Mathematics Faculty Research Publications

Mathematics

11-1-1980

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ON THE OPTIMAL STOPPING TIME PROBLEM FOR DEGENERATE DIFFUSIONS*

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Abstract. In this paper we give a characterization of the optimal cost of a stopping time problem as the maximum solution of a variational inequality without coercivity. Some properties of continuity for the optimal cost are also given.

Introduction. Summary of main results. This article develops the proofs of results obtained in Note [12].

A. Bensoussan and J. L. Lions [3] have introduced the variational inequality approach in order to solve the optimal stopping time problem in the case of non-degenerate diffusions. A. Friedman [8] treated the same case, M. Robin [18] the optimal stopping time problem for Feller processes, and J. M. Bismut [4] the same problem for a class of more general processes. C. Bardos [1] studied partial differential equations of first order, M. I. Freidlin [7] degenerate elliptic equations, and N. V. Krylov [9] nonlinear degenerate elliptic equations.

In [14] and [17] the variational inequality associated with the deterministic optimal stopping time problem is considered, and in [11] the degenerate nonlinear variational equalities are also studied.

In this paper, the case of degenerate variational inequality associated with the optimal stopping time problem for diffusion processes is developed combining analytic and probabilistic methods.

Let $(\Omega, \mathcal{F}, P)$ be a probability space and $\{\mathcal{F}^t\}_{t \geq 0}$ be a nondecreasing right-continuous family of completed sub-σ-fields of $\mathcal{F}$.

Now let $y(t) = y_x(t, \omega)$, $t \geq 0$, $\omega \in \Omega$ be the diffusion on $\mathbb{R}^N$ with Lipschitz continuous coefficients $g(\cdot)$ and $\sigma(\cdot)$, starting at x.

Suppose that $\mathcal{O}$ is an open subset of $\mathbb{R}^N$, and that $\tau = \tau_x(\omega)$ is the first exit time of process $y(t)$ from $\bar{\mathcal{O}}$.

Next, let $f(\cdot)$, $\psi(\cdot)$ be real bounded measurable functions on $\bar{\mathcal{O}}$, and θ be any stopping time. The cost functional $J_x(\theta)$ is given by

$$(0.1) \qquad J_x(\theta) = E\left\{ \int_0^{\theta \wedge \tau} f(y(t)) e^{-\alpha t} \, dt + 1_{\theta < \tau} \psi(y(\theta)) e^{-\alpha \theta} \right\},$$

where α is a positive constant.

Our purpose is to characterize the optimal cost

$$(0.2) \qquad \hat{u}(x) = \inf \{J_x(\theta)/\theta \text{ stopping time}\},$$

and to obtain an optimal stopping time.

We denote by A_0 the second order differential operator associated with the Ito equation[1]

$$(0.3) \qquad A_0 = -\tfrac{1}{2} \operatorname{tr} \left(\sigma \sigma^* \frac{\partial^2}{\partial x^2} \right) - g \frac{\partial}{\partial x},$$

and $A = A_0 + \alpha$.

---

[1] If B is a matrix, then B^* denotes the transpose of B and tr (B) the trace of B.

We define Γ_0 as the set of regular points

$$(0.4) \qquad \Gamma_0 = \{x \in \partial\mathcal{O} / P(\tau_x > 0) = 0\},$$

and we give the following integral formulation of the operator A, inspired by D. W. Stroock and S. R. S. Varadhan [19], for any real bounded measurable function on $\bar{\mathcal{O}}$, u and v.

$Au \leqq v$ in $\bar{\mathcal{O}} \backslash \Gamma_0$ if the process

$$(0.5)^2 \qquad X_t = \int_0^{t \wedge \tau} v(y(s)) e^{-\alpha s} ds + u(y(t \wedge \tau)) e^{-\alpha(t \wedge \tau)}$$

is a strong submartingale for each $x \in \bar{\mathcal{O}} \backslash \Gamma_0$.

Finally, we introduce the problem: To find a real bounded measurable function on $\bar{\mathcal{O}}$, $u(x)$ such that

$$(0.6) \qquad \begin{aligned} u &= 0 \quad \text{on } \Gamma_0, \\ u &\leqq \psi \quad \text{in } \bar{\mathcal{O}} \backslash \Gamma_0, \\ Au &\leqq f \quad \text{in } \bar{\mathcal{O}} \backslash \Gamma_0. \end{aligned}$$

We obtain the following characterization.

THEOREM 0.1. *Assume that g, σ are Lipschitz continuous and that f, ψ are Borel measurable and bounded. Suppose also*

$$(0.7) \qquad \psi(x) \geqq 0 \quad \forall x \in \Gamma_0, \quad \psi \text{ upper semicontinuous.}$$

Then, the problem (0.6) has a maximum solution $\hat{u}$ given explicitly by (0.2). Moreover, if ψ is continuous, the stopping time $\hat{\theta} = \hat{\theta}_x$ defined by

$$(0.8) \qquad \hat{\theta} = \inf \{t \in [0, \tau] / \hat{u}(y(t)) = \psi(y(t))\}$$

is optimal.

We have also the following regularity result.

THEOREM 0.2. *Let the assumptions be as in Theorem 0.1. Suppose that*

$$(0.9) \qquad \Gamma_0 \text{ is a closed set.}$$

Then if the functions f and ψ are upper semicontinuous (or continuous) the optimal cost $\hat{u}$ is also upper semicontinuous (or continuous).

Now in order to use the variational inequality approach, we assume that the open set $\mathcal{O}$ is bounded, with smooth boundary Γ verifying

$$(0.10) \qquad \Gamma = \{x \in \Gamma / |\sigma(x)n(x)| > 0\} \cup \{x \in \Gamma / 2g(x)n(x) < -\text{tr}\,[\sigma\sigma^*(x)]\},$$

where $n(x)$ denotes the inner normal. We remark that (0.10) implies $\Gamma_0 = \Gamma$ (cf. D. W. Stroock and S. R. S. Varadhan [19]).

Denote by $(\cdot, \cdot)$ the duality between $H^{-1}(\mathcal{O})$ and $H_0^1(\mathcal{O})$, and by A the differential operator (0.3).

Let us consider the following degenerate variational inequality associated with the stopping time problem

$$(0.11) \qquad \begin{aligned} u &\in H_0^1(u), \qquad u \leqq \psi \\ (Au, v-u) &\geqq (f, v-u) \quad \forall v \in H_0^1(\mathcal{O}), \quad v \geqq \psi. \end{aligned}$$

---

^{2} $t \wedge \tau$ denotes the minimum between t and τ.

