On the Ohsawa-Takegoshi-Manivel L^2 extension theorem

Jean-Pierre Demailly

Université de Grenoble I
Laboratoire de Mathématiques, UMR 5582 du CNRS
Institut Fourier, BP74, 38402 Saint-Martin d'Hères, France

en l'honneur de Monsieur Pierre Lelong, à l'occasion de son 85ème anniversaire

Résumé. Un des buts de ce travail est d'illustrer de diverses manières l'efficacité des outils fondamentaux introduits par Pierre Lelong dans l'étude de l'Analyse Complexe et de la Géométrie analytique ou algébrique. Nous donnons d'abord une présentation détaillée du théorème d'extension L^2 de Ohsawa-Takegoshi, avec le même point de vue géométrique que celui introduit par L. Manivel. Ce faisant, nous simplifions la démarche de Ohsawa-Takegoshi et Manivel, et mettons en évidence une difficulté (non encore surmontée) dans l'argument invoqué par Manivel pour la régularité en bidegré $(0, q)$, $q \geqslant 1$. Nous donnons ensuite quelques applications frappantes du théorème d'extension, en particulier un théorème d'approximation des fonctions plurisouharmoniques par des logarithmes de fonctions holomorphes, préservant autant que possible les singularités et nombres de Lelong de la fonction plurisosuharmonique donnée. L'étude des singularités de fonctions plurisousharmoniques se poursuit par un théorème de type Briançon-Skoda nouveau pour les faisceaux d'idéaux multiplicateurs de Nadel. En utilisant ce résultat et des idées de R. Lazarsfeld, nous donnons finalement une preuve nouvelle d'un résultat récent de T. Fujita: la croissance du nombre des sections des multiples d'un fibré en droites gros sur une variété projective est donnée par la puissance d'intersection de plus haut degré de la partie numériquement effective dans la décomposition de Zariski du fibré.

Abstract. One of the goals of this work is to demonstrate in several different ways the strength of the fundamental tools introduced by Pierre Lelong for the study of Complex Analysis and Analytic or Algebraic Geometry. We first give a detailed presentation of the Ohsawa-Takegoshi L^2 extension theorem, inspired by a geometric viewpoint introduced by L. Manivel in 1993. Meanwhile, we simplify the original approach of the above authors, and point out a difficulty (yet to be overcome) in the regularity argument invoked by Manivel in bidegree $(0, q)$, $q \geqslant 1$. We then derive some striking consequences of the L^2 extension theorem. In particular, we give an approximation theorem of plurisubharmonic functions by logarithms of holomorphic functions, preserving as much as possible the singularities and Lelong numbers of the given function. The study of plurisubharmonic singularities is pursued, leading to a new Briançon-Skoda type result concerning Nadel's multiplier ideal sheaves. Using this result and some ideas of R. Lazarsfeld, we finally give a new proof of a recent result of T. Fujita: the growth of the number of sections of multiples of a big line bundle is given by the highest power of the first Chern class of the numerically effective part in the line bundle Zariski decomposition.

Contents

0. Introduction

The Ohsawa-Takegoshi-Manivel L^2 extension theorem addresses the following basic problem.

Problem. *Let Y be a complex analytic submanifold of a complex manifold X ; given a holomorphic function f on Y satisfying suitable L^2 conditions on Y, find a holomorphic extension F of f to X, together with a good L^2 estimate for F on X.*

The first satisfactory solution has been obtained by Ohsawa-Takegoshi [OT87, Ohs88, Ohs94, Ohs95]. We follow here a more geometric approach due to Manivel [Man93], which provides a more general extension theorem in the framework of vector bundles and higher cohomology groups. The first goal of this notes is to simplify further Manivel's approach, as well as to point out a technical difficulty in Manivel's proof. This difficulty occurs in the regularity argument for $(0, q)$ forms, when $q \geqslant 1$; it does not look to be very serious, so we strongly hope that it will be overcome in a new future !

As in Ohsawa-Takegoshi's fundamental paper, the main idea is to use a modified Bochner-Kodaira-Nakano inequality. Such inequalities were originally introduced in the work of Donnelly-Fefferman [DF83] and Donnelly-Xavier [DX84]. The main a priori inequality we are going to use is a simplified (and slightly extended) version of the original Ohsawa-Takegoshi a priori inequality, as proposed recently by Ohsawa [Ohs95]; see also Berndtsson [Ber96] for related calculations in the special case of domains in $\mathbb{C}^n$.

We then describe how the Oshawa-Takegoshi-Manivel extension theorem can be applied to solve several important problems of complex analysis or geometry. The first of these is an approximation theorem for plurisubharmonic functions. It is known since a long time that every plurisubharmonic function can be written as a limit of logarithms of the modulus of holomorphic functions, multiplied by suitable small positive numbers (see Bremermann [Bre54] and Lelong [Lel72]). Here, we show that the approximation can be made with a uniform convergence of the Lelong numbers of the holomorphic functions towards those of the given plurisubharmonic function. This result contains as a special case Siu's theorem [Siu74] on the analyticity of Lelong number sublevel sets. A geometric (more or less equivalent) form of the result is the existence of approximations of an arbitrary closed positive current of type $(1, 1)$ of rational cohomology class by effective rational divisors. Somewhat surprisingly, the proof of all the above only uses the 0-dimensional case of the L^2 extension theorem !

By combining some of the results provided by the proof of that approximation theorem with Skoda's L^2 estimates for the division of holomorphic functions, we obtain a Briançon-Skoda type theorem for Nadel's multiplier ideal sheaves. A weak form of it says that $\mathcal{I}(\ell\varphi) \subset \mathcal{I}(\varphi)^{\ell-n}$ for every plurisubharmonic function on an open set of $\mathbb{C}^n$. This result can in its turn be used to prove Fujita's "asymptotic Zariski decomposition result". The result tells us that if we write a big $\mathbb{Q}$-divisor D as a sum $D = E + A$ with E effective and A ample, then the value of the supremum of A^n is determined by the growth of the number of sections, and equal to $\limsup \frac{n!}{k^n} h^0(X, kL)$ where $n = \dim X$.

I thank wholeheartedly Pierre Dolbeault, Andrei Iordan and Henri Skoda for giving me the opportunity to present this work on the occasion of Pierre Lelong's 85th birthday celebration in September 1997.

1. Notation and general setting

Let X be a complex n-dimensional manifold equipped with a hermitian metric ω, viewed as a positive $(1,1)$-form

$$\omega = \mathrm{i} \sum_{1\leqslant j,k\leqslant n} \omega_{jk}(z)\, dz_j \wedge d\overline{z}_k.$$

The bundle of (p,q)-forms $\Lambda^{p,q}T^\star_X = \Lambda^p T^\star_X \otimes \Lambda^q \overline{T}^\star_X$ then inherits a natural hermitian metric. If (E,h) is a hermitian vector bundle, and $u,v : X \to \Lambda^{p,q}T^\star_X \otimes E$ are (p,q)-forms with values in E with measurable coefficients, we set

$$\|u\|^2 = \int_X |u|^2 dV_\omega, \qquad \langle\!\langle u,v\rangle\!\rangle = \int_X \langle u,v\rangle dV_\omega \tag{1.1}$$

where $|u| = |u|_{\omega,h}$ is the pointwise norm induced by ω and h on $\Lambda^{p,q}T^\star_X \otimes E$, and $dV_\omega = \frac{1}{n!}\omega^n$ is the hermitian volume element. In this way, we obtain a Hilbert space $L^2(X, \Lambda^{p,q}T^\star_X \otimes E)$ of sections, containing the space $\mathcal{D}(X, \Lambda^{p,q}T^\star_X \otimes E)$ of smooth compactly supported sections as a dense subspace.

If we assume that the metric h on E is smooth, there is a unique smooth connection $D = D' + D''$ on E (the so-called *Chern connection* of (E,h)) acting on forms with values in E, such that:

- D' is of pure type $(1,0)$ and D'' is of pure type $(0,1)$;
- D'' coincides with the $\overline{\partial}$ operator;
- D is compatible with h, that is, D satisfies the Leibniz rule

$$d\{u,v\} = \{Du,v\} + (-1)^{\deg u}\{u,Dv\}$$

where $\{\ ,\ \} : \Lambda^{p,q}T^\star_X \otimes E \times \Lambda^{r,s}T^\star_X \otimes E \to \Lambda^{p+s,q+r}T^\star_X$ is the sesquilinear product which combines the the wedge product $(u,v) \mapsto u \wedge \overline{v}$ on scalar valued forms with the hermitian inner product on E.

As usual one can view D', D'' as closed and densely defined operators on the Hilbert space $L^2(X, \Lambda^{p,q}T^\star_X \otimes E)$; the domain of D'', for example, is the set of all $u \in L^2$ such that $D''u$ calculated in the sense of distributions satisfies $D''u \in L^2$. The formal adjoints $D'^\star$, $D''^\star$ also have closed extensions in the sense of distributions, which do not necessarily coincide with the Hilbert space adjoints in the sense of Von Neumann, since the latter ones may have strictly smaller domains. It is well known, however, that the domains coincide if the hermitian metric ω is (geodesically) complete. The complex *Laplace Beltrami operators* are defined by

$$\Delta' = [D', D'^\star] = D'D'^\star + D'^\star D', \qquad \Delta'' = [D'', D''^\star] = D''D''^\star + D''^\star D'' \tag{1.2}$$

where $[A,B] = AB - (-)^{\deg A\,\deg B}BA$ is the graded commutator bracket of operators. Other important operators of hermitian geometry are $Lu := \omega \wedge u$ and its adjoint Λ. Under the assumption that ω is *Kähler*, i.e. $d\omega = 0$, we have the following basic commutation identities:

$$(1.3)\qquad \begin{aligned} &[D''^\star, L] = \mathrm{i}D', && [D'^\star, L] = -\mathrm{i}D'',\\ &[\Lambda, D''] = -\mathrm{i}D'^\star, && [\Lambda, D'] = \mathrm{i}D''^\star. \end{aligned}$$

From there, one gets the fundamental *Bochner-Kodaira-Nakano identity*

$$(1.4)\qquad \Delta'' = \Delta' + [\Lambda, \mathrm{i}\Theta(E)],$$

where $\Theta(E) = D^2 = D'D'' + D''D' \in C^\infty(X, \Lambda^{1,1}T_X^\star \otimes \mathrm{Hom}(E,E))$ is the curvature tensor of E.

2. Basic a priori inequality

The standard L^2 estimates for solutions of $\overline{\partial}$ equations (Andreotti-Vesentini [AV65], Hörmander [Hör65, 66]) are based on a direct application of the Bochner-Kodaira-Nakano identity (1.4). In this setting, the curvature integrals are spread over X and everything goes through in a rather straightforward manner. For the application to the L^2 extension theorem, however, one has to "concentrate" the effect of the curvature around the subvariety from which the extension is to be made. For this, a modified a priori inequality is required, involving "bump functions" in the weight of the L^2 integrals. The following is an improved version, due to Ohsawa [Ohs95] of the original a priori inequality used by Ohsawa-Takegoshi [OT87, Ohs88]. Earlier similar estimates had been used in a different context by Donnelly-Fefferman [DF83] and Donnelly-Xavier [DX84].

(2.1) Lemma ([Ohs95]. *Let E be a hermitian vector bundle on a complex manifold X equipped with a Kähler metric ω. Let η, $\lambda > 0$ be smooth functions on X. Then for every form $u \in \mathcal{D}(X, \Lambda^{p,q}T_X^\star \otimes E)$ with compact support we have*

$$\begin{aligned} &\|(\eta^{\frac12} + \lambda^{\frac12})D''^\star u\|^2 + \|\eta^{\frac12} D'' u\|^2 + \|\lambda^{\frac12} D' u\|^2 + 2\|\lambda^{-\frac12} d'\eta \wedge u\|^2\\ &\qquad\qquad \geqslant \langle\!\langle [\eta\, \mathrm{i}\Theta(E) - \mathrm{i}\, d'd''\eta - \mathrm{i}\lambda^{-1} d'\eta \wedge d''\eta, \Lambda]u, u\rangle\!\rangle. \end{aligned}$$

Proof. Let us consider the "twisted" Laplace-Beltrami operators

$$\begin{aligned} D'\eta D'^\star + D'^\star\eta D' &= \eta[D', D'^\star] + [D', \eta]D'^\star + [D'^\star, \eta]D'\\ &= \eta\Delta' + (d'\eta)D'^\star - (d'\eta)^* D',\\ D''\eta D''^\star + D''^\star\eta D'' &= \eta[D'', D''^\star] + [D'', \eta]D''^\star + [D''^\star, \eta]D''\\ &= \eta\Delta'' + (d''\eta)D''^\star - (d''\eta)^* D'', \end{aligned}$$

where η, $(d'\eta)$, $(d''\eta)$ are abbreviated notations for the multiplication operators $\eta\bullet$, $(d'\eta)\wedge\bullet$, $(d''\eta)\wedge\bullet$. By subtracting the above equalities and taking into account the Bochner-Kodaira-Nakano identity $\Delta'' - \Delta' = [\mathrm{i}\Theta(E), \Lambda]$, we get

$$(2.2)\qquad \begin{aligned} &D''\eta D''^\star + D''^\star\eta D'' - D'\eta D'^\star - D'^\star\eta D'\\ &\qquad = \eta[\mathrm{i}\Theta(E), \Lambda] + (d''\eta)D''^\star - (d''\eta)^\star D'' + (d'\eta)^\star D' - (d'\eta)D'^\star. \end{aligned}$$

Moreover, the Jacobi identity yields

$$[D'', [d'\eta, \Lambda]] - [d'\eta, [\Lambda, D'']] + [\Lambda, [D'', d'\eta]] = 0,$$

whilst $[\Lambda, D''] = -\mathrm{i}D'^\star$ by the basic commutation relations 7.2. A straightforward computation shows that $[D'', d'\eta] = -(d'd''\eta)$ and $[d'\eta, \Lambda] = \mathrm{i}(d''\eta)^\star$. Therefore we get

$$\mathrm{i}[D'', (d''\eta)^\star] + \mathrm{i}[d'\eta, D'^\star] - [\Lambda, (d'd''\eta)] = 0,$$

that is,

$$[\mathrm{i}\, d'd''\eta, \Lambda] = [D'', (d''\eta)^\star] + [D'^\star, d'\eta] = D''(d''\eta)^\star + (d''\eta)^\star D'' + D'^\star(d'\eta) + (d'\eta)D'^\star.$$

After adding this to (2.2), we find

$$\begin{aligned} &D''\eta D''^\star + D''^\star\eta D'' - D'\eta D'^\star - D'^\star\eta D' + [\mathrm{i}\, d'd''\eta, \Lambda] \\ &\qquad = \eta[\mathrm{i}\Theta(E), \Lambda] + (d''\eta)D''^\star + D''(d''\eta)^\star + (d'\eta)^\star D' + D'^\star(d'\eta). \end{aligned}$$

We apply this identity to a form $u \in \mathcal{D}(X, \Lambda^{p,q}T_X^\star \otimes E)$ and take the inner bracket with u. Then

$$\langle\!\langle (D''\eta D''^\star)u, u\rangle\!\rangle = \langle\!\langle \eta D''^\star u, D''^\star u\rangle\!\rangle = \|\eta^{\frac12} D''^\star u\|^2,$$

and likewise for the other similar terms. The above equalities imply

$$\begin{aligned} &\|\eta^{\frac12} D''^\star u\|^2 + \|\eta^{\frac12} D'' u\|^2 - \|\eta^{\frac12} D' u\|^2 - \|\eta^{\frac12} D'^\star u\|^2 = \\ &\qquad \langle\!\langle [\eta\, \mathrm{i}\Theta(E) - \mathrm{i}\, d'd''\eta, \Lambda]u, u\rangle\!\rangle + 2\,\mathrm{Re}\,\langle\!\langle D''^\star u, (d''\eta)^\star u\rangle\!\rangle + 2\,\mathrm{Re}\,\langle\!\langle D'u, d'\eta\wedge u\rangle\!\rangle. \end{aligned}$$

By neglecting the negative terms $-\|\eta^{\frac12} D'u\|^2 - \|\eta^{\frac12} D'^\star u\|^2$ and adding the squares

$$\begin{aligned} \|\lambda^{\frac12} D''^\star u\|^2 + 2\,\mathrm{Re}\,\langle\!\langle D''^\star u, (d''\eta)^\star u\rangle\!\rangle + \|\lambda^{-\frac12}(d''\eta)^\star u\|^2 &\geqslant 0, \\ \|\lambda^{\frac12} D'u\|^2 + 2\,\mathrm{Re}\,\langle\!\langle D'u, d'\eta\wedge u\rangle\!\rangle + \|\lambda^{-\frac12} d'\eta\wedge u\|^2 &\geqslant 0 \end{aligned}$$

we get

$$\begin{aligned} &\|(\eta^{\frac12} + \lambda^{\frac12})D''^\star u\|^2 + \|\eta^{\frac12} D''u\|^2 + \|\lambda^{\frac12} D'u\|^2 + \|\lambda^{-\frac12} d'\eta\wedge u\|^2 + \|\lambda^{-\frac12}(d''\eta)^\star u\|^2 \\ &\qquad\qquad \geqslant \langle\!\langle [\eta\, \mathrm{i}\Theta(E) - \mathrm{i}\, d'd''\eta, \Lambda]u, u\rangle\!\rangle. \end{aligned}$$

Finally, we use the identities

$$\begin{aligned} (d'\eta)^\star(d'\eta) - (d''\eta)(d''\eta)^\star = \mathrm{i}[d''\eta, \Lambda](d'\eta) + \mathrm{i}(d''\eta)[d'\eta, \Lambda] = [\mathrm{i}d''\eta\wedge d'\eta, \Lambda], \\ \|\lambda^{-\frac12} d'\eta\wedge u\|^2 - \|\lambda^{-\frac12}(d''\eta)^\star u\|^2 = -\langle\!\langle [\mathrm{i}\lambda^{-1} d'\eta\wedge d''\eta, \Lambda]u, u\rangle\!\rangle, \end{aligned}$$

The inequality asserted in Lemma 2.1 follows by adding the second identity to our last inequality. □

In the special case of (n, q)-forms, the forms $D'u$ and $d'\eta\wedge u$ are of bidegree $(n+1, q)$, hence the estimate takes the simpler form

$$(2.3)\quad \|(\eta^{\frac12} + \lambda^{\frac12})D''^\star u\|^2 + \|\eta^{\frac12} D''u\|^2 \geqslant \langle\!\langle [\eta\, \mathrm{i}\Theta(E) - \mathrm{i}\, d'd''\eta - \mathrm{i}\lambda^{-1}\, d'\eta\wedge d''\eta, \Lambda]u, u\rangle\!\rangle.$$

3. L^2 existence theorem

By essentially repeating the Hilbert space techniques already used by Kohn [Ko63, 64] and Hörmander [Hör65, 66] in our context, we now derive from (2.3) the following existence theorem.

