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ON THE OPERATION OF A TUNABLE ELECTROMAGNETIC DETECTOR
FOR GRAVITATIONAL WAVES

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Abstract

The interaction of a gravitational wave with an oscillating e.m. field inside a detector is analyzed in the frame in which the detector's walls are at rest. The interaction is described in terms of a time dependent dielectric tensor and the energy transfer is derived from a generalized Poynting theorem. The energy transfer is maximum when the g.w. frequency matches one of the frequencies of quadrupole moment of the e.m. field. Provided the Q-value of the detector is sufficiently high this method seems particularly suitable for observing g.w.'s emitted by periodic sources.

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I. Introduction

The first experimental attempt to detect gravitational waves (g.w.) (Weber 1969) was sensitive to frequencies of the order of a kHz which are supposed to be present only in the bursts of gravitational energy emitted during supernova collapses. Essentially the same frequency range has been investigated in subsequent experiments conducted in the Soviet Union (Braginskii et al. 1972), in the United Kingdom (Drever et al. 1973), in the United States (Tyson 1973; Levine and Garwin 1973), in Western Europe (Bramanti and Maischberger 1972; Billing et al. 1975), and in Japan (Hirakawa and Narihara 1975).

The detectors used in all these experiments are mechanical quadrupole antennas which can resonate at the gravitational frequency. The power absorbed is proportional to the gravitational energy flux. Other types of detectors, based on the interaction between the g.w. and the electromagnetic field, have been suggested by Braginskii and Menskii (1971). The power absorbed by them is proportional to the gravitational energy flux, or to its square root, depending on the initial state of the e.m. field (Braginskii et al. 1974). The obvious advantage of the electromagnetic detectors is that they can easily be tuned on a wider range of frequencies than the mechanical ones and can thus be made to respond either to the relatively high frequencies present in the gravitational bursts emitted during stellar collapses or

to the low frequencies emitted by continuous periodic sources such as rotating neutron stars and close binary systems. The fluxes in the latter case are considerably smaller than those expected from stellar collapses but, since the emission is continuous, one can take advantage of the possibility of repeating the experiment to reduce the noise to signal ratio.

To compare different sources it is convenient to define the dimensionless quantity ⁽¹⁾

$$h = \frac{1}{\Omega} \sqrt{32 \pi G I} \quad (1.1)$$

where G is the gravitational constant, I is the energy flux and Ω the angular frequency of the g.w. Another way of writing h is

$$h = \frac{G M_{eq}}{r} \quad (1.2)$$

i.e. as the gravitational potential produced by an equivalent mass at the distance r of the source. The equivalent mass M_{eq} is given in terms of the quadrupole moment D of the source by

$$M_{eq} = \frac{2}{3} \left(\frac{2}{5} \right)^{\frac{1}{2}} D \Omega^2 \quad (1.3)$$

(1) We shall set the velocity of light equal to one.

The values of h and I for a few typical sources are given in Table 1 (see, e.g., Press and Thorne 1972; Drever 1977).

The maximum value of h for the Crab pulsar corresponds to the unpalatable assumption that the gravitational braking be of the same order of the electromagnetic braking. The actual value can be several orders of magnitude smaller. The two binaries in the table have h 's of the same order of magnitude of the extragalactic supernovae, but frequencies at least five orders of magnitude smaller.

In the following sections we shall analyze the possibility of detecting these g.w.'s through their effect on the electromagnetic field enclosed inside the detector. We will describe the change of the e.m. field due to the combined effect of the modification of the boundary conditions and of the direct interaction between the e.m. field and the g.w.'s in terms of a dielectric and magnetic susceptibility tensor ϵ which is related to the metric tensor. Moreover, we shall concentrate on the detectors in which the power exchanged between the detector and the g.w. is proportional to the square root of the incoming gravitational flux. It will be shown that the power exchanged is proportional to the e.m. energy W_0 in the detector and is given by:

$$P = \Lambda \text{tr} (\dot{a} \Pi) W_0 \quad (1.4)$$

Here: Λ is a dimensionless coefficient which depends upon the

mechanical properties of the detector, a is the correction to the metric tensor due to the g.w. in the transverse traceless gauge; Π is the dimensionless quadrupole moment associated with the configuration of the e.m. field and the dot denotes time derivative.

The configurations of the detectors that we will discuss consist of two arms at right angles between which the energy oscillates at the frequency of the g.w., Ω . In this way we obtain a resonance between the frequency of the e.m. energy and the time variation of a . Since \dot{a} is proportional to $h\Omega$, the leading contribution to the energy absorbed by the detector during time τ is

$$\Delta W \sim \Lambda W_0 h \Omega \tau \quad (1.5)$$

where the maximum value of τ is of the order of the detector's relaxation time.

Equations (1.4) and (1.5) are valid provided $P \ll IL^2$ where L is the characteristic size of the detector. This condition is easily satisfied in all cases we shall consider.

