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ON THE OPERATOR EQUATION $TX - XV = A$

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ABSTRACT. A characterization of the operators A for which the equation $TX - XV = A$ is solvable is given, where T is a fixed right invertible operator and V is a fixed unilateral shift.

The aim of this note is to give a characterization of the solutions of the equation

$$TX - XV = A,$$

where T, V, A are given operators acting in a Hilbert space, and V is a unilateral shift. As a by-product we also give a sufficient condition for A to be expressed in the form

$$A = VX - XV = V^* Y - YV^* = V^* Z - ZV$$

with V a unilateral shift.

The problems studied throughout the paper originate from a question of C. Foiaş.

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Let H be a complex Hilbert space and denote by $L(H)$ the algebra of all bounded linear operators acting in H. For any $T, S, A \in \mathcal{L}(H)$ put

$$d_n(A; T, S) = \sum_{j+k=n} T^j A S^k, \quad n \geqslant 0.$$

It is easy to see what we have

$$Td_n(A; T, S) - d_n(A; T, S)S = d_n(TA - AS; T, S) = T^{n+1}A - AS^{n+1},$$

$$d_{n+1}(A; T, S) = Td_n(A; T, S) + AS^{n+1} = d_n(A; T, S)S + T^{n+1}A.$$

1. PROPOSITION. *Let $B \in \mathcal{L}(H)$ and let V be a unilateral shift such that $TB - BV = 0$, $B(I - VV^*) = 0$. Then we have $B = 0$.*

PROOF. By induction we derive $BV^n = T^n B$, so $BV^n(I - VV^*) = 0$, $n \geqslant 0$, and this implies $B = 0$.

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2. LEMMA. *Let $T, A \in \mathfrak{L}(H)$ be given and let V be a unilateral shift. Then the following conditions are equivalent*:

(i) *there exists $B \in \mathfrak{L}(H)$ such that $TB - BV = A$, $B(I - VV^*) = 0$,*

(ii) *there exists $c > 0$, such that for any sequence $\{x_k\}_{k=0}^{\infty} \subset \operatorname{Ker} V^*$ and for any $n \geqslant 0$, we have*

$$\left\| \sum_{k=0}^{n} d_k(A; T, V) x_k \right\| \leqslant c \left(\sum_{k=0}^{n} \|x_k\|^2 \right)^{1/2},$$

(iii) *the series $\sum_{k=0}^{\infty} d_k(A; T, V)(I - VV^*)V^{*k}$ is strongly convergent.*

PROOF. (i) $\Rightarrow$ (ii). Let $\{x_k\}_{k=0}^{\infty} \subset \operatorname{Ker} V^*$ and take $c = \|B\|$. We have

$$\left\| \sum_{k=0}^{n} d_k(A; T, V) x_k \right\|^2 = \left\| \sum_{k=0}^{n} d_k(TB - BV; T, V) x_k \right\|^2$$

$$= \left\| \sum_{k=0}^{n} (T^{k+1}B - BV^{k+1}) x_k \right\|^2 = \left\| -B \sum_{k=0}^{n} V^{k+1} x_k \right\|^2$$

$$\leqslant \|B\|^2 \left\| \sum_{k=0}^{n} V^{k+1} x_k \right\|^2 = c^2 \sum_{k=0}^{n} \|x_k\|^2.$$

(ii) $\Rightarrow$ (iii). For any $x \in H$ we have

$$\left\| \sum_{k=m}^{m+n} d_k(A; T, V)(I - VV^*)V^{*k} x \right\|^2 \leqslant c^2 \sum_{k=m}^{m+n} \|(I - VV^*)V^{*k} x\|^2$$

$$= c^2 \sum_{k=m}^{m+n} \|V^k(I - VV^*)V^{*k} x\|^2 = c^2 \|V^{m+n} V^{*m+n} x - V^{m+1} V^{*m+1} x\|^2$$

and because $\{V^{*k}\}_{k=0}^{\infty}$ tends strongly to 0, the strong convergence of the series $\sum_{k=0}^{\infty} d_k(A; T, V)(I - VV^*)V^{*k}$, follows.