(3.1) Proposition. *Let X be a complete Kähler manifold equipped with a (non necessarily complete) Kähler metric ω, and let E be a hermitian vector bundle over X. Assume that there are smooth and bounded functions η, $\lambda > 0$ on X such that the (hermitian) curvature operator $B = B^{n,q}_{E,\omega,\eta} = [\eta\,\mathrm{i}\Theta(E) - \mathrm{i}\,d'd''\eta - \mathrm{i}\lambda^{-1}d'\eta\wedge d''\eta, \Lambda_\omega]$ is positive definite everywhere on $\Lambda^{n,q}T^\star_X\otimes E$, for some $q\geqslant 1$. Then for every form $g\in L^2(X,\Lambda^{n,q}T^\star_X\otimes E)$ such that $D''g=0$ and $\int_X\langle B^{-1}g,g\rangle\,dV_\omega<+\infty$, there exists $f\in L^2(X,\Lambda^{n,q-1}T^\star_X\otimes E)$ such that $D''f=g$ and*

$$\int_X(\eta+\lambda)^{-1}|f|^2\,dV_\omega\leqslant 2\int_X\langle B^{-1}g,g\rangle\,dV_\omega.$$

Proof. Let $v\in L^2(X,\Lambda^{n,q}T^\star_X\otimes E)$ be an arbitrary element. Assume first that ω is complete, so that $(\operatorname{Ker}D'')^\perp=\overline{\operatorname{Im}D''^\star}\subset\operatorname{Ker}D''^\star$. Then, by using the decomposition $v=v_1+v_2\in(\operatorname{Ker}D'')\oplus(\operatorname{Ker}D'')^\perp$ and the fact that $g\in\operatorname{Ker}D''$, we infer from Cauchy-Schwarz the inequality

$$|\langle g,v\rangle|^2=|\langle g,v_1\rangle|^2\leqslant\int_X\langle B^{-1}g,g\rangle\,dV_\omega\int_X\langle Bv_1,v_1\rangle\,dV_\omega.$$

We have $v_2\in\operatorname{Ker}D''^\star$, hence $D''^\star v=D''^\star v_1$, and (2.3) implies

$$\int_X\langle Bv_1,v_1\rangle\,dV_\omega\leqslant\|(\eta^{\frac12}+\lambda^{\frac12})D''^\star v_1\|^2+\|\eta^{\frac12}D''v_1\|^2=\|(\eta^{\frac12}+\lambda^{\frac12})D''^\star v\|^2$$

provided that $v\in\operatorname{Dom}D''^\star$. Combining both, we find

$$|\langle g,v\rangle|^2\leqslant\Big(\int_X\langle B^{-1}g,g\rangle\,dV_\omega\Big)\|(\eta^{\frac12}+\lambda^{\frac12})D''^\star v\|^2.$$

This shows the existence of an element $w\in L^2(X,\Lambda^{n,q}T^\star_X\otimes E)$ such that

$$\|w\|^2\leqslant\int_X\langle B^{-1}g,g\rangle\,dV_\omega\qquad\text{and}$$
$$\langle\!\langle v,g\rangle\!\rangle=\langle\!\langle(\eta^{\frac12}+\lambda^{\frac12})D''^\star v,w\rangle\!\rangle\qquad\forall g\in\operatorname{Dom}D''\cap\operatorname{Dom}D''^\star.$$

As $(\eta^{1/2}+\lambda^{\frac12})^2\leqslant 2(\eta+\lambda)$, it follows that $f=(\eta^{1/2}+\lambda^{\frac12})w$ satisfies $D''f=g$ as well as the desired L^2 estimate. If ω is not complete, we set $\omega_\varepsilon=\omega+\varepsilon\widehat\omega$ with some complete Kähler metric $\widehat\omega$. The final conclusion is then obtained by passing to the limit and using a monotonicity argument (the integrals are monotonic with respect to ε). The technique is quite standard and entirely similar to the approach described in [Dem82a], so we will not give any detail here. □

(3.2) Remark. We will also need a variant of the L^2-estimate, so as to obtain approximate solutions with weaker requirements on the data: given $\delta>0$ and

$g \in L^2(X, \Lambda^{n,q}T^\star_X \otimes E)$ such that $D''g = 0$ and $\int_X \langle (B+\delta I)^{-1}g, g\rangle\, dV_\omega < +\infty$, there exists an approximate solution $f \in L^2(X, \Lambda^{n,q-1}T^\star_X \otimes E)$ and a correcting term $h \in L^2(X, \Lambda^{n,q}T^\star_X \otimes E)$ such that $D''f + \delta^{1/2}h = g$ and

$$\int_X (\eta+\lambda)^{-1}|f|^2\, dV_\omega + \int_X |h|^2\, dV_\omega \leqslant 2\int_X \langle (B+\delta I)^{-1}g, g\rangle\, dV_\omega.$$

The proof is almost unchanged, we rely instead on the estimates

$$|\langle g, v_1\rangle|^2 \leqslant \int_X \langle (B+\delta I)^{-1}g, g\rangle\, dV_\omega \int_X \langle (B+\delta I)v_1, v_1\rangle\, dV_\omega,$$

and

$$\int_X \langle (B+\delta I)v_1, v_1\rangle\, dV_\omega \leqslant \|(\eta^{\frac{1}{2}} + \lambda^{\frac{1}{2}})D''^\star v\|^2 + \delta\|v\|^2. \qquad \square$$

4. The Ohsawa-Takegoshi-Manivel L^2 extension theorem

We now derive the basic L^2 extension theorem, by using a variant of the original "weight bumping technique" of Ohsawa-Takegoshi. At this point, our approach is closer to Manivel's exposition [Man93].

(4.1) Theorem. *Let X be a weakly pseudoconvex n-dimensional complex manifold equipped with a Kähler metric ω, let L (resp. E) be a hermitian holomorphic line bundle (resp. a hermitian holomorphic vector bundle of rank r over X), and s a global holomorphic section of E. Assume that s is generically transverse to the zero section, and let*

$$Y = \big\{x \in X\,;\ s(x) = 0, \Lambda^r ds(x) \neq 0\big\}, \qquad p = \dim Y = n - r.$$

Moreover, assume that the $(1,1)$-form $\mathrm{i}\Theta(L) + r\,\mathrm{i}\,d'd''\log|s|^2$ is semipositive and that there is a continuous function $\alpha \geqslant 1$ such that the following two inequalities hold everywhere on X :

a) $\quad \mathrm{i}\Theta(L) + r\,\mathrm{i}\,d'd''\log|s|^2 \geqslant \alpha^{-1}\dfrac{\{\mathrm{i}\Theta(E)s, s\}}{|s|^2}$,

b) $\quad |s| \leqslant e^{-\alpha}$.

Then for every holomorphic section f of $\Lambda^n T^\star_X \otimes L$ over Y, such that

$$\int_Y |f|^2 |\Lambda^r(ds)|^{-2} dV_\omega < +\infty,$$

there exists a holomorphic extension F to X such that $F_{\restriction Y} = f$ and

$$(\star) \qquad \int_X \frac{|F|^2}{|s|^{2r}(-\log|s|)^2}\, dV_{X,\omega} \leqslant C_r \int_Y \frac{|f|^2}{|\Lambda^r(ds)|^2} dV_{Y,\omega}\,,$$

where C_r is a numerical constant depending only on r.

We further state a conjecture for the extension $(0,q)$ forms, a variant of which was claimed as a theorem by L. Manivel [Man93]. The proof given by Manivel is indeed essentially correct, apart from a minor regularity argument which is incorrectly settled in [Man93]. Although we have been unable to fix the difficulty, we strongly hope that it will be overcome in a near future.

(4.2) Conjecture. *If f is a smooth $\overline{\partial}$-closed $(0,q)$-form over Y stisfying the same L^2 condition as above, there exists a locally square integrable extension $(0,q)$-form F over X which is an extension of f, is smooth on $X \smallsetminus \{s = \Lambda^r(ds) = 0\}$, and satisfies $(\star)$.*

(4.3) Remark. Observe that the differential ds (which is intrinsically defined only at points where s vanishes) induces a vector bundle isomorphism $ds : T_X/T_Y \to E$ along Y, hence a non vanishing section $\Lambda^r(ds)$, taking values in

$$\Lambda^r(T_X/T_Y)^\star \otimes \det E \subset \Lambda^r T_X^\star \otimes \det E.$$

The norm $|\Lambda^r(ds)|$ is computed here with respect to the metrics on $\Lambda^r T_X^\star$ and $\det E$ induced by the Kähler metric ω and by the given metric on E. Also notice that hypothesis a) is the only one that really matters: if a) is satisfied for some choice of the function $\alpha \geqslant 1$, one can always achieve b) by multiplying the metric of E with a sufficiently small weight $e^{-\chi\circ\psi}$ (where ψ is a psh exhaustion on X and χ a convex increasing function; property a) remains valid after we multiply the metric of L by $e^{-(r+\alpha_0^{-1})\chi\circ\psi}$, with $\alpha_0 = \inf_{x\in X}\alpha(x)$.

We now split the proof of Theorem 4.1 in several steps, pushing forward the general case of $(0,q)$-forms as long as we can (i.e. until the check of regularity, where we unfortunately got stuck ...). By this we mean that we consider a section f of

$$\Lambda^{0,q}T_Y^* \otimes (\Lambda^n T_X^* \otimes L)_{\restriction Y}$$

with smooth coefficients on the regular part $Y_{\rm reg} \subset Y$, satisfying a further ad hoc L^2 condition on Y.

(4.4) Construction of a smooth extension $\widetilde{f}_\infty$. Let us first assume that the singularity set $\Sigma = \{s = 0\} \cap \{\Lambda^r(ds) = 0\}$ is empty, so that Y is closed and nonsingular. We claim that there exists a smooth section

$$\widetilde{f}_\infty \in C^\infty(X, \Lambda^{n,q}T_X^\star \otimes L) = C^\infty(X, \Lambda^{0,q}T_X^\star \otimes \Lambda^n T_X^\star \otimes L)$$

such that

(a) $\widetilde{f}_\infty$ coincides with f in restriction to Y,

(b) $|\widetilde{f}_\infty| = |f|$ at every point of Y,

(c) $D''\widetilde{f}_\infty = 0$ at every point of Y.

For this, consider coordinates patches $U_j \subset X$ biholomorphic to polydiscs such that $U_j \cap Y = \{z \in U_j\,;\ z_1 = \ldots = z_r = 0\}$ in the corresponding coordinates. We can certainly find a section $\widehat{f} \in C^\infty(X, \Lambda^{n,q}T_X^\star \otimes L)$ which achieves a) and b), since the restriction map $(\Lambda^{0,q}T^*_X)_{\restriction Y} \to \Lambda^{0,q}T^*_Y$ can be viewed as an orthogonal projection onto a C^∞-subbundle of $(\Lambda^{0,q}T^*_X)_{\restriction Y}$. It is enough to extend this subbundle from $U_j \cap Y$ to U_j (e.g. by extending each component of a frame), and then to extend f globally via local smooth extensions and a partition of unity. For any such extension $\widehat{f}$ we have

$$(D''\widehat{f})_{\restriction Y} = (D''\widehat{f}_{\restriction Y}) = D''f = 0.$$

It follows that we can divide $D''\widehat{f} = \sum_{1\leqslant\lambda\leqslant r} g_{j,\lambda}(z) \wedge d\overline{z}_\lambda$ on $U_j \cap Y$, with suitable smooth $(0,q)$-forms $g_{j,\lambda}$ which we also extend arbitrarily from $U_j \cap Y$ to U_j. Then

$$\widetilde{f}_\infty := \widehat{f} - \sum_j \theta_j(z) \sum_{1\leqslant\lambda\leqslant r} \overline{z}_\lambda g_{j,\lambda}(z)$$

coincides with $\widehat{f}$ on Y and satisfies (c).

(4.5) Construction of weights, using a bumping technique. Since we do not know about $\widetilde{f}_\infty$ far away from Y, we will consider a truncation $\widetilde{f}_\varepsilon$ of $\widetilde{f}_\infty$ with support in a small tubular neighborhood $|s| < \varepsilon$ of Y, and solve the equation $D''u_\varepsilon = D''\widetilde{f}_\varepsilon$ with the constraint that u_ε should be 0 on Y. As $\operatorname{codim} Y = r$, this will be the case if we can guarantee that $|u_\varepsilon|^2|s|^{-2r}$ is locally integrable near Y. For this, we apply Proposition 3.1 with a suitable choice of the functions $\eta = \eta_\varepsilon$ and $\lambda = \lambda_\varepsilon$, and an additional weight $|s|^{-2r}$ in the metric of L. The functions η_ε and λ_ε will present carefully constructed "bumps", taking effect on the tubular neighborhood $|s| < \varepsilon$.