We have

THEOREM 0.3. *Let the assumptions be as in Theorem 0.1. Suppose that f, ψ are Lipschitz continuous, and that conditions (0.10),*

$$(0.12) \qquad\qquad A\psi \in L^\infty(\mathcal{O}),$$

are satisfied.[3] Then, there exists one and only one solution u of the variational inequality (0.11) which is given as the optimal cost (0.2). Moreover, the solution u is Lipschitz continuous and verifies (0.12).

Remark 0.1. A weak formulation of the variational inequality (0.11) is also considered, and the case of an unbounded domain $\mathcal{O}$ is studied.

This work is divided into four sections. The first section gives some useful lemmas. In § 2 we study the penalized problem, and in § 3 we solve the initial problem. Finally, in the last section, we treat the variational inequality.

1. Preliminary lemmas. Let $(\Omega, \mathcal{F}, P)$ be a probability space, $\{\mathcal{F}^t\}_{t \geq 0}$ be a non-decreasing right continuous family of completed sub-σ-fields of $\mathcal{F}$, and $w(t)$ be a Brownian motion in $\mathbb{R}^N$ with respect to $\mathcal{F}^t$.

Suppose we are given two Lipschitz continuous functions $g(x)$ and $\sigma(x)$ on $\mathbb{R}^N$, taking values in $\mathbb{R}^N$ and $\mathbb{R}^N \otimes \mathbb{R}^N$ respectively, $g = (g_i)$, $\sigma = (\sigma_{ij})$,

$$(1.1)^4 \qquad\qquad \frac{\partial g_i}{\partial x_k}, \frac{\partial \sigma_{ij}}{\partial x_k} \in B(\mathbb{R}^N), \qquad i, j, k = 1, \cdots, N.$$

We consider the diffusion $y(t) = y_x(t, \omega)$, $t \geq 0$, $\omega \in \Omega$, and $x \in \mathbb{R}^N$ described by the Ito equation

$$
\begin{aligned}
(1.2) \qquad & dy(t) = g(y(t))\, dt + \sigma(y(t))\, dw(t), \qquad t \geq 0, \\
& y(0) = x.
\end{aligned}
$$

We have

LEMMA 1.1. *Suppose (1.1), and let θ be any stopping time with respect to $\mathcal{F}^t$. Then there exists a constant γ depending only on the Lipschitz constants of g and σ such that*

$$(1.3) \qquad E\{|y_x(\theta) - y_{x'}(\theta)|^2\, e^{-\gamma\theta}\} \leqq |x - x'|^2 \quad \forall x, x' \in \mathbb{R}^N.$$

Proof. We set

$$
\begin{aligned}
(1.4) \qquad \gamma = \sup\Bigg\{ &\mathrm{tr}\left[\frac{(\sigma(x) - \sigma(x'))(\sigma(x) - \sigma(x'))^*}{|x - x'|^2}\right] \\
&+ \frac{2(x - x')(g(x) - g(x'))}{|x - x'|^2} \Big/ x, x' \in \mathbb{R}^N \Bigg\}.
\end{aligned}
$$

Then Ito's formula applied to the function $|x|^2\, e^{-\gamma t}$ and the process $y_x(t) - y_{x'}(t)$ gives

$$
\begin{aligned}
(1.5) \qquad |y_x(t) - y_{x'}(t)|^2\, e^{-\gamma t} \leqq\ & |x - x'|^2 + 2\int_0^t (y_x(s) - y_{x'}(s)) \\
& \cdot [\sigma(y_x(s)) - \sigma(y_{x'}(s))]\, e^{-\gamma s}\, dw(s).
\end{aligned}
$$

Hence (1.3) follows. $\quad\square$

---

[3] We also assume α large enough, $\mathcal{O}$ bounded, and Γ smooth.

[4] $B(\mathbb{R}^N)$ denotes the set of all Borel measurable and bounded functions on $\mathbb{R}^N$ taking values in $\mathbb{R}$.

Remark 1.1. Using the martingale inequality

$$(1.6) \qquad E\left\{ \sup_{t \geq 0} \left| \int_0^t \phi(s) \, dw(s) \right| \right\} \leq 3E\left\{ \sqrt{\int_0^t \phi^2(s) \, ds} \right\},$$

and the same technique as in Lemma 1.1, we can obtain

$$(1.7) \qquad E\{\sup_{t \geq 0} |y_x(t) - y_{x'}(t)|^k \, e^{-\gamma t}\} \leq C |x - x'|^k \quad \forall x, x' \in \mathbb{R}^N,$$

where the constants γ and C depend only on $k > 0$ and on the Lipschitz constants of $g(x)$ and $\sigma(x)$. $\square$

Now let $\tau = \tau_x(\omega)$ and $\tau'_x = \tau'(\omega)$ be the first exit time of the process $y(t)$ from the closed set $\bar{\mathcal{O}}$ and the open set $\mathcal{O}$ respectively,

$$(1.8)^5 \qquad\qquad \tau = \inf \{t \geq 0 / y(t) \notin \bar{\mathcal{O}}\},$$

and a similar definition for τ' with $\mathcal{O}$ instead of $\bar{\mathcal{O}}$.

We have

LEMMA 1.2. *Suppose* (1.1). *Then, for any constant* $T > 0$ *and* $x \in \mathbb{R}^N$, *we have*

$$(1.9)^6 \qquad\qquad \lim_{z \to x} E\{(T \wedge \tau'_x - T \wedge \tau'_z)^+\} = 0,$$

$$(1.10) \qquad\qquad \lim_{z \to x} E\{(T \wedge \tau_z - T \wedge \tau_x)^+\} = 0.$$

Proof. Let z_n be a sequence, $z_n \to x$, and let us consider the diffusions $y_n(t)$, $y(t)$ starting respectively at z_n, x. Using Lemma 1.1, we can assume that

$$\lim \sup_{0 \leq t \leq T} |y_n(t) - y(t)| = 0 \quad \text{a.s.}$$

In order to obtain (1.9), we will prove

$$(1.11) \qquad\qquad \underline{\lim} \, \tau'_n \geq \tau' \quad \text{a.s.}$$

We assume $\omega \in \Omega$ fixed. Then, if $\tau' = 0$, (1.11) is clearly verified, and so we can suppose $0 < \delta < \tau'$ and define the set $K_\omega = \{y(t)/t \in [0, \delta]\}$ which is a compact subset of $\mathcal{O}$. Hence for n large enough, $n \geq N_\omega$,

$$\{y_n(t)/t \in [0, \delta]\} \subset \mathcal{O}.$$

Thus $\tau'_n \geq \delta$ and taking the limit,

$$\underline{\lim} \, \tau'_n \geq \delta;$$

since δ is arbitrary, we deduce (1.11).

