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ON THE OPTIMAL DESIGN OF TRUSSES UNDER ONE LOADING CONDITION*

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Abstract. The optimal design of singly loaded trusses required to satisfy allowable stress criteria has the popular structural properties of a statically determinate configuration and fully stressed members. Linear programming and the duality theorem were used to attain these properties. This paper formulates the truss problem as a nonlinear programming problem, derives the optimality criteria via the Lagrangian and through proper physical interpretation of the Lagrange multipliers demonstrates the validity of the above results.

1. Introduction. The concern for the most economic structures has led Cilley [1] (1900) and Mitchell [2] (1904) to the famous conclusion that the optimal design of singly loaded elastic trusses is a fully stressed statically determinate design. Shen and Schmit [3] regard the singly loaded truss as a special case in their discussion on lower bounds to multiply loaded trusses and demonstrate the properties of statical determinacy and global optimum. Hemp [4] and Spillers [5] used linear programming [6] with its necessary and sufficient conditions and the duality theorem to arrive at the same conclusion. They formulated the problem as a linear problem demanding that the volume, which is a linear function of the internal forces, be minimized and equilibrium satisfied. Such a procedure which neglects the constitutive relations from the general formulation is termed plastic design, formally written as

$$\begin{gathered}\text{Minimize } \sum_i A_i L_i \\ \text{Subject to } \tilde{N}F = P \quad \text{and} \quad A_i \geqslant |F_i|/\sigma_a\end{gathered} \tag{1}$$

where L_i's are the member lengths and σ_a represents the allowable stress for both negative and positive stresses. With the areas taken as $A_i = |F_i|/\sigma_a$ and the allowable member length changes introduced as $(\Delta_a)_i = \sigma_a L_i/E$, Eq. (1) becomes

$$\begin{gathered}\text{Minimize } \sum_i |F_i|(\Delta_a)_i \\ \text{Subject to } \tilde{N}F = P.\end{gathered} \tag{2}$$

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Equation (2) comprises a linear programming problem with the member forces, F_i, as unknown variables whose dual can be written as

$$\begin{aligned}&\text{Maximize } \tilde{P}\delta\\ &\text{Subject to } |(N\delta)_i| \leqslant (\Delta_a)_i.\end{aligned} \tag{3}$$

The solution to Eq. (2) is a statically determinate subtruss (defined as a truss containing p bars of an originally b-bar truss) for which the inequality constraints of Eq. (3) are satisfied as equalities and the member displacements, Δ_i, and the member forces have the same sign. Moreover it can be shown that the inequality constraint of Eq. (3) for members out of the design is not violated. The dual variables, δ, can be interpreted as nodal displacements and the results of the optimal design for singly loaded trusses satisfying allowable stress requirements only summarized as

$$\begin{aligned}&F = \begin{Bmatrix} F_T \\ 0 \end{Bmatrix}; \qquad K = \begin{bmatrix} K_T & \\ & 0 \end{bmatrix} \quad \text{where } K_T = \frac{|F_T|}{(\Delta_a)_T},\\ &\operatorname{sgn} F_T = \operatorname{sgn} \Delta_T,\\ &|\Delta_T| = (\Delta_a)_T, \quad \text{and} \quad |\Delta_L| \leqslant (\Delta_a)_L.\end{aligned} \tag{4}$$

Here a matrix with corner diagonals is a diagonal matrix. The results in Eq. (4) are given in partitioned form with the subscript T standing for the number of members in the statically determinate subtruss and the subscript L for the number of superfluous members. Note that the notation in Eqs. (1)–(4) is that of the node method as described in Sec. 2.

Nonlinear programming [7, 8] with its powerful techniques for attaining optimal solutions has had little impact on the understanding of structural behavior and physical properties at the optimum. It is the author's intent here to demonstrate its use in proving the results of Eq. (4).