Let us consider the smooth strictly convex function $\chi_0 :]-\infty, 0] \to]-\infty, 0]$ defined by $\chi_0(t) = t - \log(1-t)$ for $t \leqslant 0$, which is such that $\chi_0(t) \leqslant t$, $1 \leqslant \chi_0' \leqslant 2$ and $\chi_0''(t) = 1/(1-t)^2$. We set

$$\sigma_\varepsilon = \log(|s|^2 + \varepsilon^2), \qquad \eta_\varepsilon = \varepsilon - \chi_0(\sigma_\varepsilon).$$

As $|s| \leqslant e^{-\alpha} \leqslant e^{-1}$, we have $\sigma_\varepsilon \leqslant 0$ for ε small, and

$$\eta_\varepsilon \geqslant \varepsilon - \sigma_\varepsilon \geqslant \varepsilon - \log(e^{-2\alpha} + \varepsilon^2).$$

Given a relatively compact subset $X_c = \{\psi < c\} \Subset X$, we thus have $\eta_\varepsilon \geqslant 2\alpha$ for $\varepsilon < \varepsilon(c)$ small enough. Simple calculations yield

$$\begin{aligned}
\mathrm{i}\, d'\sigma_\varepsilon &= \frac{\mathrm{i}\{D's, s\}}{|s|^2 + \varepsilon^2},\\
\mathrm{i}\, d'd''\sigma_\varepsilon &= \frac{\mathrm{i}\{D's, D's\}}{|s|^2 + \varepsilon^2} - \frac{\mathrm{i}\{D's, s\} \wedge \{s, D's\}}{(|s|^2 + \varepsilon^2)^2} - \frac{\{\mathrm{i}\Theta_E s, s\}}{|s|^2 + \varepsilon^2}\\
&\geqslant \frac{\varepsilon^2}{|s|^2} \frac{\mathrm{i}\{D's, s\} \wedge \{s, D's\}}{(|s|^2 + \varepsilon^2)^2} - \frac{\{\mathrm{i}\Theta_E s, s\}}{|s|^2 + \varepsilon^2}\\
&\geqslant \frac{\varepsilon^2}{|s|^2} \mathrm{i} d'\sigma_\varepsilon \wedge d''\sigma_\varepsilon - \frac{\{\mathrm{i}\Theta_E s, s\}}{|s|^2 + \varepsilon^2},
\end{aligned}$$

thanks to Lagrange's inequality $\mathrm{i}\{D's, s\} \wedge \{s, D's\} \leqslant |s|^2 \mathrm{i}\{D's, D's\}$. On the other hand, we have $d'\eta_\varepsilon = -\chi_0'(\sigma_\varepsilon) d\sigma_\varepsilon$ with $1 \leqslant \chi_0'(\sigma_\varepsilon) \leqslant 2$, hence

$$
\begin{aligned}
-\mathrm{i}d'd''\eta_\varepsilon &= \chi_0'(\sigma_\varepsilon)\mathrm{i}d'd''\sigma_\varepsilon + \chi_0''(\sigma_\varepsilon)\mathrm{i}d'\sigma_\varepsilon\wedge d''\sigma_\varepsilon \\
&\geqslant \Big(\frac{1}{\chi_0'(\sigma_\varepsilon)}\frac{\varepsilon^2}{|s|^2}+\frac{\chi_0''(\sigma_\varepsilon)}{\chi_0'(\sigma_\varepsilon)^2}\Big)\mathrm{i}d'\eta_\varepsilon\wedge d''\eta_\varepsilon - \chi_0'(\sigma_\varepsilon)\frac{\{\mathrm{i}\Theta_E s,s\}}{|s|^2+\varepsilon^2}.
\end{aligned}
$$

We consider the original metric of L multiplied by the weight $|s|^{-2r}$. In this way, we get a curvature form

$$
\mathrm{i}\Theta_L + r\,\mathrm{i}d'd''\log|s|^2 \geqslant \frac{1}{2}\chi_0'(\sigma_\varepsilon)\alpha^{-1}\frac{\{\mathrm{i}\Theta_E s,s\}}{|s|^2+\varepsilon^2}
$$

by hypothesis a), thanks to the semipositivity of the left hand side and the fact that $\frac{1}{2}\chi_0'(\sigma_\varepsilon)\frac{1}{|s|^2+\varepsilon^2}\leqslant\frac{1}{|s|^2}$. As $\eta_\varepsilon\geqslant 2\alpha$ on X_c for ε small, we infer

$$
\eta_\varepsilon(\mathrm{i}\Theta_L+\mathrm{i}d'd''\log|s|^2) - \mathrm{i}d'd''\eta_\varepsilon - \frac{\chi_0''(\sigma_\varepsilon)}{\chi_0'(\sigma_\varepsilon)^2}\mathrm{i}d'\eta_\varepsilon\wedge d''\eta_\varepsilon \geqslant \frac{\varepsilon^2}{\chi_0'(\sigma_\varepsilon)|s|^2}\mathrm{i}d'\eta_\varepsilon\wedge d''\eta_\varepsilon
$$

on X_c. Hence, if $\lambda_\varepsilon = \chi_0'(\sigma_\varepsilon)^2/\chi_0''(\sigma_\varepsilon)$, we obtain

$$
\begin{aligned}
B_\varepsilon &:= \big[\eta_\varepsilon(\mathrm{i}\Theta_L+\mathrm{i}d'd''\log|s|^2) - \mathrm{i}d'd''\eta_\varepsilon - \lambda_\varepsilon^{-1}\mathrm{i}d'\eta_\varepsilon\wedge d''\eta_\varepsilon\ ,\ \Lambda\big] \\
&\geqslant \Big[\frac{\varepsilon^2}{\chi_0'(\sigma_\varepsilon)|s|^2}\mathrm{i}d'\eta_\varepsilon\wedge d''\eta_\varepsilon\ ,\ \Lambda\Big] = \frac{\varepsilon^2}{\chi_0'(\sigma_\varepsilon)|s|^2}(d''\eta_\varepsilon)(d''\eta_\varepsilon)^\star
\end{aligned}
$$

as an operator on (n,q)-forms (the last equality $[\mathrm{i}\,a\wedge\overline{a},\Lambda]=(a)(a)^\star$ for $a=a^{1,0}$ is easily checked and left as an exercise to the reader; recall that we denote $(a)=a\wedge\bullet$).

(4.6) Solving $\overline{\partial}$ with L^2 estimates, for suitably truncated forms. Let us fix a cut-off function $\theta:\mathbb{R}\to[0,1]$ such that $\theta(t)=1$ on $]-\infty,1/2]$, $\operatorname{Supp}\theta\subset\,]-\infty,1[$ and $|\theta'|\leqslant 3$. For $\varepsilon>0$ small, we consider the (n,q)-form $\widetilde{f}_\varepsilon=\theta(\varepsilon^{-2}|s|^2)\widetilde{f}_\infty$ and its D''-derivative

$$
g_\varepsilon = D''\widetilde{f}_\varepsilon = (1+\varepsilon^{-2}|s|^2)\theta'(\varepsilon^{-2}|s|^2)d''\sigma_\varepsilon\wedge\widetilde{f}_\infty + \theta(\varepsilon^{-2}|s|^2)D''\widetilde{f}_\infty
$$

[as is easily seen from the equality $1+\varepsilon^{-2}|s|^2=\varepsilon^{-2}e^{\sigma_\varepsilon}$]; our later goal is to solve the $\overline{\partial}$ equation $D''u_\varepsilon = g_\varepsilon = D''\widetilde{f}_\varepsilon$. We observe that g_ε has its support contained in the tubular neighborhood $|s|<\varepsilon$; moreover, as $\varepsilon\to 0$, the second term in the right hand side converges uniformly to 0 on every compact set; it will therefore produce no contribution in the limit. On the other hand, the first term has the same order of magnitude as $d''\sigma_\varepsilon$ and $d''\eta_\varepsilon$, and can be controlled in terms of B_ε. In fact, for any (n,q)-form u and any $(n,q+1)$-form v we have

$$
\begin{aligned}
|\langle d''\eta_\varepsilon\wedge u,v\rangle|^2 &= |\langle u,(d''\eta_\varepsilon)^\star v\rangle|^2 \leqslant |u|^2|(d''\eta_\varepsilon)^\star v|^2 = |u|^2\langle (d''\eta_\varepsilon)(d''\eta_\varepsilon)^\star v,v\rangle \\
&\leqslant \frac{\chi_0'(\sigma_\varepsilon)|s|^2}{\varepsilon^2}|u|^2\langle B_\varepsilon v,v\rangle.
\end{aligned}
$$

This implies

$$
\langle B_\varepsilon^{-1}(d''\eta_\varepsilon\wedge u),(d''\eta_\varepsilon\wedge u)\rangle \leqslant \frac{\chi_0'(\sigma_\varepsilon)|s|^2}{\varepsilon^2}|u|^2.
$$

The main term in g_ε can be written

$$
g_\varepsilon^{(1)} := (1+\varepsilon^{-2}|s|^2)\theta'(\varepsilon^{-2}|s|^2)\chi_0'(\sigma_\varepsilon)^{-1}d''\eta_\varepsilon\wedge\widetilde{f}_\infty.
$$

On $\operatorname{Supp} g_\varepsilon^{(1)} \subset \{|s| < \varepsilon\}$, since $\chi_0'(\sigma_\varepsilon) \geqslant 1$, we thus find

$$\langle B_\varepsilon^{-1} g_\varepsilon^{(1)}, g_\varepsilon^{(1)} \rangle \leqslant (1 + \varepsilon^{-2}|s|^2)^2\, \theta'(\varepsilon^{-2}|s|^2)^2 |\widetilde{f}_\infty|^2.$$

Instead of working on X itself, we will work rather on the relatively compact subset $X_c \smallsetminus Y_c$, where $Y_c = Y \cap X_c = Y \cap \{\psi < c\}$. It is easy to check that $X_c \smallsetminus Y_c$ is again complete Kähler (see e.g. [Dem82a]): a Kähler metric of the form

$$\omega_c = C\omega + \mathrm{i}d'd''\Big(\log \frac{1}{c - \psi} + \log|s| - \log(C - \log|s|)\Big), \qquad C \gg 0$$

indeed satisfies $\omega_c \geqslant |d' \log(c - \psi)|^2 + |d' \log(C - \log|s|)|^2$ and is therefore complete Kähler. In this way, we avoid the singularity of the weight $|s|^{-2r}$ along Y. We find

$$\int_{X_c \smallsetminus Y_c} \langle B_\varepsilon^{-1} g_\varepsilon^{(1)}, g_\varepsilon^{(1)} \rangle\, |s|^{-2r} dV_\omega \leqslant \int_{X_c \smallsetminus Y_c} |\widetilde{f}_\infty|^2 (1 + \varepsilon^{-2}|s|^2)^2 \theta'(\varepsilon^{-2}|s|^2)^2 |s|^{-2r} dV_\omega.$$

Now, we let $\varepsilon \to 0$ and view s as "transverse local coordinates" around Y. As $\widetilde{f}_\infty$ coincides with f on Y, it is not hard to see from (4.4 b) that the right hand side converges to $c_r \int_{Y_c} |f|^2 |\Lambda^r(ds)|^{-2} dV_{Y,\omega}$ where c_r is the "universal" constant

$$c_r = \int_{z \in \mathbb{C}^r,\, |z| \leqslant 1} (1 + |z|^2)^2 \theta'(|z|^2)^2 \frac{\mathrm{i}^{r^2} \Lambda^r(dz) \wedge \Lambda^r(d\overline{z})}{|z|^{2r}} < +\infty$$

depending only on r. The second term

$$g_\varepsilon^{(2)} = \theta(\varepsilon^{-2}|s|^2) d'' \widetilde{f}_\infty$$

in g_ε satisfies $\operatorname{Supp}(g_\varepsilon^{(2)}) \subset \{|s| < \varepsilon\}$ and $|g_\varepsilon^{(2)}| = O(|s|)$ (just look at the Taylor expansion of $d'' \widetilde{f}_\infty$ near Y). From this we easily conclude that

$$\int_{X_c \smallsetminus Y_c} \langle B_\varepsilon^{-1} g_\varepsilon^{(2)}, g_\varepsilon^{(2)} \rangle\, |s|^{-2r} dV_{X,\omega} = O(\varepsilon^2),$$

provided that B_ε remains locally uniformly bounded below near Y (this is the case for instance if we have strict inequalities in the curvature assumption a)). If this holds true, we apply Proposition 3.1 on $X_c \smallsetminus Y_c$ with the additional weight factor $|s|^{-2r}$. Otherwise, we use the modified estimate stated in Remark 3.2 to solve the approximate equation $D''u + \delta^{1/2} h = g_\varepsilon$ with $\delta > 0$ small. This yields sections $u = u_{c,\varepsilon,\delta}$, $h = h_{c,\varepsilon,\delta}$ such that

$$\int_{X_c \smallsetminus Y_c} (\eta_\varepsilon + \lambda_\varepsilon)^{-1} |u_{c,\varepsilon,\delta}|^2 |s|^{-2r}\, dV_\omega + \int_{X_c \smallsetminus Y_c} |h_{c,\varepsilon,\delta}|^2 |s|^{-2r}\, dV_\omega$$
$$\leqslant 2 \int_{X_c \smallsetminus Y_c} \langle (B_\varepsilon + \delta I)^{-1} g_\varepsilon, g_\varepsilon \rangle |s|^{-2r}\, dV_\omega,$$

and the right hand side is under control in all cases. The extra error term $\delta^{1/2} h$ can be removed at the end by letting δ tend to 0. Since there is essentially no additional difficulty involved in this process, we will assume for simplicity of exposition that we do have the required lower bound for B_ε and the estimates of $g_\varepsilon^{(1)}$ and $g_\varepsilon^{(2)}$

as above. For $\delta = 0$, the above estimate provides a solution $u_{c,\varepsilon}$ of the equation $D''u_{c,\varepsilon} = g_\varepsilon = D''\widetilde{f}_\varepsilon$ on $X_c \smallsetminus Y_c$, such that

$$\begin{aligned}\int_{X_c\smallsetminus Y_c}(\eta_\varepsilon+\lambda_\varepsilon)^{-1}|u_{c,\varepsilon}|^2|s|^{-2r}dV_{X,\omega} &\leqslant 2\int_{X_c\smallsetminus Y_c}\langle B_\varepsilon^{-1}g_\varepsilon, g_\varepsilon\rangle\,|s|^{-2r}dV_{X,\omega}\\ &\leqslant 2\,c_r\int_{Y_c}\frac{|f|^2}{|\Lambda^r(ds)|^2}dV_{Y,\omega}+O(\varepsilon).\end{aligned}$$

Here we have

$$\begin{aligned}\sigma_\varepsilon &= \log(|s|^2+\varepsilon^2) \leqslant \log(e^{-2\alpha}+\varepsilon^2) \leqslant -2\alpha + O(\varepsilon^2) \leqslant -2+O(\varepsilon^2),\\ \eta_\varepsilon &= \varepsilon - \chi_0(\sigma_\varepsilon) \leqslant (1+O(\varepsilon))\sigma_\varepsilon^2,\\ \lambda_\varepsilon &= \frac{\chi_0'(\sigma_\varepsilon)^2}{\chi_0''(\sigma_\varepsilon)} = (1-\sigma_\varepsilon)^2+(1-\sigma_\varepsilon) \leqslant (3+O(\varepsilon))\sigma_\varepsilon^2,\\ \eta_\varepsilon+\lambda_\varepsilon &\leqslant (4+O(\varepsilon))\sigma_\varepsilon^2 \leqslant (4+O(\varepsilon))\big(-\log(|s|^2+\varepsilon^2)\big)^2.\end{aligned}$$

As $\widetilde{f}_\varepsilon$ is uniformly bounded with support in $\{|s|<\varepsilon\}$, we conclude from an obvious volume estimate that

$$\int_{X_c}\frac{|\widetilde{f}_\varepsilon|^2}{(|s|^2+\varepsilon^2)^r(-\log(|s|^2+\varepsilon^2))^2}dV_{X,\omega} \leqslant \frac{\mathrm{Const}}{(\log\varepsilon)^2}.$$

Therefore, thanks to the usual inequality $|t+u|^2 \leqslant (1+k)|t|^2+(1+k^{-1})|u|^2$ applied to the sum $F_{c,\varepsilon} = \widetilde{f}_\varepsilon - u_{c,\varepsilon}$ with $k = |\log\varepsilon|$, we obtain from our previous estimates

$$\int_{X_c\smallsetminus Y_c}\frac{|F_{c,\varepsilon}|^2}{(|s|^2+\varepsilon^2)^r(-\log(|s|^2+\varepsilon^2))^2}dV_{X,\omega} \leqslant 8\,c_r\int_{Y_c}\frac{|f|^2}{|\Lambda^r(ds)|^2}dV_{Y,\omega}+O(|\log\varepsilon|^{-1}).$$

In addition to this, we have $d''F_{c,\varepsilon} = 0$ on $X_c \smallsetminus Y_c$, by construction. This equation actually extends from $X_c \smallsetminus Y_c$ to X_c because $F_{c,\varepsilon}$ is locally L^2 near Y_c. In fact, we have the following well-known lemma in $\overline{\partial}$-operator theory (see e.g. [Dem82a]).

(4.7) Lemma. *Let Ω be an open subset of $\mathbb{C}^n$ and Y a complex analytic subset of Ω. Assume that v is a $(p,q-1)$-form with L^2_{loc} coefficients and w a (p,q)-form with L^1_{loc} coefficients such that $\overline{\partial}v = w$ on $\Omega \smallsetminus Y$ (in the sense of distribution theory). Then $\overline{\partial}v = w$ on Ω.*

(4.8) Final check, and regularity of the solution. If $q = 0$, then $u_{c,\varepsilon}$ must be smooth also by the ellipticity of $\overline{\partial}$ in bidegree $(0,0)$. The non integrability of the weight $|s|^{-2r}$ along Y shows that $u_{c,\varepsilon}$ vanishes on Y, therefore

$$F_{c,\varepsilon\restriction Y} = \widetilde{f}_{\varepsilon\restriction Y} = \widetilde{f}_{\infty\restriction Y} = f.$$

The theorem and its final estimate are thus obtained by extracting weak limits, first as $\varepsilon \to 0$, and then as $c \to +\infty$. The initial assumption that $\Sigma = \{s = \Lambda^r(ds) = 0\}$ is empty can be easily removed in two steps: i) the result is true if X is Stein, since we can always find a complex hypersurface Z in X such that $\Sigma \subset \overline{Y} \cap Z \subsetneq \overline{Y}$, and then apply the extension theorem on the Stein manifold $X \smallsetminus Z$, in combination with

Lemma 11.10 ; ii) the whole procedure still works when Σ is nowhere dense in $\overline{Y}$ (and possibly nonempty). Indeed local L^2 extensions $\widetilde{f}_j$ still exist by step i) applied on small coordinate balls U_j ; we then set $\widetilde{f}_\infty = \sum \theta_j \widetilde{f}_j$ and observe that $|D''\widetilde{f}_\infty|^2|s|^{-2r}$ is locally integrable, thanks to the estimate $\int_{U_j} |\widetilde{f}_j|^2|s|^{-2r}(\log|s|)^{-2}dV < +\infty$ and the fact that $|\sum d''\theta_j \wedge \widetilde{f}_j| = O(|s|^\delta)$ for suitable $\delta > 0$ [as follows from Hilbert's Nullstensatz applied to $\widetilde{f}_j - \widetilde{f}_k$ at singular points of $\overline{Y}$].