In the simplified case when the mechanical properties of the detector are described in terms of a single frequency ω_m and a relaxation time $1/(2\gamma_m)$ the coefficient Λ of Eqs. (1.4) and (1.5) becomes

$$\Lambda \approx \frac{\Omega^2}{\Omega^2 - \omega_m^2} \quad \text{for } |\Omega - \omega_m| > \gamma_m \quad (1.6)$$

or

$$\Lambda = \frac{\omega_m}{\gamma_m} = Q_m, \text{ for } \Omega = \omega_m \quad (1.7)$$

In writing these relations we have neglected terms of order $L^2 \Omega^2$ which, for all the detectors we shall discuss, are negligibly small. As one can see from Eq. (1.7), the case of rigid walls, where $\omega_m \gg \Omega$, leads to a strong reduction of the power absorbed.

II. Power Exchange

a) The rest frame

The power exchanged between the g.w. and the e.m. field inside a detector (Eq. (1.4)) can be split into two terms: a direct one arising from the interaction of the g.w. with the e.m. field, and an indirect one due to the effect which the motion of the detector's walls has on the e.m. field through the change of the boundary conditions. The indirect interaction vanishes in the coordinate system in which the walls are at rest (rest frame). This frame depends in a complicated way upon the geometrical structure and the mechanical properties of the detector. We will only consider the simplified case in which the detector's walls are treated as a system of equal particles elastically bound to the center of mass.

We shall analyze the interaction between the g.w. and the e.m. field in the rest frame of the detector to first order in the amplitude h of the g.w. We shall assume that the size of the detector is small compared to the wave length of the g.w. and neglect terms of order $h(\Omega L)^2$ in the metric tensor in the rest frame. The relation between the definition of the rest frame and the mechanical properties of the detector is derived in the Appendix together with the corrections due to the finite size of the detector.

Consider a coordinate system $x^\mu(\lambda)$ depending on a parame-

ter λ determined in such a way as to transform away the motion of the detector's walls induced by the g.w. This frame satisfies the following conditions:

- (i) In the absence of the g.w. it reduces to the same Minkowski frame.
- (ii) The origin coincides with the center of mass of the detector.
- (iii) The wave vector k_μ of the g.w. is $k_\mu = (k_0, 0, 0, -k_3)$, where $k_0 = k_3 = \Omega$ is the g.w.'s frequency.
- (iv) The metric tensor $g_{\mu\nu}(\lambda)$ is of the form ^(*)

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}(\lambda) \quad (2.1)$$

where $\eta_{\mu\nu}$ is the Minkowski tensor and $h_{\mu\nu}(\lambda)$, which represents the contribution of the g.w., is of the form

$$h_{\mu\nu}(\lambda) = (2\lambda - 1) \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & a_{11} & a_{12} & 0 \\ 0 & a_{21} & a_{22} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + O(h \Omega^2 L^2) \quad (2.2)$$

(*) We shall use the sign convention (+, -, -, -); greek indices run from 0 to 3, latin indices from 1 to 2; summation over repeated indices is implied.

where

$$a_{ij} = h \operatorname{Re} \begin{pmatrix} \alpha & \beta \\ \beta & -\alpha \end{pmatrix} ; \quad a_{ij} = a_{ji} , \quad \operatorname{tr} a = 0 \quad (2.3)$$

with $i, j = 1, 2$ and

$$\begin{aligned} \alpha &= \alpha_0 \exp i [\Omega (x^3 - x^0) + \theta_\alpha] \\ \beta &= \beta_0 \exp i [\Omega (x^3 - x^0) + \theta_\beta] \\ \alpha_0, \beta_0 &\geq 0 , \quad \alpha_0^2 + \beta_0^2 = 1 . \end{aligned} \quad (2.4)$$

In the simplified case when the detector can be treated as a system of oscillators with frequency ω_m and relaxation time $\tau_m = \frac{1}{2} \gamma_m$, the value of λ which corresponds to the rest frame of the detector is given by (see Appendix)

$$\lambda = \frac{1}{2} \frac{\omega_m^2}{\omega_m^2 - \Omega} \quad (2.5)$$

if $\omega_m \neq \Omega$ and by

$$|\lambda| = \frac{\omega_m}{\gamma_m} = Q_m \quad (2.6)$$

if the resonance condition $\omega_m = \Omega$ is met. In the latter case the phases ϑ_α and ϑ_β in Eq.(2.4) are shifted by $\pi/2$.

b) The dielectric tensor

As is well known a g.w. propagating in vacuum induces a time dependent dielectric tensor proportional to its amplitude. This dielectric tensor depends upon the reference frame and we will calculate it in the rest frame. In this frame the indirect interaction vanishes and the effect of the g.w. on the e.m. field is entirely taken into account by the dielectric tensor.

