text{const.}}\,. \tag{11}$$

The two equations in (10c) are equivalent. The later equation of (10c) is obtained from the former by the substitution of T from (10b). The remaining field equation $G^\theta_\theta = k_0 T^\theta_\theta = G^\phi_\phi = k_0 T^\phi_\phi$ is then just a consequence of (10a) to (10d).

In all we have five unknowns, namely $T(R, r)$, $G(R, r)$, $H(R, r)$, $F(R, r)$ and $p_2(R, r)$, and three equations (10b), (10c), and (10d) relating them. In fact, the functions F and p_2 determine the form of matter and the equation of state one is dealing with. For example, for dust models $p_2 = 0 = F_{,R}$ which implies $F(R, r) \equiv F(r)$, and for a perfect fluid $p_2 = -F_{,R}/R^2$, if the fluid has the equation of state $\rho + p = 0$. In addition then $F_{,r} = 0$ implies $F(R, r) \equiv F(R)$. Therefore, one starts with a particular stress-energy tensor by selecting these two functions and then the geometry of space-time or the metric functions are determined as follows. Knowing $F(R, r)$ and $p_2(R, r)$ one determines $G = G(R, r)$ from (10c), which is a second order partial differential equation, with appropriate initial and boundary data for G. Note that Eq. (10c) is a paraoblic type (in other words a generalized heat wave type) second order partial differential equation and as such could be solved with a possible set of initial and boundary conditions given by $G(R, 0) = g(R)$, $G(a, r) = g_1(r)$ and $G(b, r) = g_2(r)$. Equation (10b) then determines $R' = T(R, r)$ as function of R and r, which on integration yields $R = R(t, r)$. The function $H = H(R, r)$ is immediate from (10d). In case $F_{,RR} + 2k_0 p_2 R \equiv 1/p = 0$ identically, then (10b) breaks down and one cannot determine $R' = T(R, r)$ from (10b). Rather it implies $(G/F_{,r})_{,R} = 0$, thus determining $G(R, r)$ instead of $T(R, r)$. The function $T(R, r)$ in such cases is determined by integrating (10c) with appropriate initial conditions by treating it as an ordinary differential equation for T in variable R and specifying initial conditions accordingly.

3. The Initial Value Problem

Consider now a spherically symmetric cloud of matter collapsing gravitationally to give rise to a space-time singularity $R = 0$ at the center $r = 0$. Our problem is to characterize the conditions under which this could be naked, and those in which the singularity is completely covered by an event horizon formed during the collapse. Thus, for example, for a homogeneous gravitational collapse of dust described by the Oppenheimer–Snyder models, the resulting singularity is fully covered by an event horizon. On the other hand, if inhomogeneities are present, strong curvature naked singularities do form in such collapse scenarios (see e.g. [2]).

We define regular initial data on a spacelike hypersurface $t = t_i$ from which the collapse starts. On the surface $t = t_i$ we require physical quantities such as density, pressures, etc. be non-singular. Matter has a compact support on $t = t_i$ and $r = r_b$ denotes the boundary of the object. One requires appropriate boundary conditions to match the interior metric of the cloud to the exterior. The exact boundary conditions will depend on what the exterior spacetime is, which could be vacuum Schwarzschild or a radiating Vaidya metric etc.

The space-time singularity occurs at the time $t = t_0(r)$, which corresponds to $R(t, r) = 0$. Let $t = t_0(0) > t_i$ be the first point of the singularity, which is the time of the singularity occurring at $r = 0$. (If $t = t_0(0)$ is not already the first point of the singularity curve $t = t_0(r)$, it could be made so by a simple translation of the coordinate r.) Thus, this implies a boundary condition that for $t_i < t < t_0(0)$, the center of the cloud $r = 0$ is a regular center. In terms of the functions above defining the gravitational collapse, this amounts to the requirement,

$$\frac{F_{,r}}{R'} < \infty \tag{12}$$

at the regular center $r = 0$ for $t_i < t < t_0(0)$. We are interested in analyzing the nature of this first shell-focusing singularity $R = 0$ which we call a central singularity when it occurs at $r = 0$. Thus, we assume that there is a neighborhood of the central singularity such that $R' > 0$ for $r > 0$.