(4.9) Remarks. Before discussing the difficulties to be overcome to reach the required regularity result for bidegrees $(0, q)$, $q \geqslant 1$, we make a few remarks.

a) When $q = 0$, the estimates provided by Theorem 4.1 are independent of the Kähler metric ω. In fact, if f and F are holomorphic sections of $\Lambda^n T^\star_X \otimes L$ over Y (resp. X), viewed as $(n, 0)$-forms with values in L, we can "divide" f by $\Lambda^r(ds) \in \Lambda^r(TX/TY)^\star \otimes \det E$ to get a section $f/\Lambda^r(ds)$ of $\Lambda^p T^\star_Y \otimes L \otimes (\det E)^{-1}$ over Y. We then find

$$|F|^2 dV_{X,\omega} = \mathrm{i}^{n^2}\{F, F\},$$

$$\frac{|f|^2}{|\Lambda^r(ds)|^2} dV_{Y,\omega} = \mathrm{i}^{p^2}\{f/\Lambda^r(ds), f/\Lambda^r(ds)\},$$

where $\{\bullet, \bullet\}$ is the canonical bilinear pairing described in § 1.

b) The hermitian structure on E is not really used in depth. In fact, one only needs E to be equipped with a *Finsler metric*, that is, a smooth complex homogeneous function of degree 2 on E [or equivalently, a smooth hermitian metric on the tautological bundle $\mathcal{O}_{P(E)}(-1)$ of lines of E over the projectivized bundle $P(E)$]. The section s of E induces a section $[s]$ of $P(E)$ over $X \setminus s^{-1}(0)$ and a corresponding section $\widetilde{s}$ of the pull-back line bundle $[s]^\star\mathcal{O}_{P(E)}(-1)$. A trivial check shows that Theorem 4.1 as well as its proof extend to the case of a Finsler metric on E, if we replace everywhere $\{\mathrm{i}\Theta(E)s, s\}$ by $\{\mathrm{i}\Theta([s]^\star\mathcal{O}_{P(E)}(-1))\widetilde{s}, \widetilde{s}\}$ (especially in hypothesis 4.1 a)). A minor issue is that $|\Lambda^r(ds)|$ is (a priori) no longer defined, since no obvious hermitian norm exists on $\det E$. A posteriori, we have the following ad hoc definition of a metric on $(\det E)^\star$ which makes the L^2 estimates work as before: for $x \in X$ and $\xi \in \Lambda^r E^\star_x$, we set

$$|\xi|^2_x = \frac{1}{c_r}\int_{z\in E_x} (1 + |z|^2)^2\theta'(|z|^2)^2\,\frac{\mathrm{i}^{r^2}\xi \wedge \overline{\xi}}{|z|^{2r}}$$

where $|z|$ is the Finsler norm on E_x [the constant c_r is there to make the result agree with the hermitian case; it is not hard to see that this metric does not depend on the choice of θ].

c) Even when $q = 0$, the regularity of $u_{c,\varepsilon,\delta}$ requires some explanations, in case $\delta > 0$. In fact, the equation

$$D''u_{c,\varepsilon,\delta} + \delta^{1/2}h_{c,\varepsilon,\delta} = g_\varepsilon = D''\widetilde{f}_\varepsilon$$

does not immediately imply smoothness of $u_{c,\varepsilon,\delta}$ (since $h_{c,\varepsilon,\delta}$ need not be smooth in general). However, if we take the pair $(u_{c,\varepsilon,\delta}, h_{c,\varepsilon,\delta})$ to be the minimal L^2 solution orthogonal to the kernel of $D'' \oplus \delta^{1/2}\,\mathrm{Id}$, then it must be in the closure of the image of the adjoint operator $D''^* \oplus \delta^{1/2}\,\mathrm{Id}$, i.e. it must satisfy the additional condition $D''^* h_{c,\varepsilon,\delta} = \delta^{1/2} u_{c,\varepsilon,\delta}$, whence $(\Delta'' + \delta\,\mathrm{Id})h_{c,\varepsilon,\delta} = (D''D''^* + \delta\,\mathrm{Id})h_{c,\varepsilon,\delta} = \delta^{1/2} D''\widetilde{f}_\varepsilon$, and therefore $h_{c,\varepsilon,\delta}$ is smooth by the ellipticity of Δ''. □

We now present a few interesting corollaries. The first one is a qualitative surjectivity theorem for restriction morphisms in Dolbeault cohomology.

(4.10) Corollary. *Let X be a weakly pseudoconvex Kähler manifold, E a holomorphic vector bundle of rank r over X, and s a holomorphic section of E which is everywhere transverse to the zero section, $Y = s^{-1}(0)$, and let L be a holomorphic line bundle such that $F = L^{1/r} \otimes E^\star$ is ample (in the sense that the associated $\mathbb{Q}$-line bundle $\pi^\star L^{1/r} \otimes \mathcal{O}_{P(E)}(1)$ is positive on the projectivized bundle $\pi : P(E) \to X$ of lines of E). Then the restriction morphism*

$$H^0(X, \Lambda^n T_X^\star \otimes L) \to H^0\big(Y, (\Lambda^n T_X^\star \otimes L)_{|Y}\big)$$

is surjective.

Note that if conjecture 4.2 were true, we would also get the surjectivity of the restriction morphism

$$H^q(X, \Lambda^n T_X^\star \otimes L) \to H^q\big(Y, (\Lambda^n T_X^\star \otimes L)_{|Y}\big), \qquad \forall q \geqslant 0,$$

as asserted in [Man93]. However, this purely qualitative result is easy to check directly [as we will see below, the real strength of Theorem 4.1 is in the quantitative L^2 estimate]. If $\operatorname{codim} Y = 1$, the hypothesis says that $L \otimes E^\star = L \otimes \mathcal{O}_X(-Y)$ is ample. We then conclude from the exact sequence

$$0 \to \mathcal{O}_X(-Y) \to \mathcal{O}_X \to (i_Y)_\star \mathcal{O}_Y \to 0$$

and the vanishing of $H^{q+1}(X, \Lambda^n T_X^\star \otimes L \otimes \mathcal{O}_X(-Y))$ by Kodaira's theorem. The case $r > 1$ can be easily reduced to the case of codimension 1 by blowing up X along Y (See also § 5 for more explanation on this strategy).

Proof. First assume for simplicity that F is Griffiths positive, i.e. that E has a hermitian metric such that

$$\frac{1}{r}\mathrm{i}\Theta(L) \otimes \mathrm{Id}_E - \mathrm{i}\Theta(E) >_{\mathrm{Grif}} 0.$$

A short computation gives

$$\begin{aligned} \mathrm{i}\,d'd'' \log |s|^2 &= \mathrm{i}\,d'\Big(\frac{\{s, D's\}}{|s|^2}\Big) \\ &= \mathrm{i}\Big(\frac{\{D's, D's\}}{|s|^2} - \frac{\{D's, s\} \wedge \{s, D's\}}{|s|^4} + \frac{\{s, \Theta(E)s\}}{|s|^2}\Big) \geqslant -\frac{\{\mathrm{i}\Theta(E)s, s\}}{|s|^2} \end{aligned}$$

thanks to Lagrange's inequality and the fact that $\Theta(E)$ is antisymmetric. Hence, if δ is a small positive constant such that

$$-\mathrm{i}\Theta(E) + \frac{1}{r}\mathrm{i}\Theta(L) \otimes \mathrm{Id}_E \geqslant_{\mathrm{Grif}} \delta\,\omega \otimes \mathrm{Id}_E > 0,$$

we find

$$\mathrm{i}\Theta(L) + r\,\mathrm{i}\,d'd''\log|s|^2 \geqslant r\delta\,\omega.$$

The compactness of X implies $\mathrm{i}\Theta(E) \leqslant C\omega \otimes \mathrm{Id}_E$ for some $C > 0$. Theorem 4.1 can thus be applied with $\alpha = r\delta/C$ and Corollary 4.10 follows. In the case when $L^{1/r} \otimes E^\star$ is just assumed to be ample, we can apply remark 4.9 b) and use the same arguments (with a Finsler metric on E rather than a hermitian metric). □

Another interesting corollary is the following special case, dealing with bounded pseudoconvex domains $\Omega \Subset \mathbb{C}^n$. Even this simple version retains highly interesting information on the behavior of holomorphic and plurisubharmonic functions.

(4.11) Corollary. *Let $\Omega \subset \mathbb{C}^n$ be a bounded pseudoconvex domain, and let $Y \subset X$ be a nonsingular complex submanifold defined by a section s of some hermitian vector bundle E with* bounded *curvature tensor on Ω. Assume that s is everywhere transverse to the zero section and that $|s| \leqslant e^{-1}$ on Ω. Then there is a constant $C > 0$ (depending only on E), with the following property: for every psh function φ on Ω, every holomorphic function f on Y with $\int_Y |f|^2 |\Lambda^r(ds)|^{-2} e^{-\varphi} dV_Y < +\infty$, there exists an extension F of f to Ω such that*

$$\int_\Omega \frac{|F|^2}{|s|^{2r}(-\log|s|)^2} e^{-\varphi} dV_\Omega \leqslant C \int_Y \frac{|f|^2}{|\Lambda^r(ds)|^2} e^{-\varphi} dV_Y.$$

Proof. We apply essentially the same idea as for the previous corollary, in the special case when $L = \Omega \times \mathbb{C}$ is the trivial bundle equipped with a weight function $e^{-\varphi - A|z|^2}$. The choice of a sufficiently large constant $A > 0$ guarantees that the curvature assumption 4.1 a) is satisfied (A just depends on the presupposed bound for the curvature tensor of E). □

(4.12) Remark. The special case when $Y = \{z_0\}$ is a point is especially interesting. In that case, we just take $s(z) = (e\,\mathrm{diam}\,\Omega)^{-1}(z - z_0)$, viewed as a section of the rank $r = n$ trivial vector bundle $\Omega \times \mathbb{C}^n$ with $|s| \leqslant e^{-1}$. We take $\alpha = 1$ and replace $|s|^{2n}(-\log|s|)^2$ in the denominator by $|s|^{2(n-\varepsilon)}$, using the inequality

$$-\log|s| = \frac{1}{\varepsilon}\log|s|^{-\varepsilon} \leqslant \frac{1}{\varepsilon}|s|^{-\varepsilon}, \qquad \forall\varepsilon > 0.$$

For any given value f_0, we then find a holomorphic function f such that $f(z_0) = f_0$ and

$$\int_\Omega \frac{|f(z)|^2}{|z - z_0|^{2(n-\varepsilon)}} e^{-\varphi(z)} dV_\Omega \leqslant \frac{C_n}{\varepsilon^2(\mathrm{diam}\,\Omega)^{2(n-\varepsilon)}} |f_0|^2 e^{-\varphi(z_0)}.$$

5. Regularity of the solution for bidegrees $(0,q)$, $q \geqslant 1$

When $q \geqslant 1$, the arguments needed to get a smooth solution necessarily involve much more delicate considerations. This is the part where the proof given by Manivel [Man93] appears to be incomplete. Actually, a natural idea is to consider the minimal L^2 solution $u_{c,\varepsilon}$ of the D'' equation considered in § 4, with respect to the weight $(\log(|s|^2+\varepsilon^2))^{-2}|s|^{-2r}$. This minimal solution satisfies

$$(5.1)\qquad D''u_{c,\varepsilon} = g_\varepsilon = D''\widetilde f_\varepsilon, \qquad D''^\star((\log(|s|^2+\varepsilon^2))^{-2}|s|^{-2r}u_{c,\varepsilon}) = 0$$

on $X_c \smallsetminus Y_c$, since $D''^\star((\log(|s|^2+\varepsilon^2))^{-2}|s|^{-2r}\bullet)$ is the adjoint of D'' for the L^2 norms involving the additional weight. The main difficulty lies in the fact that the differential system (5.1) is singular along Y. This forbids the use of a straightforward elliptic regularity argument (as we did for the case $q=0$). We nevertheless discuss a strategy which might possibly lead to C^0 or Hölder regularity – and one could then use conventional regularization techniques to obtain a smooth solution from there.

Case of codimension $r=1$**.** If $r=1$, the subvariety Y is a divisor; therefore, when we consider a D'' equation with values in the line bundle $\Lambda^nT^\star_X\otimes L$, a L^2 solution for the weight $|s|^{-2}$ can be interpreted as a L^2 solution with values in the twisted line bundle $\Lambda^nT^\star_X\otimes L\otimes\mathcal O_X(-Y)$, equipped with a smooth hermitian metric. Hence, if $r=1$, the minimal L^2 solution $u_{c,\varepsilon}$ of the D'' equation considered earlier satisfies the equations

$$D''u_{c,\varepsilon} = g_\varepsilon = D''\widetilde f_\varepsilon, \qquad D''^\star((\log(|s|^2+\varepsilon^2))^{-2}|s|^{-2}u_{c,\varepsilon}) = 0$$

on $X_c \smallsetminus Y_c$. These equations can be rewritten as

$$(5.2)\qquad D''(s^{-1}u_{c,\varepsilon}) = s^{-1}D''\widetilde f_\varepsilon, \qquad D''^\star((\log(|s|^2+\varepsilon^2))^{-2}s^{-1}u_{c,\varepsilon}) = 0$$

where $s^{-1}u_{c,\varepsilon}$ is viewed as a $(0,q)$-form with values in $\Lambda^nT^\star_X\otimes L\otimes\mathcal O_X(-Y)$. By Lemma 4.7, the equalities (5.2) are valid on X_c and not only on $X_c\smallsetminus Y_c$, for $s^{-1}u_{c,\varepsilon}$ is locally L^2 and $s^{-1}D''\widetilde f_\varepsilon$ is locally bounded. From this, we infer that $F_{c,\varepsilon}=\widetilde f_\varepsilon-u_{c,\varepsilon}$ satisfies

$$\begin{aligned}
D''(s^{-1}F_{c,\varepsilon}) &= D''(s^{-1}\widetilde f_\varepsilon) - s^{-1}D''\widetilde f_\varepsilon,\\
D''^\star((\log(|s|^2+\varepsilon^2))^{-2}s^{-1}F_{c,\varepsilon}) &= D''^\star((\log(|s|^2+\varepsilon^2))^{-2}s^{-1}\widetilde f_\varepsilon)\\
&= D''^\star\big((\log(|s|^2+\varepsilon^2))^{-2}\theta(\varepsilon^{-2}|s|^2)s^{-1}\widetilde f_\infty\big).
\end{aligned}$$

It is easy to show that $D''(s^{-1}\widetilde f) - s^{-1}D''\widetilde f$ is independent of the choice of the smooth extension $\widetilde f$ of f (whether $\widetilde f$ is D''-closed or not is irrelevant), and that it is equal to the current $D''(s^{-1})\wedge\widetilde f$ with support in Y. On the other hand, $s^{-1}\widetilde f_\infty$ is locally integrable, hence $\theta(\varepsilon^{-2}|s|^2)s^{-1}\widetilde f_\infty$ converges weakly to 0 as $\varepsilon\to0$. The uniform L^2 estimate on $F_{c,\varepsilon}$ implies that there exists a weak limit $F_{c,\varepsilon}\to F$ in $L^2_{\rm loc}((|s|\log|s|)^{-2})$. From this we easily infer that

$$4(\log(|s|^2+\varepsilon^2))^{-2}s^{-1}F_{c,\varepsilon} \to (\log|s|)^{-2}s^{-1}F$$

in the weak topology of distributions, hence

$$D''(s^{-1}F) = D''(s^{-1}) \wedge \widetilde{f}, \qquad D''^{\star}((\log|s|)^{-2}s^{-1}F) = 0$$

in the limit. This is an elliptic differential system on $X \smallsetminus Y$, therefore F is smooth on $X \smallsetminus Y$. Unfortunately, the above equations do not imply smoothness of the coefficients of F near Y. We hope that *they nevertheless imply Hölder continuity near* Y, for any Hölder exponent $\gamma < 1$.

In order to justify this, we select a smooth local extension $\widetilde{f}$ such that $D''\widetilde{f} = 0$ and $D's \lrcorner \widetilde{f} = 0$ on Y ($\lrcorner$ denotes contraction by $(1,0)$-forms, which is an operator of type $(0,-1)$). The form $\widetilde{f}$ always exists: if the second condition is not satisfied, we can replace $\widetilde{f}$ with $\widetilde{f} - D''(\overline{s}\,h)$, where h is a suitable smooth $(n,q-1)$-form on X ; the values taken by $\widetilde{f}$ on Y are then uniquely defined. We then find

$$D''(s^{-1}(F-\widetilde{f})) = 0, \qquad D''^{\star}((\log|s|)^{-2}s^{-1}(F-\widetilde{f})) = -D''^{\star}((\log|s|)^{-2}s^{-1}\widetilde{f}).$$

The main point with the choice of $\widetilde{f}$ is that no derivative of s contributes in $D''^{\star}((\log|s|)^{-2}s^{-1}\widetilde{f})$, therefore the singularity of this form along Y is at most $(\log|s|)^{-2}s^{-1}$; in particular it is in L^2 (and even a little bit better). We infer that $w := (\log|s|)^{-2}s^{-1}(F-\widetilde{f})$ satisfies

$$D''((\log|s|)^{2}w) = 0, \qquad D''^{\star}w = -D''^{\star}((\log|s|)^{-2}s^{-1}\widetilde{f}). \tag{5.3}$$

This is a smooth elliptic differential system on $X \smallsetminus Y$, satisfied in the sense of distributions on the whole of X, the section w is known to be L^2, and the principal terms in the differential system have mild singularities of the form $(\log|s|)^2$ at worse. Our hope is that one can prove from this that w has singularities of the form $O((\log|s|)^C)$. This would imply

$$F - \widetilde{f} = O\big(|s|(\log|s|)^C\big),$$

and thus F would extend to a continuous form on X, whose restriction to Y is equal to f. From this, it would not be very hard to regularize F further (by local convolution procedures) to get a smooth solution.