Now we are going to prove

$$(1.12) \qquad\qquad \overline{\lim} \, \tau_n \leq \tau \quad \text{a.s.},$$

so that (1.10) holds.

We assume $\omega \in \Omega$ fixed. Then, if $\tau = \infty$, (1.12) is clearly verified, so we can assume $\delta > \tau$. Hence, there exists $s < \delta$ such that $y(s) \notin \bar{\mathcal{O}}$. Thus for n large enough, $y_n(s) \notin \bar{\mathcal{O}}$, so $\tau_n \leq s < \delta$, and taking the limit

$$\underline{\lim} \, \tau_n \leq \delta;$$

since δ is arbitrary, we deduce (1.12). $\square$

---

[5] $\tau = +\infty$ if $y(t) \in \bar{\mathcal{O}} \; \forall t \geq 0$.

[6] If $a \in \mathbb{R}$ we denote by a^+ the maximum between a and zero.

Remark 1.2. From E. B. Dynkin [6, Theorem 10.2, p. 302] it follows that either the process $y_x(t)$ stopped at the exit of $\mathcal{O}$, or $\bar{\mathcal{O}}$ is a strong Markov process. Also observe that τ and τ' are stopping times with respect to the family $\mathcal{F}^t$.

D. W. Stroock and S. R. S. Varadhan [19] proved Lemma 1.2 in a different way.

Remark 1.3. We recall the following martingale property: Let $a(t)$ and $b(t)$ be measurable adapted and bounded processes, such that

$$M_t = a(t) + \int_0^t b(s)\, ds \quad \text{is a martingale.}$$

Then, for any arbitrary measurable adapted and bounded process $c(t)$, the process

$$a(t) \exp\left(-\int_0^t c(s)\, ds\right) + \int_0^t (b(s) + c(s)a(s)) \exp\left(-\int_0^s c(r)\, dr\right) ds$$

is the martingale

$$M_0 + \int_0^t \exp\left(-\int_0^s c(r)\, dr\right) dM_s.$$

Now, we define the set Γ_0 of regular points (cf. D. W. Stroock and S. R. S. Varadhan [19]), $\Gamma = \partial \mathcal{O}$,

(1.13) $$\Gamma_0 = \{x \in \Gamma / P(\tau_x > 0) = 0\}.$$

We have

LEMMA 1.3. *Assume* (1.1) *and that*

(1.14) $$\Gamma_0 \text{ is a closed set.}$$

Then for any constant $T > 0$, and $x \in \mathcal{O}$, we have

(1.15) $$\lim_{z \to x} E\{|T \wedge \tau_z - T \wedge \tau_x|\} = 0, \qquad z \in \bar{\mathcal{O}}.$$

Proof. Let $\tilde{\tau} = \tilde{\tau}_x(\omega)$ be the first exit time from $\bar{\mathcal{O}} \backslash \Gamma_0$ of the process $y(t)$. From the strong Markov property of the process $y(t)$ stopped at the exit of $\bar{\mathcal{O}}$, we easily deduce

(1.16) $$P(\tau \neq \tilde{\tau}) = 0.$$

Later on, we will show

(1.17) $$\lim_{z \to x} E\{(T \wedge \tilde{\tau}_x - T \wedge \tilde{\tau}_z)^+\} = 0, \qquad z \in \bar{\mathcal{O}}.$$

Indeed, we assume $\omega \in \Omega$ fixed and the notations of Lemma 1.2 with $\tilde{\tau}$ instead of τ'. Then, without loss of generality, we suppose $0 < \delta < \tilde{\tau}$, and we define the set $K_\omega = \{y(t)/t \in [0, \delta]\}$, which is a compact subset of $\mathcal{O}$ such that $K_\omega \cap \Gamma_0 = \varnothing$. Because of (1.14), for n sufficiently large, $n \geq N_\omega$, we have

$$\{y_n(t)/t \in [0, \delta]\} \subset \bar{\mathcal{O}} \backslash \Gamma_0.$$

Thus $\tilde{\tau}_n \geq \delta$, and taking the limit we obtain

$$\underline{\lim} \, \tilde{\tau}_n \geq \tilde{\tau} \quad \text{a.s.}$$

So, (1.17) follows.

Finally, by combining (1.16), (1.17) and (1.10) the lemma is proved. $\quad\square$

Remark 1.4. In D. W. Stroock and S. R. S. Varadhan [19] it is proved that, assuming (1.14), we have $\tau_x = \tau_x'$ a.s. for each $x \in \mathcal{O} \cup \Gamma_0$. Then we deduce Lemma 1.3 for the particular case $x \in \mathcal{O} \cup \Gamma_0$. Notice that Lemma 1.3 implies that the process $y(t \wedge \tau)$ is Feller continuous on the whole domain $\bar{\mathcal{O}}$.

Let us consider the differential operator A given by (0.3) where α is a constant large enough, $2\alpha \geqq \gamma$, defined in Lemma 1.1.

LEMMA 1.4. *Suppose* (1.1). *Let* $f(x)$, $\psi(x)$, *and* $\bar{u}(x)$ *be continuous real function on* $\bar{\mathcal{O}}$ *such that*

$$
\bar{u} \in C(\bar{\mathcal{O}}), \qquad \frac{\partial \bar{u}}{\partial x_i} \in L^\infty(\mathcal{O}), \qquad i = 1, \cdots, N,
$$

(1.18)

$$
\bar{u} \leqq \psi \text{ in } \mathcal{O}, \qquad \bar{u}(x) = \mathcal{O} \quad \forall x \in \Gamma_0,
$$

$$
A\bar{u} \leqq -|f| \quad \text{in } \mathcal{D}'(\mathcal{O}).
$$

Then for any nonnegative, bounded and adapted process $\delta(t) = \delta(t, \omega)$, *the following estimate holds*

(1.19)

$$
E\left\{ \int_{\tau_x \wedge \tau_{x'}}^{\tau_x} (|f(y_x(t))| - \delta(t)\psi(y_x(t))) \exp\left(-\int_0^t (\alpha - \delta(s)) \, ds \right) dt \right\}
$$

$$
\leqq \left\| \frac{\partial \bar{u}}{\partial x} \right\| |x - x'| \quad \forall x, x' \in \bar{\mathcal{O}}
$$

where $\|\partial \bar{u}/\partial x\|$ *denotes the smallest Lipschitz continuous constant of the function* $\bar{u}$.