Basically we would require certain general differentiability conditions for functions F, p_2 and R and take $F(R, r)$ and $p_2(R, r)$ to be at least C^2 for $R > 0$ and $R(t, r)$ as C^2 for all t and r.

As mentioned earlier, the choice of a particular matter distribution is made by selecting the physical quantities F and p_2. Since the data at the initial hypersurface is non-singular, this puts restrictions on the values of these functions at the initial surface $t = t_i$. That is, one has to choose F such that at the initial surface $F_{,r}/R^2R' < \infty$, $F_{,R}/R^2 < \infty$ and $p_2 < \infty$. Therefore, the condition that initial data be non-singular means a proper choice of free functions F and p_2.

To make this clear, one could use the coordinate freedom left in the rescaling of the coordinates r and t without any loss of generality, so that at $t = t_i$,

$$R(t_i, r) = r \,. \tag{13}$$

At $t = t_i$, the quantities ρ, p_1, p_2, etc. do not diverge at $r \geqq 0$. Since the initial data is to be non-singular at $r = 0$ on this surface, the first derivatives of F must have the following behavior at $r = 0$,

$$[(F_{,r})_{R=r}/r^2]_{r=0} < \infty\,, \quad [(F_{,R})_{R=r}/r^2]_{r=0} < \infty\,, \quad [p_2(r, r)]_{r=0} < \infty \,. \tag{14}$$

Similarly, requiring that $G(R, r) < \infty$ at this $R = r$ surface at $r = 0$ puts restriction on the choice of p_2 and second derivatives of F. That is, from Eq. (10b) one has

$$\left[\frac{[F_{,RR}]_{R=r} + 2k_0 p_2(r, r)r + [F_{,rR}]_{R=r}}{[F_{,r}]_{R=r}}\right]_{r=0} \neq -\infty \tag{15}$$

at $t = t_i$ and therefore $e^{2\psi} = 1/G \neq \infty$. This ensures that the initial data is nonsingular at $t = t_i$. We do not discuss further implications of the conditions such as above, but the point is the choice of a non-singular initial data and boundary conditions restrict the functions F and p_2 suitably ensuring their proper choice.

The space-time singularity appears at the point $R = 0$, $r = 0$, and therefore the behavior of various functions near the singularity is important. To examine this, we get after some simplification from Eqs. (10a) to (10d),

$$\rho = \frac{1}{k_0 u^2 X^2}\left(\Lambda_{,X} + \frac{\eta}{\beta}\right), \quad p_1 = -\frac{\Lambda_{,X}}{k_0 u^2 X^2}, \quad p_2 = p_2(X, u)\,, \tag{16a}$$

$$\beta(X, u) = P\left(\eta\frac{f_{,X}}{2(1+f)} - \eta_{,X}\right), \tag{16b}$$

$$-2\left[\frac{P}{\sqrt{1+f}}\left(\eta\frac{f_{,X}}{2(1+f)} - \eta_{,X}\right)\right]_{,X}\sqrt{1+f} + \frac{(f+\frac{\Lambda}{X})_{,X}}{f+\frac{\Lambda}{X}}\left[P\left(\eta\frac{f_{,X}}{2(1+f)} - \eta_{,X}\right) - X\right]$$

$$= u\frac{f_{,u} + \frac{\Lambda_{,u}}{X}}{f+\frac{\Lambda}{X}}\,, \tag{16c}$$

$$H = f + \frac{\Lambda}{X}\,, \tag{16d}$$

where we put for the sake of convenience $G(X, u) \equiv 1 + f(X, u)$ and introduce two variables $u = r^\alpha$ ($\alpha \geqq 1$ is a constant), $X = R/r^\alpha = R/u$ and the following notation:

$$\eta = \eta(X, u) = \frac{F_{,r}}{r^{\alpha-1}}, \quad \Lambda = \Lambda(X, u) = \frac{F(R, r)}{r^\alpha} = \frac{F}{u}\,,$$

$$P = P(X, u) = \frac{1}{\Lambda_{,XX} + 2k_0 p_2 u^2 X}, \quad \beta(X, u) = \frac{R'}{r^{\alpha-1}} = \frac{T}{r^{\alpha-1}}\,. \tag{17}$$