Case of arbitrary codimension $r > 1$. When $r > 1$, the above arguments can no longer be applied directly; one possibility to overcome the difficulty is to blow-up Y so as to deal again with the case of a divisor. We may assume that $\Sigma = \emptyset$ (otherwise, we just replace X_c with $X_c \smallsetminus \Sigma$, which is again complete Kähler). Instead of working on $X_c \smallsetminus Y_c$ as we did earlier, we work on the blow-up $\widehat{X}_c$ of X_c along Y_c. If $\mu : \widehat{X}_c \to X_c$ is the blow-up map, $\widehat{Y}_c = \mu^{-1}(Y_c)$ the exceptional divisor and γ a positive constant, we equip $\widehat{X}_c$ with the smooth Kähler metric

$$\widehat{\omega}_\gamma = \mu^\star\omega + \gamma\,(\mathrm{i}d'd''\log|s|^2 + \frac{\mathrm{i}}{r}\Theta(L)) \geqslant \mu^\star\omega.$$

Then the minimal $L^2(\omega_\gamma)$ solution $u_{c,\varepsilon,\gamma}$ satisfies the equations

$$D''u_{c,\varepsilon,\gamma} = \mu^\star g_\varepsilon = D''(\mu^\star\widetilde{f}_\varepsilon), \qquad D''^{\star}_{\omega_\gamma}(|s|^{-2r}u_{c,\varepsilon,\gamma}) = 0$$

on $\widehat{X}_c \smallsetminus \widehat{Y}_c$, and $\widehat{F}_{c,\varepsilon,\gamma} = \mu^\star\widetilde{f}_\varepsilon - u_{c,\varepsilon,\gamma}$ satisfies the L^2 estimate

$$\int_{\widehat{X}_c} \frac{|\widehat{F}_{c,\varepsilon,\gamma}|^2}{(|\widehat{s}|^2+\varepsilon^2)^r(-\log(|\widehat{s}|^2+\varepsilon^2))^2} dV_{\widehat{X}_c,\omega_\gamma} \leqslant 8\,c_r \int_{Y_c} \frac{|f|^2}{|\Lambda^r(ds)|^2} dV_{Y,\omega} + \frac{\mathrm{Const}}{(\log\varepsilon)^2}$$

where $\widehat{s} = s\circ\mu$ [one can use the fact that for every (n,q)-form u, the integrands of $\int |u|^2_\omega dV_\omega$ and $\int \langle B_\omega^{-1}u,u\rangle^2_\omega dV_\omega$ are decreasing functions of ω ; as $\omega_\gamma \geqslant \mu^\star\omega$, we then infer that the right hand side always admits the given ω-estimate as an upper bound; see e.g. [Dem82a] for details]. We can view X_c as a submanifold of the projectivized bundle $P(E)$ of lines of E, and $\mathcal{O}_{\widehat{X}_c}(-\widehat{Y}_c)$ as the restriction to X_c of the tautological line bundle $\mathcal{O}_{P(E)}(-1)$ on $P(E)$. We thus view $\widehat{s}$ as a section of $\mathcal{O}_{\widehat{Y}_c}(-\widehat{Y}_c)$ (actually, $\widehat{s}$ is a generator of that ideal sheaf). Since $|\widehat{s}|^{-2r}|u_{c,\gamma,\varepsilon}|^2$ is locally integrable by construction, we get

$$D''(\widehat{s}^{\,-r}u_{c,\varepsilon,\gamma}) = \widehat{s}^{\,-r}D''(\mu^\star\widetilde{f}_\varepsilon), \qquad D''^\star_{\omega_\gamma}(\widehat{s}^{\,-r}u_{c,\varepsilon,\gamma}) = 0$$

on $\widehat{X}_c$. Thanks to the equality

$$\mu^\star(\Lambda^n T^\star_X) = \Lambda^n T^\star_{\widehat{X}} \otimes \mathcal{O}_{\widehat{X}}(-(r-1)\widehat{Y}),$$

we see that $\mu^\star\widetilde{f}_\varepsilon$ vanishes at order $r-1$ along $\widehat{Y}$. If we view our (n,q)-forms on $\widehat{X}$ rather as $(0,q)$-forms with values in $\mu^\star(\Lambda^n T^\star_X\otimes L)$, we may consider philosophically that we cancel out a factor $\widehat{s}^{\,r-1}$ in the equations. The same proof as in the case of codimension 1 now shows that $\widehat{F}_{c,\varepsilon,\gamma}$ is smooth on $\widehat{X}_c\smallsetminus\widehat{Y}_c$ and has Hölder continuous coefficients on $\widehat{X}_c$; in particular, we have a meaningful restriction equality

$$\widehat{F}_{c,\varepsilon,\gamma\restriction\widehat{Y}_c} = \mu^\star f \qquad \text{in } \mu^\star(\Lambda^n T^\star_X\otimes L)\otimes\Lambda^{0,q}T^\star_{\widehat{X}},\ \text{over } \widehat{X}_c.$$

We now want to take the limit as ε, γ tend to 0 and c tends to $+\infty$. The trouble is that we lose control on the regularity properties as γ goes to zero ($\widehat{\omega}_\gamma$ becomes a degenerate metric on $\widehat{X}_c$ for $\gamma=0$). We can nevertheless let ε go to 0 and then c to $+\infty$. In this way we find a section $\widehat{F}_\gamma$ of $\mu^\star(\Lambda^n T^\star_X\otimes L)\otimes\Lambda^{0,q}T^\star_{\widehat{X}}$ on $\widehat{X}$ such that

$$\widehat{F}_{\gamma\restriction\widehat{Y}} = \mu^\star f \qquad \text{in } \mu^\star(\Lambda^n T^\star_X\otimes L)\otimes\Lambda^{0,q}T^\star_{\widehat{X}}, \tag{5.4}$$

$$\int_{\widehat{X}} \frac{|\widehat{F}_\gamma|^2}{|\widehat{s}|^{2r}(-\log(|\widehat{s}|^2))^2} dV_{\widehat{X},\omega_\gamma} \leqslant 8\,c_r \int_Y \frac{|f|^2}{|\Lambda^r(ds)|^2} dV_{Y,\omega}. \tag{5.5}$$

If what we have said earlier in codimension 1 holds true, then $\widehat{F}_\gamma$ is continuous and we can smooth it further to get a smooth solution on $\widehat{X}$ satisfying essentially the same L^2 estimate. We still have to push forward the solution down to X and obtain an L^2 estimate for it when $\gamma=0$ (and still without losing the regularity of the solution). For this, we observe that there is a commutative diagram

$$\begin{array}{ccc} H^q(X,\Lambda^n T^\star_X\otimes L) & \xrightarrow{\mu^\star} & H^q\big(\widehat{X},\mu^\star(\Lambda^q T^\star_X\otimes L)\big) \\ \text{restr}\big\downarrow & & \big\downarrow\text{restr} \\ H^q(Y,(\Lambda^n T^\star_X\otimes L)_{\restriction Y}) & \xrightarrow{\mu^\star} & H^q\big(\widehat{Y},\mu^\star(\Lambda^n T^\star_X\otimes L)_{\restriction\widehat{Y}}\big). \end{array} \tag{5.6}$$

The horizontal arrows are isomorphisms, thanks to Leray's spectral sequence and the fact that the higher direct sheaves $R^q\mu_\star(\mathcal{O}_{\widehat{X}})$ (resp. $R^q(\mu_{|\widehat{Y}})_\star(\mathcal{O}_{\widehat{Y}})$) are zero on X (resp. Y) for $q \geqslant 1$, by Künneth's formula and the well known cohomological properties of projective spaces. The left hand restriction arrow is surjective by what we have just proved (any $(0,q)$ section becomes L^2 with a suitable rapidly decaying weight $e^{-\chi\circ\psi}$). Hence the right hand vertical arrow is also surjective, and we infer that there is a $\overline{\partial}$-closed form

$$F \in C^\infty\big(X, (\Lambda^n T^\star_X \otimes L) \otimes \Lambda^{0,q}T^\star_X\big)$$

such that $F_{|Y} = f$. [Note: a priori F is obtained only as a cohomology class, since every coboundary form $\overline{\partial}g$ on Y extends to X, we even conclude that the extension exists as a pointwise defined form]. This is anyway enough to conclude the qualitative extension result stated after Corollary 4.10, in the case of arbitrary degree q and arbitrary codimension r. □

6. Approximation of psh functions by logarithms of holomorphic functions

We prove here, as an application of the Ohsawa-Takegoshi extension theorem, that every psh function on a pseudoconvex open set $\Omega \subset \mathbb{C}^n$ can be approximated very accurately by functions of the form $c\log|f|$, where $c > 0$ and f is a holomorphic function. The main idea is taken from [Dem92]. For other applications to algebraic geometry, see [Dem93b] and Demailly-Kollár [DK96]. Recall that the Lelong number of a function $\varphi \in \mathrm{Psh}(\Omega)$ at a point x_0 is defined to be

$$\nu(\varphi, x_0) = \liminf_{z\to x_0} \frac{\log\varphi(z)}{\log|z-x_0|} = \lim_{r\to 0_+} \frac{\sup_{B(x_0,r)}\varphi}{\log r}.$$

In particular, if $\varphi = \log|f|$ with $f \in \mathcal{O}(\Omega)$, then $\nu(\varphi, x_0)$ is equal to the vanishing order $\mathrm{ord}_{x_0}(f) = \sup\{k \in \mathbb{N}\,;\, D^\alpha f(x_0) = 0,\ \forall|\alpha| < k\}$.

(6.1) Theorem. *Let φ be a plurisubharmonic function on a bounded pseudoconvex open set $\Omega \subset \mathbb{C}^n$. For every $m > 0$, let $\mathcal{H}_\Omega(m\varphi)$ be the Hilbert space of holomorphic functions f on Ω such that $\int_\Omega |f|^2 e^{-2m\varphi}d\lambda < +\infty$ and let $\varphi_m = \frac{1}{2m}\log\sum|\sigma_\ell|^2$ where (σ_ℓ) is an orthonormal basis of $\mathcal{H}_\Omega(m\varphi)$. Then there are constants $C_1, C_2 > 0$ independent of m such that*

a) $\displaystyle \varphi(z) - \frac{C_1}{m} \leqslant \varphi_m(z) \leqslant \sup_{|\zeta-z|<r}\varphi(\zeta) + \frac{1}{m}\log\frac{C_2}{r^n}$
for every $z \in \Omega$ and $r < d(z, \partial\Omega)$. In particular, φ_m converges to φ pointwise and in L^1_{loc} topology on Ω when $m \to +\infty$ and

b) $\displaystyle \nu(\varphi, z) - \frac{n}{m} \leqslant \nu(\varphi_m, z) \leqslant \nu(\varphi, z)$ *for every $z \in \Omega$.*

Proof. Note that $\sum|\sigma_\ell(z)|^2$ is the square of the norm of the evaluation linear form $f \mapsto f(z)$ on $\mathcal{H}_\Omega(m\varphi)$. As φ is locally bounded above, the L^2 topology is actually stronger than the topology of uniform convergence on compact subsets of Ω. It follows that the series $\sum |\sigma_\ell|^2$ converges uniformly on Ω and that its sum is real analytic. Moreover we have

$$\varphi_m(z) = \sup_{f\in B(1)} \frac{1}{m} \log|f(z)|$$

where $B(1)$ is the unit ball of $\mathcal{H}_\Omega(m\varphi)$. For $r < d(z, \partial\Omega)$, the mean value inequality applied to the psh function $|f|^2$ implies

$$\begin{aligned}|f(z)|^2 &\leqslant \frac{1}{\pi^n r^{2n}/n!}\int_{|\zeta-z|<r} |f(\zeta)|^2 d\lambda(\zeta)\\ &\leqslant \frac{1}{\pi^n r^{2n}/n!}\exp\Big(2m\sup_{|\zeta-z|<r}\varphi(\zeta)\Big)\int_\Omega |f|^2 e^{-2m\varphi}d\lambda.\end{aligned}$$

If we take the supremum over all $f \in B(1)$ we get

$$\varphi_m(z) \leqslant \sup_{|\zeta-z|<r}\varphi(\zeta) + \frac{1}{2m}\log\frac{1}{\pi^n r^{2n}/n!}$$

and the second inequality in a) is proved. Conversely, the Ohsawa-Takegoshi extension theorem (estimate 4.10) applied to the 0-dimensional subvariety $\{z\} \subset \Omega$ shows that for any $a \in \mathbb{C}$ there is a holomorphic function f on Ω such that $f(z) = a$ and

$$\int_\Omega |f|^2 e^{-2m\varphi} d\lambda \leqslant C_3 |a|^2 e^{-2m\varphi(z)},$$

where C_3 only depends on n and $\operatorname{diam}\Omega$. We fix a such that the right hand side is 1. This gives the other inequality

$$\varphi_m(z) \geqslant \frac{1}{m}\log|a| = \varphi(z) - \frac{\log C_3}{2m}.$$

The above inequality implies $\nu(\varphi_m, z) \leqslant \nu(\varphi, z)$. In the opposite direction, we find

$$\sup_{|x-z|<r}\varphi_m(x) \leqslant \sup_{|\zeta-z|<2r}\varphi(\zeta) + \frac{1}{m}\log\frac{C_2}{r^n}.$$

Divide by $\log r$ and take the limit as r tends to 0. The quotient by $\log r$ of the supremum of a psh function over $B(x,r)$ tends to the Lelong number at x. Thus we obtain

$$\nu(\varphi_m, x) \geqslant \nu(\varphi, x) - \frac{n}{m}. \qquad \square$$

Theorem 6.1 implies in a straightforward manner the deep result of [Siu74] on the analyticity of the Lelong number sublevel sets.

(6.2) Corollary. *Let φ be a plurisubharmonic function on a complex manifold X. Then, for every $c > 0$, the Lelong number sublevel set*

$$E_c(\varphi) = \big\{z \in X\,;\ \nu(\varphi, z) \geqslant c\big\}$$

is an analytic subset of X*.*

Proof. Since analyticity is a local property, it is enough to consider the case of a psh function φ on a pseudoconvex open set $\Omega \subset \mathbb{C}^n$. The inequalities obtained in 6.1 b) imply that

$$E_c(\varphi) = \bigcap_{m \geqslant m_0} E_{c-n/m}(\varphi_m).$$

Now, it is clear that $E_c(\varphi_m)$ is the analytic set defined by the equations $\sigma_\ell^{(\alpha)}(z) = 0$ for all multi-indices α such that $|\alpha| < mc$. Thus $E_c(\varphi)$ is analytic as a (countable) intersection of analytic sets. □

We now translate Theorem 6.1 into a more geometric setting. Let X be a projective manifold and L a line bundle over X. A singular hermitian metric h on L is a metric such that the weight function φ of h is L^1_{loc} in any local trivialization (such that $L_{|U} \simeq U \times \mathbb{C}$ and $\|\xi\|_h = |\xi| e^{-\varphi(x)}$, $\xi \in L_x \simeq \mathbb{C}$). The curvature of L can then be computed in the sense of distributions

$$T = \frac{\mathrm{i}}{2\pi} \Theta_h(L) = \frac{\mathrm{i}}{\pi} \partial\overline{\partial}\varphi,$$

and L is said to be pseudoeffective if L admits a singular hermitian metric h such that the curvature current $T = \frac{\mathrm{i}}{2\pi}\Theta_h(L)$ is semipositive [The weight functions φ of L are thus plurisubharmonic]. Our goal is to approximate T in the weak topology by divisors which have roughly the same Lelong numbers as T. The existence of weak approximations by divisors has already been proved in [Lel72] for currents defined on a pseudoconvex open set $\Omega \subset \mathbb{C}^n$ with $H^2(\Omega, \mathbb{R}) = 0$, and in [Dem92, 93b] in the situation considered here (cf. also [Dem82b], although the argument given there is somewhat incorrect). We take the opportunity to present here a slightly simpler derivation. In what follows, we use an additive notation for $\mathrm{Pic}(X)$, i.e. kL is meant for the line bundle $L^{\otimes k}$.