(iii) $\Rightarrow$ (i). If we take $B = -(\sum_{k=0}^{\infty} d_k(A; T, V)(I - VV^*)V^{*k})V^*$, we have

$$\begin{aligned} TB - BV = {} & \sum_{k=0}^{\infty} d_k(A; T, V)(I - VV^*)V^{*k} \\ & - T \sum_{k=0}^{\infty} d_k(A; T, V)(I - VV^*)V^{*k+1} \\ = {} & \sum_{k=0}^{\infty} d_k(A; T, V)(I - VV^*)V^{*k} + A \sum_{k=0}^{\infty} V^{k+1}(I - VV^*)V^{*k+1} \\ & - \sum_{k=0}^{\infty} d_{k+1}(A; T, V)(I - VV^*)V^{*k+1} \\ = {} & d_0(A; T, V)(I - VV^*) + A \sum_{k=0}^{\infty} V^{k+1}(I - VV^*)V^{*k+1} = A. \end{aligned}$$

3. THEOREM. *Let $T, A \in \mathfrak{L}(H)$ be given and let V be a unilateral shift. Then the equations*

$$TB - BV = A, \qquad B(1 - VV^*) = 0$$

have a simultaneous solution B if and only if there exists $c > 0$ as in Lemma 2(ii); *the solution B is unique.*

PROOF. Existence follows from Lemma 2, and uniqueness from Proposition 1.

4. THEOREM. *Let $T, A' \in \mathfrak{L}(H)$ and let V be a unilateral shift. Suppose that T is right invertible. Then the equation $TB' - B'V = A'$ has a solution B' if and only if A' is of the form $A' = A + A_0$, with A fulfilling condition* (ii) *of Lemma* 2 *and $A_0(1 - VV^*) = A_0$.*

PROOF. Suppose that there exists $B' \in \mathfrak{L}(H)$ such that $TB' - B'V = A'$. Taking $A = TB'VV^* - B'V$, $A_0 = TB'(I - VV^*)$, the properties of A and A_0 stated in our theorem can be easily checked (see also Theorem 3). Conversely, if $A' = A + A_0$ and A, A_0 have the properties stated in our theorem, then taking B_0 such that $TB_0 = A_0$, $B_0V = 0$ (this is possible because T is right invertible) and taking B given by Theorem 3, the operator $B' = B + B_0$ will be a solution of the equation $T'B' - B'V = A'$. Recall now that the essential reducing spectrum of $A \in \mathfrak{L}(H)$ is the set of all complex numbers λ for which there exists an orthonormal sequence $\{e_n\} \subset H$ such that (see [2])

$$\varlimsup_{n\to\infty} \|(A - \lambda)e_n\| = \varlimsup_{n\to\infty} \|(A - \lambda)^* e_n\| = 0.$$

5. THEOREM. *Let $A \in \mathfrak{L}(H)$ have* 0 *in its essentially reducing spectrum. Then there exists a unilateral shift V such that the equations*

$$VX - XV = V^*Y - YV^* = V^*Z - ZV = A$$

are solvable.

PROOF. Let V be a unilateral shift and denote by P_n the orthogonal projection of H onto V^nH. Because 0 belongs to the essential reducing spectrum of A, we may choose V such that

$$\max\{\|P_nA\|, \|AP_n\|\} \leqslant \|A\|/2^n.$$

Let us put $Q_n = P_n - P_{n+1}$. Then we have $\sum_{j+k=0}^{\infty} \|Q_jAQ_k\| < \infty$ and the existence of X and Y follows from [1, Lemmas 1.1 and 1.3]. On the other hand, we have

$$\|d_n(A; V^*, V)\| \leqslant \sum_{j+k=0}^{n} \|P_jAP_k\| \leqslant (n+1)\|A\|/2^n;$$

thus if $\{x_k\}_{k=0}^{\infty} \subset \operatorname{Ker} V^*$ is given, we obtain

$$\begin{aligned}\left\|\sum_{k=0}^{n} d_k(A; V^*, V)x_k\right\| &\leqslant \sum_{k=0}^{n} \|d_k(A; V^*, V)\|\,\|x_k\| \\ &\leqslant \|A\|\left(\sum_{k=0}^{\infty}\frac{(k+1)^2}{2^k}\right)^{1/2}\left(\sum_{k=0}^{n}\|x_k\|^2\right)^{1/2}.\end{aligned}$$

Applying Lemma 2, we derive the existence of Z.

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