Let us define as usual (Landau and Lifschitz 1963) the electric field $\underline{E} = (E_1, E_2, E_3)$ and magnetic flux density $\underline{B} = (B_1, B_2, B_3)$ by

$$\begin{pmatrix} 0 & E_1 & E_2 & E_3 \\ -E_1 & 0 & B_3 & -B_2 \\ -E_2 & -B_3 & 0 & B_1 \\ -E_3 & B_2 & -B_1 & 0 \end{pmatrix} = F_{\mu\nu} \quad (2.7)$$

where $F_{\mu\nu}$ is the e.m. field tensor. The displacement $\underline{D} = (D^1, D^2, D^3)$ and the magnetic field $\underline{H} = (H^1, H^2, H^3)$ are defined by

$$\sqrt{g_{00}} F^{\mu\nu} = \begin{pmatrix} 0 & -D^1 & -D^2 & -D^3 \\ D^1 & 0 & H^3 & -H^2 \\ D^2 & -H^3 & 0 & H^1 \\ D^3 & H^2 & -H^1 & 0 \end{pmatrix} = \sqrt{g_{00}} g^{\mu\alpha} F_{\alpha\beta} g^{\beta\nu} \quad (2.8)$$

From (2.7), (2.8) and (2.2) it follows

$$D^i = \epsilon^{ij} E_j, \quad D^3 = E_3, \quad B_i = \mu_{ij} H^j, \quad B_3 = H^3 \quad (2.9)$$

where $i, j = 1, 2$ and

$$\begin{aligned} \epsilon^{ij} &= -[\eta^{ij} + (1-2\lambda) a^{ij}] \\ \mu_{ij} &= -[\eta_{ij} + (1-2\lambda) a_{ij}] . \end{aligned} \quad (2.10)$$

The value of each component of the dielectric tensor ϵ^{ij} is equal to the corresponding component of the magnetic susceptibility tensor μ_{ij} . The corrections to Eqs. (2.9) and (2.10) which are introduced by the finite size of the detector are given in the Appendix.

Equations (2.10) show that the dielectric tensor in the rest frame depends upon the mechanical properties of the detector: in the $\lambda = \frac{1}{2}$ frame it reduces to the identity, whereas in the free falling frame ($\lambda = 0$) we have

$$\epsilon^{ij} = -[\eta^{ij} + a^{ij}] . \quad (2.11)$$

In the case corresponding to Eq. (2.6) ($\Omega = \omega_m$) the dielectric tensor becomes

$$\epsilon^{ij} = -[\eta^{ij} + 2 Q_m a^{ij}(\theta_\alpha - \pi/2, \theta_\beta - \pi/2)] . \quad (2.12)$$

The correction to the vacuum value of the dielectric tensor, $-\eta^{ij}$, reduces in this case to the product of the mechanical quality factor Q_m of the detector times the amplitude of the g.w. Due to the resonance, the tensor a^{ij} appears in Eq. (2.12) with phases shifted by $\pi/2$ with respect to the phases θ_α and θ_β of the g.w.

c) The Poynting theorem

The fields \underline{E} , \underline{B} , \underline{D} , \underline{H} defined in sec. II b) satisfy the standard Maxwell equations (with $\underline{x} = (x^1, x^2, x^3)$, $t = x^0$). From them one derives the Poynting theorem in the usual way:

$$\operatorname{div} (\underline{E} \times \underline{H}) + \frac{1}{2} \frac{\partial}{\partial t} (\underline{E} \cdot \underline{D} + \underline{B} \cdot \underline{H}) = - \frac{1}{2} \operatorname{tr} (\dot{\epsilon} T) - 4\pi \underline{j} \cdot \underline{E} \quad (2.13)$$

Here $\underline{j} = (j^1, j^2, j^3)$ is the current density, T_{ij} are the components of the energy momentum tensor in the (1-2) plane and

$$\begin{aligned} \operatorname{tr} (\dot{\epsilon} T) &= \dot{\epsilon}^{ij} T_{ij} = -(1-2\lambda) \dot{a}^{ij} T_{ij} = \\ &= - \frac{(1-2\lambda)}{8\pi} [\dot{a}_{11} (B_1^2 + E_1^2 - B_2^2 - E_2^2) + 2 \dot{a}_{12} (B_1 B_2 + E_1 E_2)] \end{aligned} \quad (2.14)$$

In deriving Eq. (2.14) we have made use of Eqs. (2.10) which,

as shown in the Appendix, are correct up to terms of order $h(\Omega L)^2$.