(6.3) Proposition. *For any* $T = \frac{\mathrm{i}}{2\pi}\Theta_h(L) \geqslant 0$ *and any ample line bundle* F*, there is a sequence of non zero sections* $h_s \in H^0(X, p_sF + q_sL)$ *with* $p_s, q_s > 0$, $\lim q_s = +\infty$ *and* $\lim p_s/q_s = 0$*, such that the divisors* $D_s = \frac{1}{q_s}\mathrm{div}(h_s)$ *satisfy* $T = \lim D_s$ *in the weak topology and* $\sup_{x \in X} |\nu(D_s, x) - \nu(T, x)| \to 0$ *as* $s \to +\infty$*.*

(6.4) Remark. The proof will actually show, with very slight modifications, that Proposition 6.3 also holds when X is a Stein manifold and L is an arbitrary holomorphic line bundle.

Proof. We first use Hörmander's L^2 estimates to construct a suitable family of holomorphic sections and combine this with some ideas of [Lel72] in a second step. Select a smooth metric with positive curvature on F, choose $\omega = \frac{\mathrm{i}}{2\pi}\Theta(F) > 0$ as a Kähler metric on X and fix some large integer k (how large k must be will be specified later). For all $m \geqslant 1$ we define

$$w_m(z) = \sup_{1 \leqslant j \leqslant N} \frac{1}{m} \log \|f_j(z)\|,$$

where $(f_1, \ldots, f_N)$ is an orthonormal basis of the space of sections of $\mathcal{O}(kF + mL)$ with finite global L^2 norm $\int_X \|f\|^2 dV_\omega$. Let e_F and e_L be non vanishing holomorphic sections of F and L on a trivializing open set Ω, and let $e^{-\psi} = \|e_F\|$, $e^{-\varphi} = \|e_L\|$ be the corresponding weights. If f is a section of $\mathcal{O}(kF+mL)$ and if we still denote by f the associated complex valued function on Ω with respect to the holomorphic frame $e_F^k \otimes e_L^m$, we have $\|f(z)\| = |f(z)|e^{-k\psi(z)-m\varphi(z)}$; here φ is plurisubharmonic, ψ is smooth and strictly plurisubharmonic, and $T = \frac{i}{\pi}\partial\overline{\partial}\varphi$, $\omega = \frac{i}{\pi}\partial\overline{\partial}\psi$. In Ω, we can write

$$w_m(z) = \sup_{1\leqslant j\leqslant N} \frac{1}{m}\log|f_j(z)| - \varphi(z) - \frac{k}{m}\psi(z).$$

In particular $T_m := \frac{i}{\pi}\partial\overline{\partial}w_m + T + \frac{k}{m}\omega$ is a closed positive current belonging to the cohomology class $c_1(L) + \frac{k}{m}c_1(F)$.

Step 1. We claim that T_m converges to T as m tends to $+\infty$ and that T_m satisfies the inequalities

$$\nu(T, x) - \frac{n}{m} \leqslant \nu(T_m, x) \leqslant \nu(T, x) \tag{6.4}$$

at every point $x \in X$. Note that T_m is defined on Ω by $T_m = \frac{i}{\pi}\partial\overline{\partial}v_{m,\Omega}$ with

$$v_{m,\Omega}(z) = \sup_{1\leqslant j\leqslant N} \frac{1}{m}\log|f_j(z)|, \qquad \int_\Omega |f_j|^2 e^{-2k\psi-2m\varphi} dV_\omega \leqslant 1.$$

We proceed in the same way as for the proof of Theorem 6.1. We suppose here that Ω is a coordinate open set with analytic coordinates $(z_1, \ldots, z_n)$. Take $z \in \Omega' \Subset \Omega$ and $r \leqslant r_0 = \frac{1}{2}d(\Omega', \partial\Omega)$. By the L^2 estimate and the mean value inequality for subharmonic functions, we obtain

$$|f_j(z)|^2 \leqslant \frac{C_1}{r^{2n}} \int_{|\zeta-z|<r} |f_j(\zeta)|^2 d\lambda(\zeta) \leqslant \frac{C_2}{r^{2n}} \sup_{|\zeta-z|<r} e^{2m\varphi(\zeta)}$$

with constants C_1, C_2 independent of m and r (the smooth function ψ is bounded on any compact subset of Ω). Hence we infer

$$v_{m,\Omega}(z) \leqslant \sup_{|\zeta-z|<r} \varphi(\zeta) + \frac{1}{2s}\log\frac{C_2}{r^{2n}}. \tag{6.5}$$

If we choose for example $r = 1/m$ and use the upper semi-continuity of φ, we infer $\limsup_{s\to+\infty} v_{m,\Omega} \leqslant \varphi$. Moreover, if $\gamma = \nu(\varphi, x) = \nu(T, x)$, then $\varphi(\zeta) \leqslant \gamma\log|\zeta - x| + O(1)$ near x. By taking $r = |z - x|$ in (6.5), we find

$$v_{m,\Omega}(z) \leqslant \sup_{|\zeta-x|<2r} \varphi(\zeta) - \frac{n}{m}\log r + O(1) \leqslant \Big(\gamma - \frac{n}{m}\Big)\log|z - x| + O(1),$$

$$\nu(T_m, x) = \nu(v_{m,\Omega}, x) \geqslant \Big(\gamma - \frac{n}{m}\Big)_+ \geqslant \nu(T, x) - \frac{n}{m}.$$

In the opposite direction, the inequalities require deeper arguments since we actually have to construct sections in $H^0(X, kF+mL)$. Assume that Ω is chosen isomorphic to a bounded pseudoconvex open set in $\mathbb{C}^n$. By the Ohsawa-Takegoshi L^2 extension theorem (remark (4.11)), for every point $x \in \Omega$, there is a holomorphic function g on Ω such that $g(x) = e^{m\varphi(x)}$ and

$$\int_{\Omega} |g(z)|^2 e^{-2m\varphi(z)} d\lambda(z) \leqslant C_3,$$

where C_3 depends only on n and $\mathrm{diam}(\Omega)$. For $x \in \Omega'$, set

$$\sigma(z) = \theta\big(|z-x|/r\big)\, g(z)\, e_F(z)^k \otimes e_L(z)^m, \qquad r = \min\big(1, 2^{-1} d(\Omega', \partial\Omega)\big),$$

where $\theta : \mathbb{R} \to [0,1]$ is a cut-off function such that $\theta(t) = 1$ for $t < 1/2$ and $\theta(t) = 0$ for $t \geqslant 1$. We solve the global equation $\overline{\partial} u = v$ on X with $v = \overline{\partial}\sigma$, after multiplication of the metric of $kF + mL$ with the weight

$$e^{-2n\rho_x(z)}, \qquad \rho_x(z) = \theta\big(|z-x|/r\big) \log|z-x| \leqslant 0.$$

The $(0,1)$-form v can be considered as a $(n,1)$-form with values in the line bundle $\mathcal{O}(-K_X + kF + mL)$ and the resulting curvature form of this bundle is

$$\mathrm{Ricci}(\omega) + k\omega + mT + n\frac{i}{\pi}\partial\overline{\partial}\rho_x.$$

Here the first two summands are smooth, $i\partial\overline{\partial}\rho_x$ is smooth on $X \smallsetminus \{x\}$ and $\geqslant 0$ on $B(x, r/2)$, and T is a positive current. Hence by choosing k large enough, we can suppose that this curvature form is $\geqslant \omega$, uniformly for $x \in \Omega'$. By Hörmander's standard L^2 estimates [AV65, Hör65, 66], we get a solution u on X such that

$$\int_X \|u\|^2 e^{-2n\rho_x} dV_\omega \leqslant C_4 \int_{r/2<|z-x|<r} |g|^2 e^{-2k\psi - 2m\varphi - 2n\rho_x} dV_\omega \leqslant C_5\,;$$

to get the estimate, we observe that v has support in the corona $r/2 < |z-x| < r$ and that ρ_x is bounded there. Thanks to the logarithmic pole of ρ_x, we infer that $u(x) = 0$. Moreover

$$\int_{\Omega} \|\sigma\|^2 dV_\omega \leqslant \int_{\Omega' + B(0,r/2)} |g|^2 e^{-2k\psi - 2m\varphi} dV_\omega \leqslant C_6,$$

hence $f = \sigma - u \in H^0(X, kF + mL)$ satisfies $\int_X \|f\|^2 dV_\omega \leqslant C_7$ and

$$\|f(x)\| = \|\sigma(x)\| = \|g(x)\|\, \|e_F(x)\|^m \|e_L(x)\|^m = \|e_F(x)\|^k = e^{-k\psi(x)}.$$

In our orthonormal basis (f_j), we can write $f = \sum \lambda_j f_j$ with $\sum |\lambda_j|^2 \leqslant C_7$. Therefore

$$e^{-k\psi(x)} = \|f(x)\| \leqslant \sum |\lambda_j| \sup \|f_j(x)\| \leqslant \sqrt{C_7 N}\, e^{m w_m(x)},$$

$$w_m(x) \geqslant \frac{1}{m} \log(C_7 N)^{-1/2} \|f(x)\| \geqslant -\frac{1}{m}\Big(\log(C_7 N)^{1/2} + k\,\psi(x)\Big)$$

where $N = \dim H^0(X, kF + mL) = O(m^n)$. By adding $\varphi + \frac{k}{m}\psi$, we get $v_{m,\Omega} \geqslant \varphi - C_8 m^{-1} \log m$. Thus $\lim_{m\to+\infty} v_{m,\Omega} = \varphi$ everywhere, $T_m = \frac{i}{\pi}\partial\overline{\partial} v_{m,\Omega}$ converges weakly to $T = \frac{i}{\pi}\partial\overline{\partial}\varphi$, and

$$\nu(T_m, x) = \nu(v_{m,\Omega}, x) \leqslant \nu(\varphi, x) = \nu(T, x).$$

Note that $\nu(v_{m,\Omega}, x) = \frac{1}{m} \min \mathrm{ord}_x(f_j)$ where $\mathrm{ord}_x(f_j)$ is the vanishing order of f_j at x, so our initial lower bound for $\nu(T_m, x)$ combined with the last inequality gives

$$\nu(T,x) - \frac{n}{m} \leqslant \frac{1}{m} \min \mathrm{ord}_x(f_j) \leqslant \nu(T,x). \tag{6.6}$$

Step 2: *Construction of the divisors D_s.*

Select sections $(g_1, \ldots, g_N) \in H^0(X, k_0F)$ with k_0 so large that k_0F is very ample, and set

$$h_{\ell,m} = f_1^\ell g_1 + \ldots + f_N^\ell g_N \in H^0\big(X, (k_0 + \ell k)F + \ell m L\big).$$

For almost every N-tuple $(g_1, \ldots, g_N)$, Lemma 6.7 below and the weak continuity of $\partial\overline{\partial}$ show that

$$\Delta_{\ell,m} = \frac{1}{\ell m}\frac{i}{\pi}\partial\overline{\partial}\log|h_{\ell,m}| = \frac{1}{\ell m}\,\mathrm{div}(h_{\ell,m})$$

converges weakly to $T_m = \frac{i}{\pi}\partial\overline{\partial}v_{m,\Omega}$ as ℓ tends to $+\infty$, and that

$$\nu(T_m,x) \leqslant \nu\Big(\frac{1}{\ell m}\Delta_{\ell,m}, x\Big) \leqslant \nu(T,x) + \frac{1}{\ell m}.$$

This, together with the first step, implies the proposition for some subsequence $D_s = \Delta_{\ell(s),s}$, $\ell(s) \gg s \gg 1$. We even obtain the more explicit inequality

$$\nu(T,x) - \frac{n}{m} \leqslant \nu\Big(\frac{1}{\ell m}\Delta_{\ell,m}, x\Big) \leqslant \nu(T,x) + \frac{1}{\ell m}. \qquad \square$$

(6.7) Lemma. *Let Ω be an open subset in $\mathbb{C}^n$ and let $f_1, \ldots, f_N \in H^0(\Omega, \mathcal{O}_\Omega)$ be non zero functions. Let $G \subset H^0(\Omega, \mathcal{O}_\Omega)$ be a finite dimensional subspace whose elements generate all 1-jets at any point of Ω. Finally, set $v = \sup\log|f_j|$ and*

$$h_\ell = f_1^\ell g_1 + \ldots + f_N^\ell g_N, \qquad g_j \in G \smallsetminus \{0\}.$$

Then for all $(g_1, \ldots, g_N)$ in $(G \smallsetminus \{0\})^N$ except a set of measure 0, the sequence $\frac{1}{\ell}\log|h_\ell|$ converges to v in $L^1_{\mathrm{loc}}(\Omega)$ and

$$\nu(v,x) \leqslant \nu\Big(\frac{1}{\ell}\log|h_\ell|\Big) \leqslant \nu(v,x) + \frac{1}{\ell}, \qquad \forall x \in X, \quad \forall \ell \geqslant 1.$$

Proof. The sequence $\frac{1}{\ell}\log|h_\ell|$ is locally uniformly bounded above and we have

$$\lim_{\ell\to+\infty}\frac{1}{\ell}\log\big|h_\ell(z)\big| = v(z)$$

at every point z where all absolute values $|f_j(z)|$ are distinct and all $g_j(z)$ are nonzero. This is a set of full measure in Ω because the sets $\{|f_j|^2 = |f_l|^2,\ j \neq l\}$ and $\{g_j = 0\}$ are real analytic and thus of zero measure (without loss of generality, we may assume that Ω is connected and that the f_j's are not pairwise proportional). The well-known uniform integrability properties of plurisubharmonic functions then show that $\frac{1}{\ell}\log|h_\ell|$ converges to v in $L^1_{\mathrm{loc}}(\Omega)$. It is easy to see that $\nu(v,x)$ is the minimum of the vanishing orders $\mathrm{ord}_x(f_j)$, hence

$$\nu(\log|h_\ell|, x) = \mathrm{ord}_x(h_\ell) \geqslant \ell\,\nu(v,x).$$

In the opposite direction, consider the set $\mathcal{E}_\ell$ of all $(N+1)$-tuples

$$(x, g_1, \ldots, g_N) \in \Omega \times G^N$$

for which $\nu(\log|h_\ell|, x) \geqslant \ell\,\nu(v,x) + 2$. Then $\mathcal{E}_\ell$ is a constructible set in $\Omega \times G^N$: it has a locally finite stratification by analytic sets, since

$$\mathcal{E}_\ell = \bigcup_{s\geqslant 0}\Big(\bigcup_{j,\,|\alpha|=s}\{x\,;\,D^\alpha f_j(x)\neq 0\}\times G^N\Big)\cap\bigcap_{|\beta|\leqslant \ell s+1}\big\{(x,(g_j))\,;\,D^\beta h_\ell(x)=0\big\}.$$

The fiber $\mathcal{E}_\ell\cap(\{x\}\times G^N)$ over a point $x\in\Omega$ where $\nu(v,x)=\min\mathrm{ord}_x(f_j)=s$ is the vector space of N-tuples $(g_j)\in G^N$ satisfying the equations $D^\beta\big(\sum f_j^\ell g_j(x)\big)=0$, $|\beta|\leqslant \ell s+1$. However, if $\mathrm{ord}_x(f_j)=s$, the linear map

$$(0,\ldots,0,g_j,0,\ldots,0)\longmapsto\big(D^\beta(f_j^\ell g_j(x))\big)_{|\beta|\leqslant \ell s+1}$$

has rank $n+1$, because it factorizes into an injective map $J_x^1 g_j \mapsto J_x^{\ell s+1}(f_j^\ell g_j)$. It follows that the fiber $\mathcal{E}_\ell\cap(\{x\}\times G^N)$ has codimension at least $n+1$. Therefore

$$\dim\mathcal{E}_\ell \leqslant \dim(\Omega\times G^N)-(n+1)=\dim G^N - 1$$

and the projection of $\mathcal{E}_\ell$ on G^N has measure zero by Sard's theorem. By definition of $\mathcal{E}_\ell$, any choice of $(g_1,\ldots,g_N)\in G^N\smallsetminus\bigcup_{\ell\geqslant 1}\mathrm{pr}(\mathcal{E}_\ell)$ produces functions h_ℓ such that $\nu(\log|h_\ell|,x)\leqslant \ell\,\nu(v,x)+1$ on Ω. □

7. Multiplier ideal sheaves and the Briançon-Skoda theorem

In this section, we briefly recall the definition and main properties of multiplier ideal sheaves. These have been originally introduced by A. Nadel [Nad89, 90] for the study of the existence of Kähler-Einstein metrics.

(7.1) Definition. *Let φ be a psh function on an open subset $\Omega\subset X$; to φ is associated the ideal subsheaf $\mathcal{I}(\varphi)\subset\mathcal{O}_\Omega$ of germs of holomorphic functions $f\in\mathcal{O}_{\Omega,x}$ such that $|f|^2e^{-2\varphi}$ is integrable with respect to the Lebesgue measure in some local coordinates near x.*

The zero variety $V(\mathcal{I}(\varphi))$ is thus the set of points in a neighborhood of which $e^{-2\varphi}$ is non integrable. Such points occur only if φ has logarithmic poles, in virtue of the following basic Lemma due to Skoda [Sko72a].