Proof. First suppose $\bar{u} \in C^2(\mathcal{O})$. Ito's formula applied to the function $\bar{u}(x)$ and the process $y_x(t)$ gives

(1.20)

$$
E\left\{ \bar{u}(y_x(\tau_x)) \exp\left(-\int_0^{\tau_x} (\alpha + \delta(t)) \, dt \right) - \bar{u}(y_x(\tau_x \wedge \tau_{x'})) \right.
$$

$$
\left. \cdot \exp\left(-\int_0^{\tau_x \wedge \tau_{x'}} (\alpha + \delta(t)) \, dt \right) \right\}
$$

$$
= -E\left\{ \int_{\tau_x \wedge \tau_{x'}}^{\tau_x} [(A\bar{u})(y_x(t)) + \delta(t)\bar{u}(y_x(t))] \exp\left(-\int_0^\tau (\alpha + \delta(s)) \, ds \right) dt \right\}.
$$

Using

$$
\bar{u}(y_x(\tau_x)) = 0 = \bar{u}(y_{x'}(\tau_x \wedge \tau_{x'})) \quad \text{a.s. in } [\tau_{x'} \leqq \tau_x < \infty],
$$

from (1.20) we have

(1.21)

$$
E\left\{ \int_{\tau_x \wedge \tau_{x'}}^{\tau_x} (|f(y(t))| - \delta(t)\psi(y_x(t))) \exp\left(-\int_0^t (\alpha + \delta(s)) \, ds \right) dt \right\}
$$

$$
\leqq E\{|\bar{u}(y_x(\tau_x \wedge \tau_{x'})) - \bar{u}(y_{x'}(\tau_x \wedge \tau_{x'}))| \, e^{-\alpha(\tau_x \wedge \tau_{x'})}\}.
$$

Next, choosing $\theta = \tau_x \wedge \tau_{x'}$ in Lemma 1.1, we deduce from (1.21) the estimate (1.19).

Finally, if $\bar{u} \notin C^2(\mathcal{O})$, by approximating $\bar{u}$ by regular functions the lemma is proved. $\square$

Remark 1.5. Clearly, Lemma 1.4 implies

(1.22)

$$
E\{|e^{-\alpha\tau_x} - e^{-\alpha\tau_{x'}}|\} \leqq 2\alpha \left\| \frac{\partial u_0}{\partial x} \right\| |x - x'|, \qquad x, x' \in \bar{\mathcal{O}},
$$

if $2\alpha \geqq \gamma$ and u_0 is a Lipschitz continuous function on $\bar{\mathcal{O}}$, vanishing on Γ_0, such that $Au_0 \leqq -1$ in $\mathcal{D}'(\mathcal{O})$.

Remark 1.6. For instance, suppose $\mathcal{O}$ is a bounded domain given by

(1.23)
$$\mathcal{O} = \{x \in \mathbb{R}^N / \phi(x) < 0\}, \qquad \phi \in C^2,$$
$$\Gamma = \{x \in \mathbb{R}^N / \phi(x) = 0\}, \qquad |\nabla \phi(x)| \geqq 1 \quad \forall x \in \Gamma,$$

and assume

(1.24)
$$A_0 \phi \leqq -1 \quad \text{on } \Gamma.$$

Then for any continuous functions f and ψ on $\bar{\mathcal{O}}$, $\psi \in C^2(\bar{\mathcal{O}})$, $\psi \geqq 0$ on Γ; we can take $\bar{u} = \lambda \phi$, which verifies (1.18) for α and λ large enough. Clearly, applying Ito's formula to the function $\bar{u}$ and process $y(t)$ between τ' and τ, we deduce $\Gamma_0 = \Gamma$.

Now, some sufficient conditions for the existence of a Lipschitz continuous subsolution are given using barrier functions as in [11].

LEMMA 1.5. *Assume* (1.1). *Suppose also that $\mathcal{O}$ is bounded, has the uniform exterior sphere property*[7]

(1.25) *There exist $\rho > 0$ such that for each point $\xi \in \Gamma$ there is a ball $B = B(\xi^*, \rho)$ of radius ρ and center ξ^* verifying $\bar{\mathcal{O}} \cap B = \{\xi\}$,*

and

(1.26) $\Gamma = \{x \in \Gamma / |\sigma(x)n(x)| > 0\} \cup \{x \in \Gamma / 2g(x)n(x) < -\text{tr}\,[\sigma\sigma^*(x)]\}$,

$n(x)$ is the inner normal of modulus ρ.

Then $\Gamma_0 = \Gamma$, and there exists a Lipschitz continuous subsolution $u_0(x)$

(1.27)
$$u_0 \in C(\bar{\mathcal{O}}), \qquad \frac{\partial u_0}{\partial x_i} \in L^\infty(\bar{\mathcal{O}}), \qquad i = 1, \cdots, N,$$
$$Au_0 \leqq -1 \text{ in } \mathcal{D}'(\mathcal{O}), \qquad u_0 = 0 \quad \text{on } \Gamma.$$

Proof. It is necessary to prove only (1.27).