To derive the power exchanged between the g.w. and the detector we integrate Eq. (2.13) over the volume of the detector. Assuming that $\underline{E} \times \underline{B}$ vanishes on the walls, we obtain:

$$\frac{dW}{dt} = W \text{tr}(\dot{\epsilon} \Pi) - \gamma_{\text{e.m.}} W \quad (2.15)$$

where:

$$W = \frac{1}{8\pi} \int d^3 \underline{x} (\underline{E} \cdot \underline{D} + \underline{B} \cdot \underline{H}) \quad (2.16)$$

is the e.m. energy inside the detector;

$$\Pi_{11} = -\Pi_{22} = \frac{1}{2W} \int d^3 \underline{x} (B_1^2 + E_1^2 - B_2^2 - E_2^2) \quad (2.17)$$

$$\Pi_{12} = \Pi_{21} = \frac{1}{W} \int d^3 \underline{x} (B_1 B_2 + E_1 E_2)$$

and

$$\gamma_{\text{e.m.}} = 2 \frac{1}{W} \int d^3 \underline{x} \underline{j} \cdot \underline{E} \quad (2.18)$$

Equation (2.15) has been obtained by neglecting terms of order $h\Omega L$ due to the x^3 -dependence of ϵ .

The tensor Π , which in general depends upon time, represents the (dimensionless) quadrupole moment of the e.m. field in the detector, and $1/\gamma_{\text{e.m.}}$ is the e.m. relaxation time of the detector.

The second term in the right hand side of Eq. (2.15), which is due to the losses of the detector, is, in all realistic cases, much larger than the first one which arises from the interaction with the g.w.

Integrating Eq.(2.15) we get

$$W(t) = W_0 \left[1 + \int_0^t \text{tr} (\dot{\epsilon}(t') \Pi(t')) dt' \right] \exp(-\gamma_{\text{e.m.}} t) \quad (2.19)$$

where W_0 is the e.m. energy in the detector at $t = 0$. The second term in the squarebrackets, which is correct only to first order in $h\Omega/\gamma_{\text{e.m.}} \ll 1$, depends upon the intensity, the polarization, the phases and the frequency of the g.w., through $\dot{\epsilon}$ and upon the geometry of the detector through the quadrupole moment Π . This term changes sign when the configuration of the detector is rotated by $\pi/2$ around the x^3 -axis. This behaviour reflects the rotational properties of the spin two carried by the g.w.

Let us define

$$\Delta W(t) = W(t, h) - W(t, 0) = W_0 \exp(-\gamma_{\text{e.m.}} t) \int_0^t \text{tr} (\dot{\epsilon}(t') \Pi(t')) dt' \quad (2.20)$$

which represents the difference between the energy in the detector at time t with and without the g.w.

For $t > 1/\Omega$ the integral in the right hand side of Eq.(2.20) is largest when the frequency Ω of the g.w. coincides with one of the frequencies of the quadrupole moment Π . In this case ΔW can be written in the form

$$\Delta W(t) = W_0(1-2\lambda)h\Omega[\rho t + \frac{1}{\Omega}f(t)]\exp(-\gamma_{e.m.}t) \quad (2.21)$$

where

$$\rho = \lim_{T \rightarrow \infty} \frac{1}{2Th\Omega} \int_0^T \text{tr}(\dot{a}(t')\Pi(t')) dt' \quad (2.22)$$

is a parameter which measures the efficiency of the detector and $f(t)$ is an oscillating function. For $t \sim 1/\gamma_{e.m.}$ the oscillating function can be neglected provided

$$|f| < \frac{\Omega\rho}{\gamma_{e.m.}} \quad (2.23)$$

and the maximum value of ΔW becomes:

$$\Delta W = h \frac{\Omega}{\gamma_{e.m.}} W_0 \rho (1 - 2\lambda) \quad (2.24)$$

To determine the explicit form of ρ let

$$\Pi_{\Omega} = \frac{1}{2\pi} \int_{-\infty}^{\infty} dt \cos(\Omega t) \Pi(t) \quad (2.25)$$

be the Fourier component of Π at the frequency of the g.w.
From Eq. (2.22) it follows:

$$\rho = |\Pi_{\Omega}| g \quad (2.26)$$

where

$$|\Pi_{\Omega}| = \left[\frac{1}{2} \text{tr} \Pi_{\Omega}^2 \right]^{1/2} \leq 1 \quad (2.27)$$

is the invariant "length" of Π_{Ω} and g , ($|g| \leq 1$), is a function of the phases and the polarizations of the g.w.

We remark that the energy transfer given in Eq.(2.21) represents the combined effect of the direct and indirect interactions between the g.w. and the detector. As we have already stated these two contributions are entirely described, in the rest frame, by the time variation of the dielectric constant.

III. Resonant detectors

In this section we shall discuss two types of resonant e.m. detectors: a low frequency detector where the e.m. frequency is equal to $\Omega/2$, and a high frequency one, in which the e.m. field oscillates at two frequencies $\omega_+, \omega_- \gg \Omega$, such that $\omega_+ - \omega_- = \Omega$.