2. Optimal properties through nonlinear programming. The optimal elastic design of singly loaded trusses is truly a nonlinear programming problem and can be formulated as such by seeking those cross-sectional areas and nodal displacements that minimize the volume (linear with respect to the stiffnesses) and satisfy the equations of statics (as opposed to equilibrium only) and allowable stress criteria. Prior to the formal formulation this section briefly introduces the equations of statics for trusses in the form of the node method then synthesizes the problem and finally shows the validity of Eq. (4).

2.1 *The node method.* The equations of statics that govern the behavior of elastic trusses are presented below as

$$\begin{aligned}\tilde{N}F &= P \quad \text{(equilibrium)},\\ F &= K\Delta \quad \text{(constructive relations)},\\ N\delta &= \Delta \quad \text{(member/nodal displacement relations)},\end{aligned}$$

where

F—member force matrix,

P—joint load matrix,

K—primitive stiffness matrix,

Δ—member displacement matrix,

δ—joint displacement matrix,

N—generalized incidence matrix.

Note that the primitive stiffness matrix is diagonal with elements $K_{ii} = A_i E/L_i$ where A_i is the cross-sectional area of member i, E is Young's modulus and L_i is the length of member i. Reference (9) gives more detail on the node method for trusses.

2.2 *Nonlinear programming formulation.* With the equations of statics now at hand the problem of minimizing the volume of linearly elastic trusses satisfying allowable stress criteria can be posed formally as a nonlinear programming problem that takes the form of

$$\begin{aligned}&\text{Minimize } \sum_i K_i(\Delta a)_i^2\\&\text{Subject to } P - \tilde{N}KN\delta = 0,\\&\qquad |(N\delta)_i| \leqslant (\Delta_a)_i, \qquad -K_i \leqslant 0.\end{aligned} \tag{5}$$

The objective function is proportional to the volume as seen from the relation below

$$\begin{aligned}\sum_i K_i(\Delta_a)_i^2 &= \sum_i\left(\frac{A_iE}{L_i}\right)\left(\frac{\sigma_a L_i}{E}\right)^2\\&= \frac{\sigma_a^2}{E}\sum_i A_iL_i \sim \text{volume.}\end{aligned}$$

The equality constraint is a condensed form of the equations of statics and the inequality constraints require that the member displacements be smaller or equal to the allowable member displacement and stiffness of bars in the design be positive.

2.3 *Optimality conditions.* The nonlinear programming format of Eq. (5) is consistent with that described in [7] and the optimality conditions are obtained via the Lagrangian below

$$L = \sum_i K_i(\Delta_a)_i^2 + \tilde{d}(P - \tilde{N}KN\delta) + \tilde{\lambda}\left(|N\delta| - \Delta_a\right) + \tilde{\mu}(-K) \tag{6}$$

where d, λ and μ are the Lagrange multipliers.

Differentiate Eq. (6) with respect to the unknown variables K and δ to obtain

$$\frac{\partial L}{\partial K_i} = 0 \Rightarrow (\Delta_a)_i^2 - Q_i\Delta_i - \mu_i = 0 \qquad [Q_i = (Nd)_i], \tag{7}$$

$$\frac{\partial L}{\partial \delta_i} = 0 \Rightarrow -\tilde{N}KNd + \tilde{N}\lambda \operatorname{sgn} \Delta = 0. \tag{8}$$

In addition to Eqs. (7), (8) the relations below must be satisfied (necessary conditions) by a point that is a candidate for a local minimum. These relations are

$$P - \tilde{N}KN\delta = 0 \tag{9}$$

$$\lambda_i\left(|(N\delta)_i| - (\Delta_a)_i\right) = 0 \tag{10}$$

$$\mu_i K_i = 0 \tag{11}$$

$$\lambda_i, \mu_i \geqslant 0 \tag{12}$$

Equations (7)–(12) comprise the Kuhn-Tucker conditions and a desgin satisfying these conditions is a Kuhn-Tucker design. For a completion of the discussion it will suffice to show that the design obtained through linear programming as represented by Eq. (4) is a Kuhn-Tucker design and, therefore, a local minimum design. That the design is unique or global will not be apparent from the considerations that follow and no attempt is made to undertake a globality proof. Beyond this point algebraic manipulations reduce the K. T. conditions to a convenient form for proper identification of the Lagrange multipliers. From Eq. (8) obtain