(7.2) Lemma (Skoda)**.** *Let φ be a psh function on an open set Ω and let $x\in\Omega$.*

a) *If $\nu(\varphi,x)<1$, then $e^{-2\varphi}$ is integrable in a neighborhood of x, in particular $\mathcal{I}(\varphi)_x=\mathcal{O}_{\Omega,x}$.*

b) *If $\nu(\varphi,x)\geqslant n+s$ for some integer $s\geqslant 0$, then $e^{-2\varphi}\geqslant C|z-x|^{-2n-2s}$ in a neighborhood of x and $\mathcal{I}(\varphi)_x\subset\mathfrak{m}_{\Omega,x}^{s+1}$, where $\mathfrak{m}_{\Omega,x}$ is the maximal ideal of $\mathcal{O}_{\Omega,x}$.*

c) *The zero variety $V(\mathcal{I}(\varphi))$ of $\mathcal{I}(\varphi)$ satisfies*

$$E_n(\varphi) \subset V(\mathcal{I}(\varphi)) \subset E_1(\varphi)$$

where $E_c(\varphi) = \{x \in X\,;\, \nu(\varphi, x) \geqslant c\}$ is the c-sublevel set of Lelong numbers of φ. □

In fact, the ideal sheaf $\mathcal{I}(\varphi)$ is always a coherent ideal sheaf, and therefore its zero variety is an analytic set. This result is due to Nadel ([Nad89], see also [Dem93b]).

(7.3) Proposition (Nadel). *For any psh function φ on $\Omega \subset X$, the sheaf $\mathcal{I}(\varphi)$ is a coherent sheaf of ideals over Ω.* □

Proof. As the main argument will be needed hereafter, we briefly reproduce the argument. As the result is local, we may assume that Ω is a bounded pseudoconvex open set in $\mathbb{C}^n$. Let E be the set of all holomorphic functions f on Ω such that $\int_\Omega |f|^2 e^{-2\varphi}\, d\lambda < +\infty$. By the strong noetherian property of coherent sheaves, the set E generates a coherent ideal sheaf $\mathcal{J} \subset \mathcal{O}_\Omega$. It is clear that $\mathcal{J} \subset \mathcal{I}(\varphi)$; in order to prove the equality, we need only check that $\mathcal{J}_x + \mathcal{I}(\varphi)_x \cap \mathfrak{m}_{\Omega,x}^{s+1} = \mathcal{I}(\varphi)_x$ for every integer s, in view of the Krull lemma. Let $f \in \mathcal{I}(\varphi)_x$ be defined in a neighborhood V of x and let θ be a cut-off function with support in V such that $\theta = 1$ in a neighborhood of x. We solve the equation $\overline{\partial} u = g := \overline{\partial}(\theta f)$ by means of Hörmander's L^2 estimates applied with the strictly plurisubharmonic weight

$$\widetilde{\varphi}(z) = \varphi(z) + (n+s)\log|z-x| + |z|^2.$$

We get a solution u such that $\int_\Omega |u|^2 e^{-2\varphi}|z-x|^{-2(n+s)} d\lambda < \infty$, thus $F = \theta f - u$ is holomorphic, $F \in E$ and $f_x - F_x = u_x \in \mathcal{I}(\varphi)_x \cap \mathfrak{m}_{\Omega,x}^{s+1}$. This proves our contention. □

The importance of multiplier ideal sheaves stems from the following basic vanishing theorem due to Nadel [Nad89] (see also [Dem93b]), which is a direct consequence of the Andreotti-Vesentini-Hörmander L^2 estimates. If (L, h) is a pseudoeffective line bundle, we denote $\mathcal{I}(h) = \mathcal{I}(\varphi)$ where φ is the weight function of h on any trivialization open set.

(7.4) Nadel vanishing theorem. *Let (X, ω) be a compact Kähler manifold, and let L be a holomorphic line bundle over X equipped with a singular hermitian metric h such that $\mathrm{i}\Theta_h(L) \geqslant \varepsilon\omega$ for some continuous positive function ε on X. Then*

$$H^q\big(X, \mathcal{O}(K_X \otimes L) \otimes \mathcal{I}(h)\big) = 0 \qquad \text{for all } q \geqslant 1. \qquad \square$$

Our next goal is to understand somewhat better the behaviour of a multiplier ideal sheaf $\mathcal{I}(k\varphi)$ as k tends to $+\infty$. The intuition is that the ideal grows more or less "linearly". The following result provides a natural inclusion result for multiplier ideal sheaves, inspired by the classical Briançon-Skoda theorem [BS74].

(7.5) Theorem. *Let X be complex n-dimensional manifold and let φ, ψ be plurisubharmonic functions on X. Then for any integer $k \geqslant n$ we have*

$$\mathcal{I}(k\varphi + \psi) \subset \mathcal{I}(\varphi)^{k-n}\mathcal{I}(\psi).$$

Proof. Since the result is local, we can assume that $X = \Omega$ is a bounded pseudoconvex open set. In that case, possibly after shrinking Ω a little bit, the proof of (7.3) shows that $\mathcal{I}(\varphi)$ is generated by a finite number of elements $g = (g_1, \ldots, g_N) \in \mathcal{O}(\Omega)$ such that

$$\int_\Omega |g_j|^2 e^{-2\varphi} d\lambda < +\infty.$$

We set as usual $|g|^2 = |g_1|^2 + \cdots + |g_N|^2$. It is a consequence of estimate (6.1a) (and thus of the Ohsawa-Takegoshi theorem) that $\varphi \leqslant \log|g| + C$ for some constant $C > 0$. Now, let $f \in \mathcal{I}(k\varphi + \psi)_{z_0}$ be a germ of holomorphic function defined on a neighborhood V of $z_0 \in \Omega$. If V is small enough, we have

$$\int_V |f|^2 |g|^{-2k} e^{-2\psi} d\lambda \leqslant C' \int_V |f|^2 e^{-2k\varphi - 2\psi} d\lambda < +\infty.$$

By Skoda's division theorem (in its original form [Sko72b]; see also [Sko78]), this implies that f can be written as

$$f = g \cdot h = \sum_{1 \leqslant j \leqslant N} g_j h_j$$

with a N-tuple $h = (h_1, \ldots, h_N)$ of holomorphic functions h_j such that

$$\int_V |h|^2 |g|^{-2(k-1)} e^{-2\psi} d\lambda \leqslant \frac{k-n+1}{k-n} \int_V |f|^2 |g|^{-2k} e^{-2\psi} d\lambda,$$

provided that $k > n$, i.e. $k - 1 \geqslant n$. By induction, we find a multi-indexed collection $u_\ell = (u_{\ell, j_1 j_2 \ldots j_\ell})$ of holomorphic functions on V such that

$$f = g^\ell \cdot u_\ell = \sum g_{j_1} g_{j_2} \cdots g_{j_\ell} u_{\ell, j_1 j_2 \ldots j_\ell}$$

and

$$\int_V \sum_\ell |u_\ell|^2 |g|^{-2(k-\ell)} e^{-2\psi} d\lambda \leqslant \frac{k-n+1}{k-n+1-\ell} \int_V |f|^2 |g|^{-2k} e^{-2\psi} d\lambda$$

whenever $k - \ell \geqslant n$. The last L^2 inequality shows that $u_\ell \in \mathcal{I}(\psi)_{z_0}$. The theorem follows by taking $\ell = k - n$. □

8. On Fujita's approximate Zariski decomposition of big line bundles

Our goal here is to reprove a result of Fujita [Fuj93], relating the growth of sections of multiples of a line bundle to the Chern numbers of its "largest nef part". Fujita's original proof is by contradiction, using the Hodge index theorem and intersection inequalities. It turns out that Theorem 7.5 (of Briançon-Skoda type) can be used to derive a simple direct proof, based on different techniques (Theorem 8.5 below). The idea arose in the course of discussions with R. Lazarsfeld [Laz99].

Let X be a projective n-dimensional algebraic variety and L a line bundle over X. We define the *volume* of L to be

$$v(L) = \limsup_{k\to+\infty} \frac{n!}{k^n} h^0(X, kL) \in [0, +\infty[.$$

The line bundle is said to be *big* if $v(L) > 0$. If L is ample, we have $h^q(X, kL) = 0$ for $q \geqslant 1$ and $k \gg 1$ by the Kodaira-Serre vanishing theorem, hence

$$h^0(X, kL) \sim \chi(X, kL) \sim \frac{L^n}{n!} k^n$$

by the Riemann-Roch formula. Thus $v(L) = L^n$ ($= c_1(L)^n$) if L is ample. This is still true if L is nef (numerically effective), i.e. if $L\cdot C \geqslant 0$ for every effective curve C. In fact, one can show that $h^q(X, kL) = O(k^{n-q})$ in that case. The following well-known proposition characterizes big line bundles.

(8.1) Proposition. *The line bundle L is big if and only if there a multiple m_0L such that $m_0L = E + A$, where E is an effective divisor and A an ample divisor.*

Proof. If the condition is satisfied, the decomposition $km_0L = kE + kA$ gives rise to an injection $H^0(X, kA) \hookrightarrow H^0(X, km_0L)$, thus $v(L) \geqslant m_0^{-n} v(A) > 0$. Conversely, assume that L is big, and take A to be a very ample nonsingular divisor in X. The exact sequence

$$0 \longrightarrow \mathcal{O}_X(kL - A) \longrightarrow \mathcal{O}_X(kL) \longrightarrow \mathcal{O}_A(kL_{|A}) \longrightarrow 0$$

gives rise to a cohomology exact sequence

$$0 \to H^0(X, kL - A) \longrightarrow H^0(X, kL) \longrightarrow H^0(A, kL_{|A}),$$

and $h^0(A, kL_{|A}) = O(k^{n-1})$ since $\dim A = n - 1$. Now, the assumption that L is big implies that $h^0(X, kL) > ck^n$ for infinitely many k, hence $H^0(X, m_0L - A) \neq 0$ for some large integer m_0. If E is the divisor of a section in $H^0(X, m_0L - A)$, we find $m_0L - A = E$, as required. $\square$

(8.2) Lemma. *Let G be an arbitrary line bundle. For every $\varepsilon > 0$, there exists a positive integer m and a sequence $\ell_\nu \uparrow +\infty$ such that*

$$h^0\big(X, \ell_\nu(mL - G)\big) \geqslant \frac{\ell_\nu^m m^n}{n!}\big(v(L) - \varepsilon\big),$$

in other words, $v(mL - G) \geqslant m^n(v(L) - \varepsilon)$ for m large enough.

Proof. Clearly, $v(mL - G) \geqslant v(mL - (G + E))$ for every effective divisor E. We can take E so large that $G + E$ is very ample, and we are thus reduced to the case where G is very ample by replacing G with $G + E$. By definition of $v(L)$, there exists a sequence $k_\nu \uparrow +\infty$ such that

$$h^0(X, k_\nu L) \geqslant \frac{k_\nu^n}{n!}\Big(v(L) - \frac{\varepsilon}{2}\Big).$$

We take $m \gg 1$ (to be precisely chosen later), and $\ell_\nu = \big[\frac{k_\nu}{m}\big]$, so that $k_\nu = \ell_\nu m + r_\nu$, $0 \leqslant r_\nu < m$. Then

$$\ell_\nu(mL - G) = k_\nu L - (r_\nu L + \ell_\nu G).$$

Fix a constant $a \in \mathbb{N}$ such that $aG - L$ is an effective divisor. Then $r_\nu L \leqslant maG$ (with respect to the cone of effective divisors), hence

$$h^0\big(X, \ell_\nu(mL - G)\big) \geqslant h^0\big(X, k_\nu L - (\ell_\nu + am)G\big).$$

We select a smooth divisor D in the very ample linear system $|G|$. By looking at global sections associated with the exact sequences of sheaves

$$0 \to \mathcal{O}(-(j+1)D) \otimes \mathcal{O}(k_\nu L) \to \mathcal{O}(-jD) \otimes \mathcal{O}(k_\nu L) \to \mathcal{O}_D(k_\nu L - jD) \to 0,$$

$0 \leqslant j < s$, we infer inductively that

$$\begin{aligned} h^0(X, k_\nu L - sD) &\geqslant h^0(X, k_\nu L) - \sum_{0 \leqslant j < s} h^0\big(D, \mathcal{O}_D(k_\nu L - jD)\big) \\ &\geqslant h^0(X, k_\nu L) - s\, h^0\big(D, k_\nu L_{|D}\big) \\ &\geqslant \frac{k_\nu^n}{n!}\Big(v(L) - \frac{\varepsilon}{2}\Big) - s\, C k_\nu^{n-1} \end{aligned}$$

where C depends only on L and G. Hence, by putting $s = \ell_\nu + am$, we get

$$\begin{aligned} h^0\big(X, \ell_\nu(mL - G)\big) &\geqslant \frac{k_\nu^n}{n!}\Big(v(L) - \frac{\varepsilon}{2}\Big) - C(\ell_\nu + am)k_\nu^{n-1} \\ &\geqslant \frac{\ell_\nu^n m^n}{n!}\Big(v(L) - \frac{\varepsilon}{2}\Big) - C(\ell_\nu + am)(\ell_\nu + 1)^{n-1} m^{n-1} \end{aligned}$$

and the desired conclusion follows by taking $\ell_\nu \gg m \gg 1$. □

The next lemma is due to Siu and was first observed in [Siu97] for the proof of the invariance of plurigenera.

(8.3) Lemma. *There exists an ample line bundle G on X such that for every pseudoeffective line bundle (L, h), the sheaf $\mathcal{O}(G + L) \otimes \mathcal{I}(h)$ is generated by its global sections. In fact, G can be chosen as follows: pick any very ample line bundle A, and take G such that $G - (K_X + (n+1)A)$ is ample, e.g. $G = K_X + (n+2)A$.*

Proof. Let φ be the weight of the metric h on a small neighborhood of a point $z_0 \in X$. Assume that we have a local section u of $\mathcal{O}(G+L) \otimes \mathcal{I}(h)$ on a coordinate open ball $B = B(z_0, \delta)$, such that

$$\int_B |u(z)|^2 e^{-2\varphi(z)} |z - z_0|^{-2(n+1)} dV(z) < +\infty.$$

Then Skoda's division theorem [Sko72b] implies $u(z) = \sum (z_j - z_{j,0}) v_j(z)$ with

$$\int_B |v_j(z)|^2 e^{-2\varphi(z)} |z - z_0|^{-2(n+1)} dV(z) < +\infty,$$

in particular $u_{z_0} \in \mathcal{O}(G+L)\otimes\mathcal{I}(h)\otimes\mathfrak{m}_{X,z_0}$. Select a very ample line bundle A on X. We take a basis $\sigma = (\sigma_j)$ of sections of $H^0(X, G\otimes\mathfrak{m}_{X,z_0})$ and multiply the metric h of G by the factor $|\sigma|^{-2(n+1)}$. The weight of the above metric has singularity $(n+1)\log|z-z_0|^2$ at z_0, and its curvature is

$$(\star)\qquad \mathrm{i}\Theta(G) + (n+1)\mathrm{i}\partial\overline{\partial}\log|\sigma|^2 \geqslant \mathrm{i}\Theta(G) - (n+1)\Theta(A).$$

Now, let f be a local section in $H^0(B, \mathcal{O}(G+L)\otimes\mathcal{I}(h))$ on $B = B(z_0,\delta)$, δ small. We solve the global $\overline{\partial}$ equation

$$\overline{\partial}u = \overline{\partial}(\theta f) \qquad \text{on } X$$

with a cut-off function θ supported near z_0 and with the weight associated with our above choice of metric on $G+L$. Thanks to Nadel's theorem 7.4, the solution exists if the metric of $G+L-K_X$ has positive curvature. As $\Theta_h(L)\geqslant 0$ in the sense of currents, $(\star)$ shows that a sufficient condition is $G-K_X-(n+1)A>0$. We then find a smooth solution u such that $u_{z_0}\in\mathcal{O}(G+L)\otimes\mathcal{I}(h)\otimes\mathfrak{m}_{X,z_0}$, hence

$$F := \theta f - u \in H^0(X, \mathcal{O}(G+L)\otimes\mathcal{I}(h))$$

is a global section differing from f by a germ in $\mathcal{O}(G+L)\otimes\mathcal{I}(h)\otimes\mathfrak{m}_{X,z_0}$. Nakayama's lemma implies that $H^0(X,\mathcal{O}(G+L)\otimes\mathcal{I}(h))$ generates the stalks of $\mathcal{O}(G+L)\otimes\mathcal{I}(h)$. □

We further need to invoke the existence of metrics with minimal singularities (the reader will find more details about this topic in [Dem98] and [DPS99]). This result can be seen as the analytic analogue of Zariski decomposition for pseudo-effective divisors. Although algebraic Zariski decomposition does not always exist and is a very hard problem to deal with, it turns out that the existence of metrics with minimal singularities is a simple basic consequence of the existence of upper envelopes of plurisubharmonic functions, as already mentioned in Lelong's early work on the subject.

(8.4) Proposition. *Let L be a pseudoeffective line bundle X. There exists a singular hermitian metric h_0 satisfying*

(i) $\mathrm{i}\Theta_{h_0}(L)\geqslant 0$.