Our first example is a L.C.circuit oscillating at frequency ω . To simplify the analysis we assume that e.m. energy and the quadrupole moment can be calculated with sufficient approximation from the following field configuration

$$\underline{E} = E_0 \cos \omega t (\cos \varphi_E, \sin \varphi_E, 0) \quad (3.1)$$

$$\underline{B} = B_0 \sin \omega t (\cos \varphi_B, \sin \varphi_B, 0),$$

where E_0 and B_0 are constant inside the volumes V_E and V_B of the capacitor and of the coil respectively, and vanish outside them. They satisfy the condition

$$\frac{E_0^2}{B_0^2} = \frac{V_B}{V_E} \quad (3.2)$$

which expresses the equality of the time average of the electric and magnetic energies. The quadrupole moment of the field configuration specified by Eq.(3.1) is the sum of two terms, one con-

stant in time and the other oscillating at frequency 2ω :

$$\Pi(t) = \Pi_0 + \Pi_{2\omega} \cos 2\omega t \quad (3.3)$$

To maximize the energy exchange between the e.m. circuit and the g.w., $\Pi(t)$ must oscillate at the same frequency as $\dot{\epsilon}(t)$, i.e. we must have $\omega = \Omega/2$.

One can easily show that the value of the invariant length of $\Pi_{2\omega}$ for this simplified configuration is

$$|\Pi_{2\omega}| = |\sin(\varphi_E - \varphi_B)| \quad (3.4)$$

Similarly the invariant length of Π_0 is

$$|\Pi_0| = |\cos(\varphi_E - \varphi_B)| \quad (3.5)$$

The maximum efficiency ρ therefore obtains when

$\varphi_E - \varphi_B = \pm \pi/2$, i.e. when the electric field in the capacitor is perpendicular to the magnetic field in the coil. In this case $|\Pi_0| = 0$ (see Fig.1).

In a more realistic case, the electric and magnetic fields of an oscillating circuit cannot be written in the simple form (3.1). However Eq.(3.4) and (3.5) remain true if $E_3 = B_3 = 0$, with φ_E and φ_B defined by

$$\cos 2\varphi_E = \frac{\int d^3 \underline{x} (E_1^2 - E_2^2)}{\int d^3 \underline{x} (E_1^2 + E_2^2)} \quad \cos 2\varphi_B = \frac{\int d^3 \underline{x} (B_1^2 - B_2^2)}{\int d^3 \underline{x} (B_1^2 + B_2^2)} \quad (3.6)$$

For low frequency g.w.'s the resonance condition $\omega = \Omega/2$ requires large inductances and capacities which make this conceptually simple detector not very promising in practice. This difficulty can be avoided by making the g.w. resonate with the beat frequency of two coupled L.C. circuits operating at high frequencies and therefore of manageable sizes.

Let ω_+ and ω_- , $\omega_+ > \omega_-$ be the two eigenfrequencies of the two coupled oscillators and let

$$2\omega = \omega_+ + \omega_-, \quad 2\eta\omega = \omega_+ - \omega_- \quad (3.7)$$

with $\eta \ll 1$. We will distinguish with superscripts 1 and 2 the quantities referring to each circuit which we shall assume to be spatially separated. As in the previous case we will calculate the quadrupole moment with a simplified field configuration, namely:

$$\begin{aligned} \underline{E}^{(1)} &= E_0^{(1)} \cos \omega t \cos \eta \omega t (\cos \varphi_E^{(1)}, \sin \varphi_E^{(1)}, 0) \\ \underline{E}^{(2)} &= E_0^{(2)} \cos \omega t \sin \eta \omega t (\cos \varphi_E^{(2)}, \sin \varphi_E^{(2)}, 0) \\ & \hspace{15em} (3.8) \\ \underline{B}^{(1)} &= B_0^{(1)} \sin \omega t \cos \eta \omega t (\cos \varphi_B^{(1)}, \sin \varphi_B^{(1)}, 0) \\ \underline{B}^{(2)} &= B_0^{(2)} \sin \omega t \sin \eta \omega t (\cos \varphi_B^{(2)}, \sin \varphi_B^{(2)}, 0) \end{aligned}$$

As before we suppose that there is no overlap between the fields and that the time average of the electric and magnetic ener

gies are equal and equally distributed between the two circuits.

The quadrupole moment for the configuration (3.8) can be written as

$$\Pi(t) = \Pi^{(1)}(t) + \Pi^{(2)}(t) \quad (3.9)$$

where

$$\Pi^{(1)}(t) = \cos^2(\eta \omega t) \left[\Pi_0^{(1)} + \Pi_{2\omega}^{(1)} \cos 2\omega t \right] \quad (3.10)$$

$$\Pi^{(2)}(t) = \sin^2(\eta \omega t) \left[\Pi_0^{(2)} + \Pi_{2\omega}^{(2)} \cos 2\omega t \right] .$$

The quadrupole moment of each circuit is of the form (3.3) multiplied by a low frequency modulating factor. To obtain the maximum efficiency at the resonance with the beat frequency $2\eta\omega$, we choose the relative orientation of the electric and magnetic fields in each circuit in such a way that $\Pi_{2\omega}^{(1)} = \Pi_{2\omega}^{(2)} = 0$.