Here G, H, η, Λ and β are all functions of $X = R/u$ and $u = r^\alpha$. The constant $\alpha \geqq 1$ is to be chosen so that $\beta = T/r^{\alpha-1}$ does not vanish or go to infinity identically as $r \to 0$ in the limit of approach to the singularity along all $X = \text{const.}$ directions. Here $(_{,X})$ and $(_{,u})$ represent partial derivatives with respect to X and u respectively. The weak energy condition implies for these functions,

$$\frac{\eta}{\beta} \geqq 0, \quad \Lambda_{,X} + \frac{\eta}{\beta} \geqq 0, \quad k_0 u^2 X^2 p_2 + \Lambda_{,X} + \frac{\eta}{\beta} \geqq 0\,. \tag{18}$$

Equation (16c) is a second order partial differential equation for $G(X, u) = 1 + f$ and is solved by specifying the initial and boundary conditions on $G(X, u)$. Knowing G, the rest of the unknowns are then immediate from Eqs. (16a) and (16d).

The values $G(X_0, 0) = 1 + f(X_0, 0)$, $\beta(X_0, 0)$ for some positive $X = X_0$ are of significance in our analysis. The function $\beta(X, 0)$ can be calculated from (16b) once $f(X, 0)$ is fixed except as mentioned earlier in the case where $1/P = \Lambda_{,XX} + 2k_0 p_2 u^2 X = 0$ identically (for example, this will hold for the case of dust). In such a case the function $\beta(X, 0)$ could not be calculated from (16b), rather it implies $1 + f(X, u) = \eta g(u)$ where $g(u)$ is an arbitrary function of u. Thus $f(X, u)$ is determined from (16b) instead of β. Then $f(X, 0)$ is again chosen (by selecting $g(u)$) and $\beta(X, u)$ is then determined by integrating an ordinary differential equation (knowing $G = 1 + f$ actually (10c) becomes an equation for β) to find $\beta(X, u)$ given below

$$-2\beta_{,X} + \beta\frac{f_{,X}}{1+f} + (\beta - X)\frac{(f + \frac{\Lambda}{X})_{,X}}{f + \frac{\Lambda}{X}} = u\frac{f_{,u} + \frac{\Lambda_{,u}}{X}}{f + \frac{\Lambda}{X}}\,. \tag{19}$$

Now, $\beta(X, 0)$ can be determined from integration of the above.

The initial values of $G(X, u) \equiv 1 + f$ and $\beta(X, u)$ at $X = X_0, u = 0$ have to be chosen while selecting a particular model of spherically symmetric gravitational collapse and this has to be a physically reasonable choice. In fact (16c), which determines f is a parabolic type second order partial differential equation and as such could be solved with a given set of initial data which could be of the form $f(X, 0), f(a, u)$ and $f(b, u)$. Similarly in the special case when $1/P = 0$ identically, one can choose $\beta_0 \equiv \beta(X_0, 0)$ as part of the initial condition to solve the resulting ordinary differential equation determining $\beta(X, u)$. Hence the spacetime subject to the chosen form of matter is determined by the solutions of either the second order parabolic partial differential equation (16c) or the second order ordinary differential equation (19) as the case may be. The input to these equations comes from the prechosen form of matter by the way of functions P, Λ and η (i.e. F and p_2). The initial data is given as a set of appropriate values a, b, c, λ, μ for $X, u, f, f_{,X}, f_{,u}$ in Eq. (16c) and $a_1, b_1, \beta_1, \beta_2$ for $X, u, \beta, \beta_{,X}$ in case of Eq. (19). The existence of solutions of equations such as (16c) or (19) (as the case may be) has been studied and these have been shown to exist with a suitable arbitrary choice of initial and boundary data under quite general conditions. In fact, under fairly general conditions solutions exist with arbitrary choice of $f(X, 0)$ and $f_{,X}(X, 0)$ in case of the parabolic partial differential equation (16c) and similarly with arbitrary choice of $\beta(X_0, 0)$ in case of the ordinary differential equation (19).