(ii) *h_0 has minimal singularities among all metrics h with $\mathrm{i}\Theta_h(L)\geqslant 0$, i.e. $h_0\leqslant Ch$, where C is a constant, for every such h.*

Moreover, for every nonnegative integer k, there is a natural isomorphism

$$\text{(iii)}\qquad H^0(X,\mathcal{O}(kL)\otimes\mathcal{I}(h_0^k)) \hookrightarrow H^0(X,\mathcal{O}(kL)).$$

Proof. Let us fix once for all a smooth hermitian metric h_∞ on L (with a curvature form $\theta_\infty = \mathrm{i}\Theta_{h_\infty}(L)$ of arbitrary signature). We write any other singular hermitian metric as $h = e^{-\psi}h_\infty$ where $\psi\in L^1_{\mathrm{loc}}(X)$. Since L is pseudoeffective, it is possible to find ψ such that $\mathrm{i}\Theta_h(L) = \mathrm{i}\partial\overline{\partial}\psi + \theta_\infty \geqslant 0$, and we will always assume that this is the case. In particular, ψ is almost plurisubharmonic and locally bounded from above. Also, we only deal with h up to equivalence of singularities, so we may adjust ψ by a constant in such a way that $\sup_X \psi \leqslant 0$. We define

$$\psi_0(x) = \sup\big\{\psi(x)\,;\ \psi \text{ almost psh, } \mathrm{i}\partial\overline{\partial}\psi + \theta_\infty \geqslant 0,\ \sup_X \psi \leqslant 0\big\}.$$

By the well known properties of plurisubharmonic functions (see Lelong [Lel68]), the upper semicontinuous regularization $\psi_0^\star$ is almost plurisubharmonic and satisfies $\mathrm{i}\partial\overline{\partial}\psi_0^\star + \theta_\infty \geqslant 0$, $\sup \psi_0^\star \leqslant 0$. We see that $\psi_0^\star$ concurs in the supremum defining ψ_0, hence $\psi_0^\star = \psi_0$. The metric $h_0 = e^{-\psi_0}\theta_\infty$ is by definition a metric with minimal singularities which satisfies $\mathrm{i}\Theta_{h_0}(L) \geqslant 0$. If $\sigma \in H^0(X, kL)$, we define a corresponding singular hermitian metric on L by

$$\|\xi\|_h = \Big|\frac{\xi^{\otimes k}}{\sigma(x)}\Big|^{1/k}, \qquad \xi \in L_x.$$

Its curvature is $\Theta_h(L) = \frac{1}{k}[Z_\sigma] \geqslant 0$ where Z_σ is the zero divisor of σ, hence (up to a multiplicative constant) $h := e^{-\psi}h_\infty$ also concurs in the definition of ψ_0. From this, we infer $\psi \leqslant \psi_0 + C$, hence

$$\|\sigma\|_{h_0^{\otimes k}} \leqslant e^{kC}\|\sigma\|_{h^{\otimes k}} = e^{kC}.$$

A fortiori σ is a L^2 section with respect to the metric h_0, i.e.

$$\sigma \in H^0(X, \mathcal{O}(kL) \otimes \mathcal{I}(h_0^{\otimes k})),$$

therefore $H^0(X, \mathcal{O}(kL) \otimes \mathcal{I}(h_0^{\otimes k})) = H^0(X, \mathcal{O}(kL))$, as desired. □

We are now ready to prove Fujita's decomposition theorem.

(8.5) Theorem (Fujita). *Let L be a big line bundle. Then for every $\varepsilon > 0$, there exists a modification $\mu : \widetilde{X} \to X$ and a decomposition $\mu^\star L = E + A$, where E is an effective $\mathbb{Q}$-divisor and A an ample $\mathbb{Q}$-divisor, such that $A^n > v(L) - \varepsilon$.*

(8.6) Remark. Of course, if $\mu^\star L = E + A$ with E effective and A nef, we get an injection

$$H^0(\widetilde{X}, kA) \hookrightarrow H^0(\widetilde{X}, kE + kA) = H^0(\widetilde{X}, k\mu^\star L) = H^0(X, kL)$$

for every integer k which is a multiple of the denominator of E, hence $A^n \leqslant v(L)$.

(8.7) Remark. Once Theorem 8.4 is proved, the same kind of argument easily shows that

$$v(L) = \lim_{k\to+\infty} \frac{n!}{k^n} h^0(X, kL),$$

because the formula is true for every ample line bundle A.

Proof of Theorem 8.5. It is enough to prove the theorem with A being a big and nef divisor. In fact, Proposition 8.1 then shows that we can write $A = E' + A'$ where E' is an effective $\mathbb{Q}$-divisor and A' an ample $\mathbb{Q}$-divisor, hence

$$E + A = E + \varepsilon E' + (1-\varepsilon)A + \varepsilon A'$$

where $A'' = (1-\varepsilon)A + \varepsilon A'$ is ample and the intersection number A''^n approaches A^n as closely as we want. Let G be as in Lemma 8.3. Lemma 8.2 implies that $v(mL-G) > m^n(v(L)-\varepsilon)$ for m large. By Proposition 8.4, there exists a hermitian metric h_m of weight φ_m on $mL-G$ such that

$$H^0\big(X, \ell(mL-G)\big) = H^0\big(X, \ell(mL-G)\otimes\mathcal{I}(\ell\varphi_m)\big)$$

for every $\ell \geqslant 0$. We take a smooth modification $\mu : \widetilde{X} \to X$ such that

$$\mu^\star\mathcal{I}(\varphi_m) = \mathcal{O}_{\widetilde{X}}(-E)$$

is an invertible ideal sheaf in $\mathcal{O}_{\widetilde{X}}$. This is possible by taking the blow-up of X with respect to the ideal $\mathcal{I}(\varphi_m)$ and by resolving singularities (Hironaka [Hir64]). Lemma 8.3 applied to $L' = mL - G$ implies that $\mathcal{O}(mL)\otimes\mathcal{I}(\varphi_m)$ is generated by its global sections, hence its pull-back $\mathcal{O}(m\,\mu^\star L - E)$ is also generated. This implies

$$m\,\mu^\star L = E + A$$

where E is an effective divisor and A is a nef (semi-ample) divisor in $\widetilde{X}$. We find

$$\begin{aligned} H^0(\widetilde{X}, \ell A) &= H^0\big(\widetilde{X}, \ell(m\,\mu^\star L - E)\big) \\ &\supset H^0\big(\widetilde{X}, \mu^\star\big(\mathcal{O}(\ell mL)\otimes\mathcal{I}(\varphi_m)^\ell\big)\big) \\ &\supset H^0\big(\widetilde{X}, \mu^\star\big(\mathcal{O}(\ell mL)\otimes\mathcal{I}((\ell+n)\varphi_m)\big)\big), \end{aligned}$$

thanks to the inclusion of sheaves $\mathcal{I}((\ell+n)\varphi_m) \subset \mathcal{I}(\varphi_m)^\ell$ implied by the Briançon-Skoda theorem. Moreover, the direct image $\mu_\star\mu^\star\mathcal{I}(\ell\varphi_m)$ coincides with the integral closure of $\mathcal{I}(\ell\varphi_m)$, hence with $\mathcal{I}(\ell\varphi_m)$, because a multiplier ideal sheaf is always integrally closed. From this we infer

$$\begin{aligned} H^0(\widetilde{X}, \ell A) &\supset H^0\big(X, \mathcal{O}(\ell mL)\otimes\mathcal{I}((\ell+n)\varphi_m)\big) \\ &\supset H^0\big(X, \mathcal{O}((\ell+n)(mL-G))\otimes\mathcal{I}((\ell+n)\varphi_m)\big) \\ &= H^0\big(X, \mathcal{O}((\ell+n)(mL-G))\big). \end{aligned}$$

The final equality is given by (8.4 iii), whereas the second inclusion holds true if $(\ell+n)G - nmL$ is effective. This is certainly the case if $\ell \gg m$. By Lemma 8.2, we find

$$h^0(\widetilde{X}, \ell A) \geqslant \frac{(\ell+n)^n}{n!} m^n\big(v(L)-\varepsilon\big)$$

for infinitely many ℓ, therefore $v(A) = A^n \geqslant m^n(v(L)-\varepsilon)$. Theorem 8.5 is proved, up to a minor change of notation $E \mapsto \frac{1}{m}E$, $A \mapsto \frac{1}{m}A$. $\square$

References

[AN54] **Akizuki, Y., Nakano, S.** *Note on Kodaira-Spencer's proof of Lefschetz theorems,* Proc. Jap. Acad. **30** (1954) 266–272.

[AV65] **Andreotti, A., Vesentini, E.** *Carleman estimates for the Laplace-Beltrami equation in complex manifolds,* Publ. Math. I.H.E.S. **25** (1965) 81–130.

[Ber96] **Berndtsson, B.** *The extension theorem of Ohsawa-Takegoshi and the theorem of Donnelly-Fefferman,* Ann. Inst. Fourier **14** (1996) 1087–1099.

[Boc48] **Bochner, S.** *Curvature and Betti numbers (I) and (II),* Ann. of Math. **49** (1948) 379–390 ; **50** (1949) 77–93.

[Bom70] **Bombieri, E.** *Algebraic values of meromorphic maps,* Invent. Math. **10** (1970) 267–287 and *Addendum*, Invent. Math. **11** (1970), 163–166.

[Bre54] **Bremermann, H.** *Über die Äquivalenz der pseudokonvexen Gebiete und der Holomorphiegebiete im Raum von n komplexen Veränderlichen,* Math. Ann. **128** (1954) 63–91.

[BS74] **Briançon, J., Skoda, H.** *Sur la clôture intégrale d'un idéal de germes de fonctions holomorphes en un point de* $\mathbb{C}^n$, C. R. Acad. Sci. série A **278** (1974) 949–951.

[Dem82a] **Demailly, J.-P.** *Estimations* L^2 *pour l'opérateur* $\overline{\partial}$ *d'un fibré vectoriel holomorphe semi-positif au dessus d'une variété kählérienne complète,* Ann. Sci. Ecole Norm. Sup. **15** (1982) 457–511.

[Dem82b] **Demailly, J.-P.** *Courants positifs extrêmaux et conjecture de Hodge,* Invent. Math. **69** (1982) 347–374.

[Dem92] **Demailly, J.-P.** *Regularization of closed positive currents and intersection theory,* J. Alg. Geom. **1** (1992) 361–409.

[Dem93a] **Demailly, J.-P.** *Monge-Ampère operators, Lelong numbers and intersection theory,* Complex Analysis and Geometry, Univ. Series in Math., edited by V. Ancona and A. Silva, Plenum Press, New-York, (1993), 115–193.

[Dem93b] **Demailly, J.-P.** *A numerical criterion for very ample line bundles,* J. Differential Geom. **37** (1993) 323–374.

[Dem96a] **Demailly, J.-P.** *Effective bounds for very ample line bundles,* Invent. Math. **124** (1996) 243–261.

[Dem96b] **Demailly, J.-P.** *Complex analytic and differential geometry,* preliminary draft, Institut Fourier, $\simeq 650$ p.

[DK96] **Demailly, J.-P., Kollár, J.** *Semicontinuity of complex singularity exponents and Kähler-Einstein metrics on Fano orbifolds,* Manuscript, July 1996.

[DPS99] **Demailly, J.-P., Peternell, T., Schneider, M.** *Pseudo-effective line bundles on projective varieties,* Manuscript in preparation, (1999).

[DF83] **Donnelly H., Fefferman C.** L^2*-cohomology and index theorem for the Bergman metric,* Ann. Math. **118** (1983) 593–618.

[DX84] **Donnelly H., Xavier F.** *On the differential form spectrum of negatively curved Riemann manifolds,* Amer. J. Math. **106** (1984) 169–185.

[Fuj94] **Fujita, T.** *Approximating Zariski decomposition of big line bundles,* Kodai Math. J. **17** (1994) 1–3.

[Gri66] **Griffiths, P.A.** *The extension problem in complex analysis II; embeddings with positive normal bundle,* Amer. J. Math. **88** (1966) 366–446.

[Gri69] **Griffiths, P.A.** *Hermitian differential geometry, Chern classes and positive vector bundles,* Global Analysis, papers in honor of K. Kodaira, Princeton Univ. Press, Princeton (1969) 181–251.

[Hir64] **Hironaka, H.** *Resolution of singularities of an algebraic variety over a field of characteristic zero,* Ann. of Math. **79** (1964) 109–326.

[Hör65] **Hörmander, L.** L^2 *estimates and existence theorems for the* $\overline{\partial}$ *operator,* Acta Math. **113** (1965) 89–152.

[Hör66] **Hörmander, L.** *An introduction to Complex Analysis in several variables,* 1st edition, Elsevier Science Pub., New York, 1966, 3rd revised edition, North-Holland Math. library, Vol 7, Amsterdam (1990).

[Kod53a] **Kodaira, K.** *On cohomology groups of compact analytic varieties with coefficients in some analytic faisceaux,* Proc. Nat. Acad. Sci. U.S.A. **39** (1953) 868–872.

[Kod53b] **Kodaira, K.** *On a differential geometric method in the theory of analytic stacks,* Proc. Nat. Acad. Sci. U.S.A. **39** (1953). 1268–1273

[Kohn63] **Kohn, J.J.** *Harmonic integrals on strongly pseudo-convex manifolds I,* Ann. Math. (2), **78** (1963). 206-213

[**Kohn64**] **Kohn, J.J.** *Harmonic integrals on strongly pseudo-convex manifolds II,* Ann. Math. **79** (1964). 450-472

[**Kod54**] **Kodaira, K.** *On Kähler varieties of restricted type,* Ann. of Math. **60** (1954) 28–48.

[**Laz99**] **Lazarsfeld, R.** *Private communication,* Ann Arbor University, April 1999.

[**Lel68**] **Lelong, P.** *Fonctions plurisousharmoniques et formes différentielles positives,* Dunod, Paris, Gordon &Breach, New York (1968).

[**Lel72**] **Lelong, P.** *Eléments extrêmaux sur le cône des courants positifs fermés,* Séminaire P. Lelong (Analyse), année 1971/72, Lecture Notes in Math., Vol. 332, Springer-Verlag, Berlin, (1972) 112–131.

[**Man93**] **Manivel, L.** *Un théorème de prolongement L^2 de sections holomorphes d'un fibré vectoriel,* Math. Zeitschrift **212** (1993) 107–122.

[**Nak55**] **Nakano, S.** *On complex analytic vector bundles,* J. Math. Soc. Japan **7** (1955) 1–12.

[**Nak73**] **Nakano, S.** *Vanishing theorems for weakly* 1*-complete manifolds,* Number Theory, Algebraic Geometry and Commutative Algebra, in honor of Y. Akizuki, Kinokuniya, Tokyo (1973) 169–179.

[**Nak74**] **Nakano, S.** *Vanishing theorems for weakly* 1*-complete manifolds II,* Publ. R.I.M.S., Kyoto Univ. **10** (1974) 101–110.

[**Ohs88**] **Ohsawa, T.** *On the extension of L^2 holomorphic functions, II,* Publ. RIMS, Kyoto Univ. **24** (1988) 265–275.

[**Ohs94**] **Ohsawa, T.** *On the extension of L^2 holomorphic functions, IV: A new density concept,* Mabuchi, T. (ed.) et al., Geometry and analysis on complex manifolds. Festschrift for Professor S. Kobayashi's 60th birthday. Singapore: World Scientific, (1994) 157–170.

[**Ohs95**] **Ohsawa, T.** *On the extension of L^2 holomorphic functions, III: negligible weights,* Math. Zeitschrift **219** (1995) 215–225.

[**OT87**] **Ohsawa, T., Takegoshi, K.** *On the extension of L^2 holomorphic functions,* Math. Zeitschrift **195** (1987) 197–204.

[**Siu74**] **Siu, Y.T.** *Analyticity of sets associated to Lelong numbers and the extension of closed positive currents,* Invent. Math. **27** (1974) 53–156.

[**Sko72a**] **Skoda, H.** *Sous-ensembles analytiques d'ordre fini ou infini dans $\mathbb{C}^n$,* Bull. Soc. Math. France, **100** (1972) 353–408.

[**Sko72b**] **Skoda, H.** *Application des techniques L^2 à l'étude des idéaux d'une algèbre de fonctions holomorphes avec poids,* Ann. Sci. Ecole Norm. Sup. **5** (1972) 545–579.

[**Sko76**] **Skoda, H.** *Estimations L^2 pour l'opérateur $\overline{\partial}$ et applications arithmétiques,* Séminaire P. Lelong (Analyse), année 1975/76, Lecture Notes in Math., Vol. 538, Springer-Verlag, Berlin (1977) 314–323.

[**Sko78**] **Skoda, H.** *Morphismes surjectifs de fibrés vectoriels semi-positifs,* Ann. Sci. Ecole Norm. Sup. **11** (1978) 577–611.

Jean-Pierre Demailly
Université de Grenoble I
Institut Fourier, BP74
38400 Saint-Martin d'Hères, France
demailly@fourier.ujf-grenoble.fr

(version of June 25, 1999, printed on January 1, 2010, 18:36)