This implies [see Eq.(3.4)]

$$\varphi_E^{(1)} - \varphi_B^{(1)} = \varphi_E^{(2)} - \varphi_B^{(2)} = n\pi \quad (3.11)$$

We can thus rewrite Eq.(3.9) as:

$$\Pi(t) = \Pi_0 + \Pi_{2\eta\omega} \cos(2\eta\omega t) \quad (3.12)$$

where

$$\begin{aligned}\Pi_0 &= \frac{1}{2} \left[\Pi_0^{(1)} + \Pi_0^{(2)} \right] \\ \Pi_{2\eta\omega} &= \frac{1}{2} \left[\Pi_0^{(1)} - \Pi_0^{(2)} \right] .\end{aligned}\tag{3.13}$$

By the same argument used before we recognize that the maximum value of $|\Pi_{2\eta\omega}|$ obtains when

$$\varphi_E^{(1)} - \varphi_E^{(2)} = \pm \pi/2\tag{3.14}$$

i.e. when the electric fields in two circuits are at right angles. The geometry corresponding to conditions (3.11) and (3.14) is sketched in Fig. 2 .

A more practical high frequency detector can be realized with two cavities at right angles coupled in such a way that the e.m. energy oscillates between them at the frequency of the g.w.

The calculation of Π is in this case more complicated since the e.m. fields inside the cavity cannot be described by the simple formula (3.8).

In terms of the e.m. quality factor $Q_{e.m.} = \omega/\gamma_{e.m.}$ the energy transfer [Eq.(2.24)] can be written for both the low and the high frequency detectors in the form

$$\Delta W = h\rho \frac{\Omega}{\omega} Q_{e.m.} W_0 (1 - 2\lambda)\tag{3.15}$$

For the low frequency detector, $\Omega/\omega = 2$, whereas for the high frequency one has $\Omega/\omega = 2\eta \ll 1$. In the latter case the energy transfer ΔW can be split into the sum of two terms

$$\Delta W = \Delta W_+ + \Delta W_- \quad (3.16)$$

representing the energy transferred by the g.w. to the two eigenmodes of the detector with frequencies ω_+ , ω_- respectively.

It can be shown that

$$\frac{d}{dt} \left(\frac{\Delta W_+}{\omega_+} + \frac{\Delta W_-}{\omega_-} \right) = (\text{oscillating terms}) \quad (3.17)$$

where the oscillating terms are of order \hbar and have frequencies much larger than Ω . From Eqs.(3.15) and (3.17) it follows:

$$|\Delta W_+| \cong |\Delta W_-| \cong \frac{1}{\eta} |\Delta W| \quad (3.18)$$

Therefore the energy transferred to each eigenmode is much larger than the total energy transfer.

We remark that also the microwave detector suggested by Braginskii and Menskii [1971] can be described in terms of Eq. (2.15). Indeed one can show that the quadrupole moment Π of a wave packet with wave number $\underline{k} = \underline{k}(t)$ and central frequency ω is of the form

$$\Pi_{ij} = \frac{k_i k_j}{\omega^2} \quad (3.19)$$

The energy W of the wave packet is

$$W = \omega N \quad (3.20)$$

where N is the number of photons in the packet. In the geometrical optics approximation N is an adiabatic invariant and therefore Eq. (2.15), in the absence of absorption, reduces to

$$\frac{d\omega}{dt} = \omega \operatorname{tr} \left(\frac{\dot{\epsilon}_{\mathbf{k}} \otimes \mathbf{k}}{\omega^2} \right) \quad (3.21)$$

The effect of the gravitational wave is in this case to change the average frequency of the wave packet. Eq. (3.21) can be shown to follow directly from the geodesic equation for a light ray.

IV. Conclusion

In this paper we have analyzed the principle of a detector whose response is linear in h . The advantage of the e.m. detectors is to be easily tunable over a wide range of frequencies. Since, to increase their sensitivity one is forced to use very high Q -values, the band-width of these detectors is correspondingly small. They are therefore ideal for studying monochromatic sources of known frequency. They can however also be used to investigate the broad spectra expected from supernova collapses by sweeping a large frequency interval. A limitation of a detector linear in h is the difficulty in measuring $\Delta W/W$ in the presence of a large energy in the two eigenmodes. With the present technology one can hope to achieve an accuracy of 10^{-6} in $\Delta W/W$. With a Q -value of 10^{11} which has already been achieved this leads to a minimum detectable $h \sim 10^{-17}$, of the same order of magnitude as the one obtained with mechanical detectors. An improvement in Q by two orders of magnitude would make this detector competitive with the mechanical ones now under study (see e.g. Drever 1977).