Therefore, the point is that for a given form of matter (i.e. given F and p_2), the values of $\beta(X_0, 0)$ (or $f_{,X}(X_0, 0)$) and $f(X_0, 0)$ at some positive value of $X = X_0$ are part of the initial data and the solution to field equations with these initial values represents one of the spacetimes with the above specified form of matter. Hence, one can make a suitable choice of these functions. One such reasonable choice could be as follows. If the usual Lorentz–Minkowskian geometry is to be valid in an infinitesimal neighborhood of the regular center $r = 0$, then one must require that the circumference $2\pi R$ of an infinitesimal sphere about the center be just the 2π times its proper radius $e^{\psi}dr$. In other words,

$$e^{2\psi} = (R')^2 \quad \text{at} \quad r = 0 \Rightarrow G = 1 \quad \text{at} \quad r = 0 \,. \tag{20}$$

Hence, $f(X) = G(X, 0) - 1 = 0$ is one of the many possible initial conditions.

4. The Existence of Naked Singularity

The existence of a naked singularity in space-time is characterized by the presence of outgoing families of future directed non-spacelike geodesics, which are past incomplete and terminate in the past at the singularity.

Radial null geodesics for a spherically symmetric space-time (3) are given by $ds^2 = 0$,

$$\frac{dt}{dr} = \frac{dt/dk}{dr/dk} = \frac{K^t}{K^r} = e^{\psi - \nu} \,, \tag{21}$$

$$\frac{d}{dk}[e^{\psi}K^r] + e^{2\psi}(K^r)^2[e^{-\nu}\dot{\psi} - e^{-\psi}\nu'] = 0 \,, \tag{22}$$

where K^t and K^r are the only non-zero components of the tangent vector K^a ($K^\theta = 0, K^\varphi = 0$) and k is an affine parameter along the null geodesics. The singularity appears at the point $R(t, r) = 0$, $r = 0$, therefore if there are outgoing future directed radial null geodesics terminating at the singularity in past, then $R \to 0$ as $r \to 0$ along these geodesics.

We have from the above,

$$\frac{dR}{du} = \frac{1}{\alpha r^{\alpha - 1}}\left(R' + \dot{R}\frac{dt}{dr}\right) = \frac{(1 - \frac{A}{X})\sqrt{G}}{(\sqrt{G} + \sqrt{H})T} = \left(1 - \sqrt{\frac{f + \frac{A}{X}}{1 + f}}\right)\frac{\beta}{\alpha} \equiv U(X, u) \,. \tag{23}$$

Note that in the case of collapse, since $\dot{R} < 0$, dR/du becomes negative if $R' < 0$. Hence, in such a case the geodesics are all ingoing and the singularity is censored. For an outgoing geodesic dR/du must be positive and hence we require that $R' \geqq 0$ at the singularity. It further follows that if dR/du is negative (for $R < F$, dR/du is negative) geodesics become ingoing (in the sense that the area coordinate R starts decreasing). Note that $F(R, 0) > 0$ implies $dR/du \to -\infty$ at the singularity and so the geodesics are all ingoing, which corresponds to a Schwarzschild type situation where mass is already present at the center $r = 0$. When $F = 0$ at the first point of the singularity, the situation may correspond either to a black hole or a naked singularity. For example, in homogeneous dust collapse, $F \propto r^3$ and $F = 0$ at the first point which is covered by horizon. The point $R = 0$, $u = 0$ is a singularity of the differential equation (23), and hence in order to determine whether geodesics do terminate at the singularity or not one has to analyze the behavior of characteristic curves in the vicinity of the singular point. If radial null geodesics do terminate at the singularity then we have

$$X_0 = \lim_{R \to 0, u \to 0} \left(\frac{R}{u}\right) = \lim_{R \to 0, u \to 0} \left(\frac{dR}{du}\right) = U(X_0, 0)\,. \tag{24}$$