Introducing the barrier functions $k > 0$, $\xi \in \Gamma$, $x \in \bar{\mathcal{O}}$,

$$v(x, \xi) = \exp\,(-k|x - \xi^*|^2) - \exp\,(-k\rho^2),$$

we have from (1.26) $A_0 v(x, \xi) \leqq -2\beta < 0$, if $x = \xi$ and k is sufficiently large independent of ξ. Hence, by continuity, we have for some $\delta > 0$,

(1.28) $A_0 v(x, \xi) \leqq -\beta < 0 \quad \forall x \in U_\xi = \{x \in \mathcal{O} / |x - \xi| < \delta\};$

now using the fact that $v(x, \xi) \leqq -\gamma < 0 \; \forall x \in \bar{\mathcal{O}} \setminus U_\xi$, er deduce, for α large enough,

(1.29) $Av(x, \xi) \leqq -\beta < 0 \quad \forall x \in \bar{\mathcal{O}}.$

Finally, remarking that $v(x, \xi)$ are equi-Lipschitz continuous in $x \in \bar{\mathcal{O}}$, we set

(1.30)
$$u_0(x) = \frac{1}{\beta} \sup\,\{v(x, \xi) / \xi \in \Gamma\}.$$

Hence, $Au_0 \leqq -1$ in the martingale sense (0.5) and in the distribution sense. $\square$

Remark 1.7. Suppose u_0 given as in Lemma 1.5. Then for any f, $\psi \in C(\bar{\mathcal{O}})$ such that

(1.31) $\psi \geqq 0$ on Γ and $A\psi \in L^\infty(\mathcal{O})$,

and taking $\bar{u} = \lambda u_0$, where $\lambda \geqq \|f\| + \|A\psi\|$,[8] we deduce (1.18).

---

[7] The constant α is supposed large enough.

[8] $\|\cdot\|$ denotes the L^∞-norm in $\mathcal{O}$.

Remark 1.8. Clearly, using other barrier functions, different sufficient conditions for the existence of a Lipschitz continuous subsolutions may be obtained.

2. Penalized problem. Before studying the stopping time problem we will start with an intermediate stochastic control problem.

We call an admissible control ν a scalar measurable adapted process such that $0 \le \nu(t; \omega) \le 1$, $t \ge 0$.

Let $f(x)$, $\psi(x)$ be functions such that

$$(2.1) \qquad\qquad\qquad f, \psi \in B(\bar{\mathcal{O}}),$$

and let α be a positive constant. We define the functional J_x^ε, $\varepsilon > 0$,

$$(2.2) \qquad J_x^\varepsilon(\nu) = E\left\{ \left[\int_0^\tau f(y(t)) + \frac{1}{\varepsilon} \nu(t) \psi(y(t)) \right] \exp\left(-\int_0^t \left(\alpha + \frac{1}{\varepsilon} \nu(s) \right) ds \right) dt \right\},$$

and we wish to characterize the optimal penalized cost,

$$(2.3) \qquad\qquad u_\varepsilon(x) = \inf \{ J_x^\varepsilon(\nu)/\nu \text{ any admissible control} \}.$$

The integral formulation of operator A (cf. D. W. Stroock and S. R. S. Varadhan [19]) is given for $u, v \in B(\bar{\mathcal{O}})$ by

$$Au = v \text{ in } \bar{\mathcal{O}} \backslash \Gamma_0 \quad \text{if the process}$$

$$(2.4) \qquad\qquad X_t = \int_0^{t \wedge \tau} v(y(s)) e^{-\alpha s} ds + u(y(t \wedge \tau)) e^{-\alpha(t \wedge \tau)}$$

is a martingale for each $x \in \bar{\mathcal{O}} \backslash \Gamma_0$.

We remark that if $Au = v$ in the sense of (2.4), then we also have $Au = v$ in the distribution in $\mathcal{O}$ for σ smooth.

Next, the following problem is considered: To find a function $u_\varepsilon(x)$ such that

$$(2.5) \qquad\qquad u_\varepsilon \in B(\bar{\mathcal{O}}), \qquad u_\varepsilon(x) = 0 \quad \forall x \in \Gamma_0,$$

$$(2.6) \qquad\qquad Au_\varepsilon = f - \frac{1}{\varepsilon}(u_\varepsilon - \psi)^+ \quad \text{in } \bar{\mathcal{O}} \backslash \Gamma_0 \text{ [in the martingale sense]}.$$

Remark 2.1. Let $\Phi(t)$ be the semigroup in $B(\bar{\mathcal{O}})$ given by

$$(2.7) \qquad\qquad \Phi(t)h = E\{ h(y(t \wedge \tau)) e^{-\alpha(t \wedge \tau)} \},$$

and $\mathscr{X}$ be the characteristic function of the set $\bar{\mathcal{O}} \backslash \Gamma_0$.

Then, using the strong Markov property of process $y(t)$ stopped at the exit of $\mathcal{O}$, we show that (2.5) and (2.6) and the condition $u_\varepsilon \in B(\bar{\mathcal{O}})$,

$$(2.8) \qquad u_\varepsilon = \int_0^t \Phi(s) \left(f\mathscr{X} - \frac{1}{\varepsilon}(u_\varepsilon - \psi)^+ \mathscr{X} \right) ds + \Phi(t)(u_\varepsilon \mathscr{X}) \quad \forall t \ge 0,$$

are equivalent. Moreover, the condition

$$(2.9) \qquad\qquad u_\varepsilon = \int_0^\infty \Phi(t) \left(f\mathscr{X} - \frac{1}{\varepsilon}(u_\varepsilon - \psi)^+ \mathscr{X} \right) dt$$

is also equivalent to (2.8).

Remark 2.2. The semigroup formulation (2.8) is used by A. Bensoussan [2] for the nondegenerate case. Here, if we assume that the set of regular points Γ_0 is closed, the

stopping process is Feller continuous (because of Lemma 1.3). Then, a semigroup formulation can also be studied as in M. Robin [18].

This section is divided in three parts. First we solve problem (2.5), (2.6). Next, we consider the case where the set of regular points Γ_0 is closed. Finally, we give some complementary results.

2.1. Existence and semicontinuity results. We have

THEOREM 2.1. *Assume* (1.1) *and* (2.1). *Then problem* (2.5), (2.6) *has one and only one solution* u_ε *which is given by* (2.3).