The same kind of e.m. detector discussed in this paper can be operated in a different way by loading the cavity in only one of the two eigenmodes. In this case the energy ΔW_+ transferred to the empty eigenmode is proportional to $h^2 Q_{e.m.}^2 W$. The advantage of this method of operation is that the minimum detectable energy ΔW_+ can be

as small as 10^{-9} erg. With this sensitivity and with an energy $W_{-} \sim 10^7$ erg one can reach $h = 10^{-19} - 10^{-21}$ with $Q_{em} = 10^{11} - 10^{13}$ respectively. Only a detailed analysis of the noise and of the feasibility of the detector will decide whether this sensitivity in h can actually be achieved.

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Appendix

If one takes into account the finite size of the detector, the correction $h_{\mu\nu}(\lambda)$ produced by the g.w. takes the form

$$h_{\mu\nu}(\lambda) = (2\lambda - 1) \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & a_{11} & a_{12} & 0 \\ 0 & a_{21} & a_{22} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} +$$

$$-\lambda [x^i(\lambda) \ddot{a}_{ij} x^j(\lambda)] \begin{pmatrix} -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 \end{pmatrix} \quad (\text{A.1})$$

where $i, j = 1, 2$ and a_{ij} is defined by Eq. (2.3).

The first term in the right hand side of Eq. (A.1) is of order h , whereas the second is of order $h(\Omega L)^2$, where L is the characteristic size of the detector. The latter has been neglected in Eq. (2.2).

The coordinates $x^\mu(\lambda_1)$ and $x^\mu(\lambda_2)$ are related, to first order in $(\lambda_2 - \lambda_1)h$, by the transformations:

$$\begin{aligned} x^0(\lambda_2) &= x^0(\lambda_1) + \frac{1}{2}(\lambda_2 - \lambda_1) x^i \dot{a}_{ij} x^j \\ x^i(\lambda_2) &= x^i(\lambda_1) - (\lambda_2 - \lambda_1) a_{ij}^i x^j \\ x^3(\lambda_2) &= x^3(\lambda_1) + \frac{1}{2}(\lambda_2 - \lambda_1) x^i \dot{a}_{ij} x^j \end{aligned} \quad (\text{A.2})$$

where, $a_j^i = \eta^{ik} a_{kj}$. To the same approximation there is no need to distinguish in the second term in the right hand side of Eq. (2.5) between $x^i(\lambda_1)$ and $x^i(\lambda_2)$. With the same argument we can drop the label λ in the term $x^i(\lambda) \ddot{a}_{ij} x^j(\lambda)$ in Eq. (A.1). We remark that in Eqs. (A.2) the second term in the equations for x^0 and x^3 is of order $h\Omega L$, whereas in the equation for x^i the corresponding term is of order h .

For every λ the geodesic equation of the center of mass is $x^i = 0$, $i = 1, 2$, $x^3 = 0$. The square of the spatial distance between two particles in the (x^1, x^2) -plane with coordinates x^i and $x^i + \Delta x^i$ is:

$$\Delta \ell^2 = \Delta x^i [\eta_{ij} + (2\lambda - 1) a_{ij}] \Delta x^j \quad (\text{A.3})$$

Due to the time dependence of a_{ij} , even if two particles are at rest, their spatial distance varies with time for all λ 's different from $\frac{1}{2}$.

Two values of λ are of special significance:

(a) $\lambda = 0$: in this case a free particle at rest in the absence of the g.w., remains at rest even when the g.w. is turned on. This means that the system $\lambda = 0$ is the free falling system; $g_{\mu\nu}(\lambda = 0)$ is thus the metric tensor in the transverse traceless gauge.

(b) $\lambda = \frac{1}{2}$: in this case $h_{\mu\nu}$ is of order $h(\Omega L)^2$ and the space distance between two points at rest in the (x^1, x^2) -plane is constant. This is the proper reference frame (Misner, Thorne and Wheeler 1973) attached to an infinitely rigid detector. In this coordinate frame a free particle, which would be at rest in the absence of the g.w., appears to oscillate.

Let us now consider an idealized detector whose walls are formed by a system of equal particles elastically bound to the center of mass. The system in which the "walls" of this detector are at rest will be called its rest system and the corresponding value of λ depends upon the mechanical properties of the detector. In the limit of vanishing binding we have $\lambda = 0$, whereas for an infinitely rigid binding we have $\lambda = \frac{1}{2}$. To find the value of λ appropriate to an oscillator with frequency ω_m we consider a particle of mass m , elastically bound to the center of mass of the detector, moving under the influence of the g.w. We assume that in the absence of the g.w. the particle is at rest in the $(x^1 - x^2)$ -plane at $x^i = X^i$. Let ℓ^2 be the spatial distance from the origin as defined in Eq. (A.3) of a point x^i and let $\ell^i(x)$ be defined by

$$\ell^i(x) \eta_{ij} \ell^j(x) = \ell^2(x) \quad . \quad (A.4)$$

Then, to order h ,

$$\varrho^i(x) = x^i + \frac{2\lambda - 1}{2} a_j^i x^j \quad (\text{A.5})$$

and

$$\varrho^i(x^i = X^i, h=0) = X^i \quad (\text{A.6})$$

Under the action of the g.w. the particle is displaced by an amount $\Delta \varrho^i = \varrho^i - X^i$ of order h . The restoring force is $-m\omega_m^2 \Delta \varrho^i$, where ω_m is the frequency of the oscillator and

$$\Delta \varrho^i = \xi^i + \frac{2\lambda - 1}{2} a_j^i X^j, \quad \xi^i = x^i - X^i \quad (\text{A.7})$$