If a real and a positive value of X_0 satisfies the above equation then the singularity could be naked. On the other hand, if the above has no real positive roots, clearly the singularity is not naked with no families of non-spacelike trajectories coming out. Therefore, the necessary condition for the singularity to be naked is $V(X) = 0$ has a real positive root $X = X_0$, where

$$V(X) \equiv \left(1 - \sqrt{\frac{f(X,0) + \frac{\Lambda(X,0)}{X}}{1 + f(X,0)}}\right)\frac{\beta(X,0)}{\alpha} - X = 0\,. \tag{25}$$

As pointed out earlier, one could select $f(X, 0)$ (or $\beta(X, 0)$ as the case may be) as an initial data and rest of the unknowns in the above equation, namely $\beta(X, 0) > 0$ is implied by the field equation. That $V(X) = 0$ has a real positive root is a necessary condition for the singularity to be naked, but need not be a sufficient condition. To examine this, consider the equation of radial null geodesics in the form $u = u(X)$ given by

$$\frac{dX}{du} = \frac{1}{u}\left(\frac{dR}{du} - X\right) = \frac{U(X,u) - X}{u}\,. \tag{26}$$

Integration of the above yields radial null geodesics in the form $u = u(X)$. Let $X = X_0$ be a simple real positive root of $V(X) = 0$. If geodesics are to terminate at the singularity $R = 0$, $u = 0$, then $u \to 0$ as $X \to X_0$ along the same. We could then decompose $V(X)$ as

$$V(X) \equiv (X - X_0)(h_0 - 1) + h(X)\,, \tag{27}$$

where $h(X)$ is chosen such that is contains higher order terms in $X - X_0$,

$$h(X_0) = \left[\frac{dh}{dX}\right]_{X = X_0} = 0\,. \tag{28}$$

Using Eqs. (27), (16b) to (16d), (17), and (28) we get for h_0,

$$h_0 = \frac{X_0^2 \beta_0 (\alpha - 1)}{\eta_0 (\beta_0 - \alpha X_0)^2}\left(\frac{\beta_0}{P_0} + \gamma_0 - \frac{\eta_0 q_0}{\beta_0}\left(\frac{1 - \frac{\alpha X_0}{2\beta_0}}{X_0 - \Lambda_0}\right)\right) - q_0 \frac{1 - \frac{\alpha X_0}{2\beta_0}}{X_0 - \Lambda_0} - \frac{X_0 M_0}{2\beta_0}\,, \tag{29}$$

where

$$\eta_0 = \eta(X_0, 0), \quad \Lambda_0 = \Lambda(X_0, 0), \quad P_0 = P(X_0, 0)\,, \tag{30}$$

$$\gamma_0 = [\eta, X]_{u=0, X=X_0}, \; q_0 = [X\Lambda_{,X} - \Lambda]_{u=0, X=X_0}, \; M_0 = \left[\frac{u H_{,u}}{H}\right]_{u=0, X=X_0} \tag{31}$$