$$d = (\tilde{N}KN)^{-1}N\lambda \operatorname{sgn} \Delta. \tag{13}$$

Premultiply Eq. (13) by N and obtain an expression for B

$$B = Nd = N(\tilde{N}KN)^{-1}\tilde{N}\lambda \operatorname{sgn} \Delta. \tag{14}$$

From Eq. (7) get

$$B_i = (\Delta_a)_i^2/\Delta_i - \mu_i/\Delta_i. \tag{15}$$

Equate Eqs. (14) and (15) to obtain

$$N(\tilde{N}KN)^{-1}\tilde{N}\lambda \operatorname{sgn} \Delta = (\Delta_a)_i^2/\Delta_i - \mu_i/\Delta_i$$

or

$$N(\tilde{N}KN)^{-1}\tilde{N}\lambda \operatorname{sgn} \Delta + \mu_i/\Delta_i = (\Delta_a)_i^2/\Delta_i$$

or in matrix form

$$\left[N(\tilde{N}KN)^{-1}\tilde{N} \left[\frac{1}{|\Delta_i|}\right]\right]\begin{Bmatrix}\lambda \operatorname{sgn} \Delta_i \\ \mu \operatorname{sgn} \Delta_i\end{Bmatrix} = \begin{Bmatrix}\dfrac{(\Delta_a)_i^2}{|\Delta_i|} \operatorname{sgn} \Delta_i\end{Bmatrix}. \tag{16}$$

If Eq. (4) is to be a solution of the system of Eqs. (9)–(12), (16) then there must exists λ's and μ's that satisfy the above same equations. Let the Lagrange multiplier λ be interpreted as the absolute value of the member force matrix i.e.

$$\lambda = \begin{Bmatrix} K_T(\Delta_a)_T \\ 0 \end{Bmatrix} \geqslant 0 \tag{17}$$

and the Lagrange multiplier μ as the difference of the squares of the allowable member displacements and the actual member displacements i.e.

$$\mu = \begin{Bmatrix} 0 \\ (\Delta_a)_L^2 - \Delta_L^2 \end{Bmatrix} \geqslant 0. \tag{18}$$

At this point the question is whether Eqs. (4), (17), (18) satisfy Eqs. (9)–(12), (16). Since Eq. (4) is a feasable design Eq. (9) is satisfied. Eqs. (10)–(12) are also satisfied. It thus, remains to satisfy Eq. (16). Let $R = (\tilde{N}KN)^{-1}$. Equation (16) can now be partitioned as

$$\begin{bmatrix} \begin{bmatrix} N_T R \tilde{N}_T & N_T R \tilde{N}_L \\ N_L R \tilde{N}_T & N_L R \tilde{N}_L \end{bmatrix} \begin{bmatrix} \dfrac{1}{|\Delta_T|} \end{bmatrix} & \\ & \begin{bmatrix} \dfrac{1}{\Delta_L} \end{bmatrix} \end{bmatrix} \begin{Bmatrix} \lambda_T \operatorname{sgn} \Delta_T \\ \lambda_L \operatorname{sgn} \Delta_L \\ \mu_T \operatorname{sgn} \Delta_T \\ \mu_L \operatorname{sgn} \Delta_L \end{Bmatrix} = \begin{Bmatrix} (\Delta_a)_T^2 \operatorname{sgn} \Delta_T \\ (\Delta_a)_L^2 \operatorname{sgn} \Delta_L \end{Bmatrix}. \tag{19}$$