The displacement $\Delta \varrho^i$ is the sum of two terms: the first one represents the difference between the coordinate of the particle and that of the equilibrium position; the second is due to the change of the distance of the equilibrium position X^i from the center of mass. By neglecting terms of order $h v$, where v is the velocity of the particle under the influence of the g.w., the equation of motion for the particle reads:

$$\ddot{\xi}^i = A^i - \omega_m^2 \Delta \varrho^i \quad (\text{A.8})$$

where

$$A^i = -\Gamma_{00}^i \Big|_{x^i=X^i} = -\lambda \ddot{a}_j^i X^j \quad (\text{A.9})$$

and Γ_{00}^i is a Christoffel index. Then Eq. (A.8) becomes:

$$\ddot{\xi}^i + \omega_m^2 \xi^i = [\lambda \Omega^2 - (\lambda - \frac{1}{2}) \omega_m^2] a_j^i X^j \quad (A.10)$$

If $\omega_m \neq \Omega$, the right hand side of Eq.(A.10) vanishes for

$$\lambda = \frac{1}{2} \frac{\omega_m^2}{\omega_m^2 - \Omega^2} \quad (A.11)$$

and therefore in the system with λ given by (A.11) a particle which was originally at rest remains at rest even in the presence of the g.w. To determine the value of λ which transforms away the forced motion due to the g.w. when $\omega_m = \Omega$, one must introduce in the right hand side of Eq.(A.8) a damping term $-2\gamma_m \Delta \xi^i$ and generalize the above discussion to the case where λ is complex. The absolute value of λ at resonance can easily be seen to be

$$|\lambda| = \frac{\omega_m}{\gamma_m} = Q_m \quad (A.12)$$

Due to the finite size of the detector the relations between (E,B) and (D,H) are also modified. Instead of Eqs. (2.9) and (2.10) we get

$$\begin{aligned} D^i &= -\eta^{ij} (1 + \frac{\lambda}{2} x^k \ddot{a}_{k\ell} x^\ell) E_j - \lambda x^k \ddot{a}_{k\ell} x^k \ddot{a}_{k\ell} x^\ell \sigma^{ij} B_j - (1-2\lambda) a^{ij} E_j \\ D^3 &= (1 - \frac{\lambda}{2} x^k \ddot{a}_{k\ell} x^\ell) E_3 \\ H^i &= -\eta^{ij} (1 - \frac{3}{2} x^k \ddot{a}_{k\ell} x^\ell) B_j - \lambda x^k \ddot{a}_{k\ell} x^k \ddot{a}_{k\ell} x^\ell \sigma^{ij} E_j + (1-2\lambda) a^{ij} B_j \\ H^3 &= (1 - \frac{\lambda}{2} x^k \ddot{a}_{k\ell} x^\ell) B_3 \quad i, j, k, \ell = 1, 2 \end{aligned} \quad (A.13)$$

where

$$\sigma^{ij} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad (\text{A.14})$$

and

$$a^{ij} = \eta^{ik} a_{kl} \eta^{\ell j} \quad (\text{A.15})$$

We remark that Eqs. (2.10) are obtained from Eqs. (A.13) by neglecting terms of order $h(\Omega L)^2$.

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FIGURE CAPTIONS

Fig. 1 - Low frequency L.C. resonator

Fig. 2 - High frequency L.C. resonator.

TABLE 1

Source	h	Ω (rad/sec)	I(erg/cm ² sec)
Extragalactic supernovae	2×10^{-22} - 10^{-20}	10^3 - 10^5	10^4 - 10
Crab pulsar	10^{-24}	377	3×10^{-7}
Iota Bootes	6×10^{-21}	5.4×10^{-4}	1.8×10^{-10}
HZ 29	3.6×10^{-22}	1.2×10^{-2}	3×10^{-10}

FIG. 1

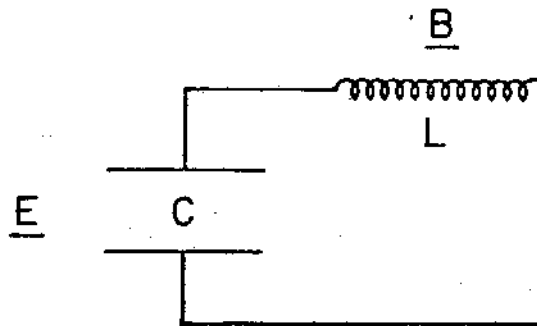


FIG. 2