Proof. First we prove that problem (2.5), (2.6) has one and only one solution $w_\varepsilon(x)$. Indeed, from the equality

$$-\frac{1}{\varepsilon}(w_\varepsilon - \psi)^+ = -\frac{1}{\varepsilon}w_\varepsilon + \frac{1}{\varepsilon}(w_\varepsilon \wedge \psi),$$

and applying Remark 1.3 for

$$a(t) = w_\varepsilon(y(\tau \wedge \tau))\, e^{-\alpha(t \wedge \tau)}, \qquad c(t) = \frac{1}{\varepsilon},$$

$$b(t) = \begin{cases} f(y(t))\, e^{-\alpha t} & \text{if } t \le \tau, \\ 0 & \text{otherwise,} \end{cases}$$

we deduce that the conditions (2.5), (2.6) are equivalent to (2.5),

$(2.10)^9$
$$\left(A + \frac{1}{\varepsilon}\right)w_\varepsilon = f + \frac{1}{\varepsilon}(w_\varepsilon \wedge \psi).$$

So, using the strong Markov property, we only need to find a unique solution of the equation,

$$(2.11) \qquad w_\varepsilon = E\left\{ \int_0^\tau \left[f(y(t)) + \frac{1}{\varepsilon}(w_\varepsilon \wedge \psi)(y(t)) \right] \exp\left(-\alpha t - \frac{1}{\varepsilon}t\right) dt \right\}.$$

Thus, we define the operator T_ε in $B(\bar{\mathcal{O}})$ by

$$(2.12) \qquad T_\varepsilon w = E\left\{ \int_0^\tau \left[f(y(t)) + \frac{1}{\varepsilon}(w \wedge \psi)(y(t)) \right] \exp\left(-\alpha t - \frac{1}{\varepsilon}t\right) dt \right\},$$

and we have[10]

$$\left\| T_\varepsilon v - T_\varepsilon w \right\| \le \frac{1}{1 + \alpha\varepsilon} \left\| v - w \right\|.$$

Hence, T_ε is a contraction in $B(\bar{\mathcal{O}})$ and so the equation (2.11) has one and only one solution.

Next, we are going to show that the solution of problem (2.5), (2.6) is given by (2.3). Indeed, let w_ε be the solution for (2.5), (2.6). Then using Remark 1.3 with $\delta(t) = (1/\varepsilon)\nu(t)$, $\nu(t)$ any admissible control, we obtain

$$w_\varepsilon = E\left\{ \int_0^\tau \left[f - \frac{1}{\varepsilon}(w_\varepsilon - \psi)^+ + \frac{1}{\varepsilon}\nu(t)w_\varepsilon \right](y(t)) \exp\left(-\int_0^t \left(\alpha + \frac{1}{\varepsilon}\nu(s)\right) ds\right) dt \right\}.$$

---

[9] w_ε instead of u_ε.

[10] $\|\cdot\|$ denotes the supremum norm in $\bar{\mathcal{O}}$.

Since

$$-(w_\varepsilon - \psi)^+ + \nu w_\varepsilon \leqq \nu\psi \quad \text{if } 0 \leqq \nu \leqq 1,$$

we have

(2.13) $w_\varepsilon(x) \leqq J_x^\varepsilon(\nu), \quad \nu$ any admissible control,

and for

$$\hat{\nu}(t) = \begin{cases} 1 & \text{if } w_\varepsilon(y(t)) > \psi(y(t)), \\ 0 & \text{if } w_\varepsilon(y(t)) \leqq \psi(y(t)), \end{cases}$$

(2.14) $w_\varepsilon(x) = J_x^\varepsilon(\hat{\nu}).$

Thus, (2.13) and (2.14) give $w_\varepsilon = u_\varepsilon$. $\square$

Remark 2.3. If u_ε and $\tilde{u}_\varepsilon$ denote the functions given by (2.3) with f, ψ and $\tilde{f}, \tilde{\psi}$ respectively, the following estimate is true,

(2.15) $\|u_\varepsilon - \tilde{u}_\varepsilon\| \leqq \dfrac{1}{\alpha}\|f - \tilde{f}\| + \|\psi - \tilde{\psi}\|,$

where $\|\cdot\|$ denotes the norm of supremum over $\bar{\mathcal{O}}$.

It is possible to consider the case with τ' instead of τ and to obtain analogous results.

Now we study properties of continuity on u_ε. We have

THEOREM 2.2. *Let the conditions* (1.1), (2.1) *hold. Then if f and ψ are nonnegative upper semicontinuous on $\bar{\mathcal{O}}$, so is u_ε defined by* (2.3).

Proof. Letting T be a positive constant, we define

(2.16) $J_x^\varepsilon(\nu, T) = E\left\{\displaystyle\int_0^{T \wedge \tau} \left[f(y)(t)) + \dfrac{1}{\varepsilon}\nu(t)\psi(y(t)) \right] \exp\left(-\int_0^t \left(\alpha + \dfrac{1}{\varepsilon}\nu(s)\right) ds\right) dt\right\}$

and

(2.17) $u_\varepsilon^T(x) = \inf\{J_x^\varepsilon(\nu, T)/\nu$ any admissible control$\}.$

We have the estimate

(2.18) $\|u_\varepsilon^T - u_\varepsilon\| \leqq \left(\dfrac{1}{\alpha}\|f\| + \|\psi\|\right) e^{-\alpha T}.$

So it is sufficient to consider u_ε^T instead of u_ε.

Then, we start with

$$u_\varepsilon^T(z) - u_\varepsilon^T(x) \leqq \sup\{[J_z^\varepsilon(\nu, T) - J_x^\varepsilon(\nu, T)]/\nu \text{ any admissible control}\}.$$

Next it follows that

$$u_\varepsilon^T(z) - u_\varepsilon^T(x) \leqq \left(\|f\| + \dfrac{1}{\varepsilon}\|\psi\|\right) E\{(T \wedge \tau_z - T \wedge \tau_x)^+\}$$

(2.19)
$$+ E\left\{\int_0^{T \wedge \tau_z \wedge \tau_x} [f(y_z(t)) - f(y_x(t))]^+ e^{-\alpha t} dt\right\}$$

$$+ \dfrac{1}{\varepsilon} E\left\{\int_0^{T \wedge \tau_z \wedge \tau_x} [\psi(y_z(t)) - \psi(y_x(t))]^+ e^{-\alpha t} dt\right\}.$$

Thus taking the limit in (2.19) and using (1.3), (1.10), the theorem is proved. $\square$

Remark 2.4. Let $u'_\varepsilon(x)$ be the optimal cost in the open set $\mathcal{O}$; that is, u'_ε is defined by (2.3) with τ' instead of τ. Then a similar theorem of regularity is proved: If f and ψ are nonnegative lower semicontinuous on $\bar{\mathcal{O}}$, so is u'_ε. Notice that the function u'_ε is the solution of (2.5), (2.6) with Γ, τ' instead of Γ_0, τ.

2.2. Regular case. In this part we assume that

(2.20) $\qquad\qquad \Gamma_0$ given by (1.13) is a closed set,

so we have

THEOREM 2.3. *Suppose* (1.1), (2.1), *and* (2.20) *hold. Then if f and ψ are upper (lower) semicontinuous on $\bar{\mathcal{O}}$, so is u_ε given by* (2.3).