With substitution of Eqs. (17), (18), Eq. (19) condenses to

$$\begin{bmatrix} \left[N_T(\tilde{N}KN)^{-1}\tilde{N}_T\right] & 0 \\ \left[N_L(\tilde{N}KN)^{-1}\tilde{N}_T\right] & \begin{bmatrix} \dfrac{1}{|\Delta_L|} \end{bmatrix} \end{bmatrix} \begin{Bmatrix} K_T(\Delta_a)_T \operatorname{sgn} \Delta_T \\ \left((\Delta_a)_L^2 - \Delta_L^2\right) \operatorname{sgn} \Delta_L \end{Bmatrix} = \begin{Bmatrix} (\Delta_a)_T \operatorname{sgn} \Delta_T \\ \dfrac{(\Delta_a)_L^2}{\Delta_L} \operatorname{sgn} \Delta_L \end{Bmatrix}. \tag{20}$$

Using Eq. (4) it is seen that Eq. (20) is satisfied since

$$\begin{aligned} N_T(\tilde{N}KN)^{-1}\tilde{N}_T K_T(\Delta_a)_T \operatorname{sgn} \Delta_T &= N_T(\tilde{N}KN)^{-1}(\tilde{N}_T K N_T)\delta \\ &= N_T\delta = (\Delta_a)_T \operatorname{sgn} \Delta_T \end{aligned}$$

and

$$\begin{aligned} &N_L(\tilde{N}KN)^{-1}\tilde{N}_T K_T \operatorname{sgn} \Delta_T + \begin{bmatrix} \dfrac{1}{|\Delta_L|} \end{bmatrix} \left((\Delta_a)_L^2 - \Delta_L^2\right) \operatorname{sgn} \Delta_L \\ &= \begin{bmatrix} \dfrac{1}{|\Delta_L|} \end{bmatrix} (\Delta_a)_L^2 \operatorname{sgn} \Delta_L = \frac{(\Delta_a)_L^2}{|\Delta_L|} \operatorname{sgn} \Delta_L. \end{aligned}$$

Hence the solution represented by Eq. (4) is a Kuhn-Tucker point.

Concluding remarks. This paper has dealt with the basic question of understanding physical properties of optimal solutions. The approach taken was one of an "intelligent guess" as to the physical meaning of the Lagrange multipliers as they appear in the optimality conditions. For the simple singly loaded truss problem it was shown that for the optimal design the multipliers pertaining to the members in the design represented the absolute value of the member forces and that for members out of the design the Lagrange multipliers represented the difference of the squares of the allowable member displacements and the member displacements. A somewhat similar approach was taken in the, rather, more complex problem of the design for two loading using prestress [10].

REFERENCES

[1] F. H. Cilley, *The exact design of statically indeterminant frameworks, An exposition of its possibility, But Futility*, Transactions, ASCE, **43** (June 1900)
[2] A. G. M. Mitchell, *The limits of economy of material in framed structures*, Phil. Mag., **6** 8 (1904)
[3] C. Y. Shen and L. A. Schmit, *Minimum weight design of elastic redundant trusses under multiple static loading conditions*, AIAA Journal, **10**, 2 (Feb. 1972)
[4] W. S. Hemp, *Optimum structures*, Oxford University Press, 1973
[5] W. R. Spillers, *Iterative structural design*, North Holland, Amsterdam, 1976
[6] G. Hadley, *Linear programming*, Addison Wesley, Reading, Mass., 1962
[7] M. S. Bazaara and C. M. Shetty, *Nonlinear programming*, John Wiley, New York, 1979
[8] D. G. Luenberger, *Introduction to linear and nonlinear programming*, Addison-Wesley, Reading, Mass., 1973
[9] W. R. Spillers, *Automated structural analysis: An introduction*, Pergamon Press, New York, 1972
[10] W. R. Spillers and Robert Levy, *Truss design: Two loading conditions with prestress*, Journal of Structural Engineering, ASCE, Volume 110, No. 4 (April 1984)