Writing $S = S(X, u) = U(X, u) - U(X, 0) + h(X)$, we could write (26) as

$$\frac{dX}{du} - (X - X_0)\frac{h_0 - 1}{u} = \frac{S}{u}\,. \tag{32}$$

Note that because of the way $S(X, u)$ is defined $S(X_0, 0) = 0$, i.e. in the limit $u \to 0$, $X \to X_0$ we have $S \to 0$. Integration of the above is straightforward by multiplication of an integrating factor u^{-h_0+1} and we get

$$X - X_0 = Du^{h_0 - 1} + u^{h_0 - 1} \int S u^{-h_0 + 1} du \,. \tag{33}$$

Here D is a constant which labels different geodesics. If geodesics described by the above equation do terminate at the singularity, $u \to 0$ as $X = X_0$ in the above. To see this, note that as $X \to X_0$, $u \to 0$ the last term of the above always vanishes near the singularity since $S \to 0$ as $u \to 0$, $X \to X_0$. The first term, i.e. Du^{h_0-1} vanishes only if $h_0 > 1$. It follows that the integral curve (radial null geodesic) $D = 0$ always terminates at the singularity $R = 0$, $u = 0$ with $X = X_0$ as tangent. Further, if $h_0 > 1$, a family of outgoing radial null geodesics terminates at the singularity in past, each curve given by a different value of the constant D.

It follows that if $V(X) = 0$ has a real positive root, then the gravitational collapse would terminate in a singularity which would at least be locally naked. As we have discussed earlier, such a condition basically corresponds to the choice of initial data for the differential equation (16c) in the form of the choice of $f(X_0) = G(X_0, 0) - 1$, or in the choice of $\beta(X_0, 0)$ in (19) as the case may be. Our analysis here implies that for all the presently known naked singular spherically symmetric examples with equations of state such as dust or perfect fluid, etc., there is a similar naked singular spacetime for all nearby equations of state in the sense defined by the existence of solutions of the parabolic differential equation discussed above. It is thus clear that for a wide range of spherically symmetric gravitational collapse, irrespective of the form of the matter, or a particular equation of state, a naked singularity would form in the above sense.

Next, we determine the curvature strength of the naked singularity. This is determined in terms of the curvature growth in the limit of approach to the naked singularity.

Consider the scalar quantity

$$\Psi = R_{ab} K^a K^b \,. \tag{34}$$

For the space-times (3), using (21) and (22) and the fact that K^a is a null vector, we get

$$\Psi = T^a_b K^b K_a = T^t_t K^t K_t + T^r_r K^r K_r = \left[\frac{F'}{R'} - \frac{\dot{F}}{\dot{R}}\right] \frac{e^{2\psi}(K^r)^2}{R^2} = \frac{\eta e^{2\psi}(K^r)^2}{\beta R^2} \,. \tag{35}$$

The singularity is said to be a strong curvature singularity [10] if

$$\lim_{k \to 0} k^2 \Psi \neq 0 \,. \tag{36}$$

We therefore have

$$\lim_{k \to 0} k^2 \Psi = \lim_{k \to 0} \frac{k^2 \eta e^{2\psi}(K^r)^2}{\beta R^2} \,. \tag{37}$$

Using the equations above and l'Hospitals rule we get

$$\lim_{k \to 0} k^2 \Psi \propto \frac{\eta_0}{\beta_0^2} \,. \tag{38}$$

Therefore, as long as $\eta_0 \neq 0$ the strong curvature condition is satisfied. In other words, in all the situations the singularity would be strong if the energy density (i.e. $\rho + p_1 = \eta/\beta u^2 X^2$) does not vanish in the neighborhood of the singularity.

5. Concluding Remarks

In this paper we have shown that the phenomena of naked singularity is dependent on the initial values chosen for solving the field equations, in that for all sets of regular initial values which produce at least one positive root of the equation $V(X) = 0$, the singularity would be naked.

It follows that a naked singularity could develop in a generic situation involving spherically symmetric collapse of matter from non-singular initial data. Therefore, in order to preserve the cosmic censorship hypothesis one has to avoid all such initial data and hence a deeper analysis of Eq. (25) is required in order to determine such initial data and the kind of physical parameters they would specify. This would, in other words, classify the range of physical parameters to be avoided for a particular form of matter. More importantly, it would also pave the way for the black hole physics to use only those ranges of allowed parameter values which would produce black holes, thus putting the black hole physics on a more reasonable footing.

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