Proof. The proof is similar to Theorem 2.2 from

$$u_\varepsilon^T(z) - u_\varepsilon^T(x) \le \left\{ \|f\| + \frac{1}{\varepsilon}\|\psi\| \right\} E\{|T \wedge \tau_z - T \wedge \tau_x|\}$$

$$+ E\left\{ \int_0^{T \wedge \tau_z \wedge \tau_x} [f(y_z(t)) - f(y_x(t))]^+ \, e^{-\alpha t} \, dt \right\}$$

$$+ \frac{1}{\varepsilon} E\left\{ \int_0^{T \wedge \tau_z \wedge \tau_x} [\psi(y_z(t)) - \psi(y_z(t))]^+ \, e^{-\alpha t} \, dt \right\};$$

using (1.3) and (1.15) gives the result. $\square$

Remark 2.5. Let $\mathcal{O}$ be smooth and $n(x)$ be the inner normal of boundary $\Gamma = \partial\mathcal{O}$. Suppose that

(2.21)
$$\sigma(x) = 0 \qquad \forall x \in \mathbb{R}^N \setminus \mathcal{O},$$
$$g(x)n(x) \geqq 0 \quad \forall x \in \Gamma;$$

then $\Gamma_0 = \varnothing$, so (2.20) is true. Clearly, if $\mathcal{O} = \mathbb{R}^N$, (2.20) can be removed.

Now we are going to obtain some a priori estimates.

THEOREM 2.4. *Assume* (1.1), (2.1), $\mathcal{O} = \mathbb{R}^N$, *and*

(2.22) $\qquad \dfrac{\partial f}{\partial x_i}, \dfrac{\partial \psi}{\partial x_i} \in L^\infty(\mathbb{R}^N), \qquad i = 1, \cdots, N.$

Then u_ε is Lipschitz continuous and verifies

(2.23)[11] $\qquad \left\| \dfrac{\partial u_\varepsilon}{\partial x} \right\| \leqq \dfrac{1}{\alpha - \gamma_0} \left\| \dfrac{\partial f}{\partial x} \right\| + \left\| \dfrac{\partial \psi}{\partial x} \right\|.$

Proof. Let T_ε be the operator defined by (2.12). From Theorem 2.1, u_ε is the fixed point of the contraction T_ε. Suppose w is a Lipschitz continuous function on $\mathbb{R}^N$, and denote $\alpha_0 = \alpha - \frac{1}{2}\gamma > 0$; then from (1.3) it follows that

(2.24)[12] $\qquad \left\| \dfrac{\partial T_\varepsilon w}{\partial x} \right\| \leqq \dfrac{\varepsilon}{1 + \varepsilon\alpha_0} \left\| \dfrac{\partial f}{\partial x} \right\| + \dfrac{1}{1 + \varepsilon\alpha_0} \left(\left\| \dfrac{\partial w}{\partial x} \right\| \vee \left| \dfrac{\partial \psi}{\partial x} \right| \right).$

Thus, (2.24) implies

$$\left\| \dfrac{\partial T_\varepsilon^k w}{\partial x} \right\| \leqq \varepsilon \left\| \dfrac{\partial f}{\partial x} \right\| \sum_{i=1}^k (1 + \varepsilon\alpha_0)^{-i} + \dfrac{1}{1 + \varepsilon\alpha_0} \left(\left\| \dfrac{\partial w}{\partial x} \right\| \vee \left\| \dfrac{\partial \psi}{\partial x} \right\| \right).$$

---

[11] γ is given by (1.4), and $\|\partial f/\partial x\|$ denotes the smallest Lipschitz of f.

[12] If $a, b \in R$, then $a \vee b$ denotes the maximum between a and b.

Hence

$$(2.25) \qquad \left\| \frac{\partial T_\varepsilon^k w}{\partial x} \right\| \leqq \frac{1}{\alpha_0} \left\| \frac{\partial f}{\partial x} \right\| + \left(\left\| \frac{\partial w}{\partial x} \right\| \vee \left\| \frac{\partial \psi}{\partial x} \right\| \right),$$

and taking $w = 0$ and letting $k \to \infty$ in (2.25) we prove (2.23). □

THEOREM 2.5. *Let the assumptions* (1.1) *and* (2.1) *hold. Suppose that there exists a Lipschitz continuous subsolution, i.e.,*

$$\bar{u} \in C(\bar{\mathcal{O}}), \qquad \frac{\partial u}{\partial x_i} \in L^\infty(\mathcal{O}), \qquad i = 1, \cdots, N,$$

$$(2.26) \qquad \bar{u} \leqq \psi \text{ in } \mathcal{O}, \qquad \bar{u}(x) = 0 \quad \forall x \in \Gamma_0,$$

$$A\bar{u} \leqq -|f| \quad \text{in } \mathcal{D}'(\mathcal{O}),$$

and

$$(2.27) \qquad \frac{\partial f}{\partial x_i}, \ \frac{\partial \psi}{\partial x_i} \in L^\infty(\mathcal{O}), \qquad i = 1, \cdots, N.$$

Then u_ε is Lipschitz continuous on $\bar{\mathcal{O}}$, and verifies

$$(2.28) \qquad \left\| \frac{\partial u_\varepsilon}{\partial x} \right\| \leqq \frac{1}{\alpha - \gamma} \left\| \frac{\partial f}{\partial x} \right\| + \left\| \frac{\partial \psi}{\partial x} \right\| + \left\| \frac{\partial \bar{u}}{\partial x} \right\|.$$

Proof. Starting at

$$u_\varepsilon(x) - u_\varepsilon(x') = \sup_{\nu'} \inf_\nu [J_x^\varepsilon(\nu) - J_{x'}^\varepsilon(\nu)],$$

and taking

$$\nu(t) = \begin{cases} \nu'(t) & \text{if } t \in [0, \tau_x \wedge \tau_{x'}], \\ 0 & \text{otherwise}, \end{cases}$$

we have

$$u_\varepsilon(x) - u_\varepsilon(x') \leqq E \left\{ \int_0^{\tau_x \wedge \tau_{x'}} |f(y_x(t)) - f(y_{x'}(t))| \, e^{-\alpha t} \, dt \right\}$$

$$+ \sup_{\nu'} E \left\{ \int_0^{\tau_x \wedge \tau_{x'}} \frac{1}{\varepsilon} \nu'(t) |\psi(y_x(t)) - \psi(y_{x'}(t))| \right.$$

$$\exp \left(-\int_0^t \left(\alpha + \frac{1}{\varepsilon} \nu'(s) \right) ds \right) dt$$

$$+ \int_{\tau_x \wedge \tau_{x'}}^{\tau_{x'}} \left[f^-(y_{x'}(t)) - \frac{1}{\varepsilon} \nu'(t) \psi(y_{x'}(t)) \right] \exp \left(-\int_0^t \left(\alpha + \frac{1}{\varepsilon} \nu'(s) \right) ds \right) dt$$

$$\left. + \int_{\tau_x \wedge \tau_x}^{\tau_x} f^+(y_x(t)) \exp \left(-\int_0^t \left(\alpha + \frac{1}{\varepsilon} \nu'(s) \right) ds \right) dt \right\}.$$

Next, using Lemmas 1.1 and 1.4 we obtain

$$(2.29) \qquad u_\varepsilon(x) - u_\varepsilon(x') \leqq \left[\frac{1}{\alpha - \gamma_0} \left\| \frac{\partial f}{\partial x} \right\| + \left\| \frac{\partial \psi}{\partial x} \right\| + \left\| \frac{\partial \bar{u}}{\partial x} \right\| \right] |x - x'|.$$

Clearly, from (2.29) the theorem is proved. □

Remark 2.6. Notice that condition (2.26) implies (2.20). Indeed, from Remark 1.5, the function $x \to E\{\exp(-\alpha \tau_x)\}$ is continuous on $\bar{\mathcal{O}}$. Then, using the fact that $\Gamma_0 = \{x \in \bar{\mathcal{O}} / E\{\exp(-\alpha \tau_x)\} = 1\}$, we reach our conclusions.

2.3. Complementary results. Now, we consider u_ε as a distribution in $\mathcal{O}$. Let A be the differential operator

$$(2.30) \qquad A = -\tfrac{1}{2} \operatorname{tr}\left(\sigma\sigma^* \frac{\partial^2}{\partial x^2}\right) - g \frac{\partial}{\partial x} + \alpha.$$

Assume

$$(2.31) \qquad \frac{\partial^2 \sigma\sigma^*}{\partial x^2} \in L^1_{\text{loc}}(\mathcal{O}).$$

So we can define Au for $u \in B(\bar{\mathcal{O}})$, as a distribution in $\mathcal{O}$, by

$$(2.32) \qquad \langle Au, \phi \rangle = \int_{\mathcal{O}} u A^* \phi \, dx \quad \forall \phi \in \mathscr{D}(\mathcal{O}),$$

where A^* is the operator

$$(2.33) \qquad A^* \phi = -\tfrac{1}{2} \operatorname{tr}\left(\frac{\partial^2 \sigma\sigma^* \phi}{\partial x}\right) + g \frac{\partial \phi}{\partial x} + \alpha \phi.$$

Then we have

THEOREM 2.6. *Let the conditions* (1.1), (2.1), *and* (2.3) *hold. Suppose that the boundary* Γ *is smooth. Then the optimal cost* u_ε *given by* (2.3) *satisfies*

$$(2.34) \qquad Au_\varepsilon + \frac{1}{\varepsilon}(u_\varepsilon - \psi)^+ = f \quad in \ \mathscr{D}'(\mathcal{O}).$$

Moreover, if

$$(2.35)^{[13]} \qquad there \ exists \ w \in B(\bar{\mathcal{O}}) \ such \ that \ A\psi = w \ in \ \bar{\mathcal{O}} \backslash \Gamma_0,$$

$$(2.36) \qquad \psi(x) \geqq 0 \quad \forall x \in \Gamma_0,$$

the following estimate is true.

$$(2.37) \qquad \|Au_\varepsilon\| \leqq \|f\| + \|(f - A\psi)^+\|.$$

Proof. Equation (2.34) is obtained by regularization, or as in D. W. Stroock and S. R. S. Varadhan [19] using an argument of monotone class. In order to get (2.37) we will show that

$$(2.38) \qquad \|(u_\varepsilon - \psi)^+\| \leqq \varepsilon \|(f - A\psi)^+\|.$$

Indeed, from (2.35) and Remark 1.3 we have

$$\psi = E\left\{\int_0^\tau \left[A\psi(y(t)) + \frac{1}{\varepsilon}\psi(y(t))\right] \exp\left(-\alpha t - \frac{1}{\varepsilon}t\right) dt\right\}$$

$$+ E\left\{1_{\tau < \infty} \psi(y(\tau)) \exp\left(-\alpha \tau - \frac{1}{\varepsilon}\tau\right)\right\}.$$

---

[13] In the martingale sense of (2.4).

Since

$$u_\varepsilon - \psi = E\left\{\int_0^\tau [f(y(t)) - A\psi(y(t))]\exp\left(-\alpha t - \frac{1}{\varepsilon}t\right)dt\right\}$$

$$-\frac{1}{\varepsilon}E\left\{\int_0^\tau [\psi(y(t)) - u_\varepsilon(y(t))]^+ \exp\left(-\alpha t - \frac{1}{\varepsilon}t\right)dt\right\}$$

$$-E\left\{1_{\tau<\infty}\psi(y(\tau))\exp\left(-\alpha\tau - \frac{1}{\varepsilon}\tau\right)\right\},$$

and because $y(\tau)\in\Gamma_0$ a.s. if $\tau<\infty$, we obtain

$$u_\varepsilon - \psi \leq \|(f - A\psi)^+\| E\left\{\int_0^\tau \exp\left(-\alpha t - \frac{1}{\varepsilon}t\right)dt\right\}.$$

Hence, (2.38) follows.   □

Remark 2.7. Notice that (2.38) remains true even if Γ is not smooth. Also, if, for instance, $\psi\in C(\Gamma_0)$ and $A\psi\in L^\infty(\mathcal{O})$, then from D. W. Stroock and S. R. S. Varadhan [19] the assumption (2.35) is satisfied.

Remark 2.8. A result analogous to Theorem 2.6 can be proved for the optimal cost u_ε' in the open set $\mathcal{O}$.

We also have monotonicity in ε.

THEOREM 2.7. *Assume* (1.1) *and* (2.1). *Then if* $0<\varepsilon\leq\varepsilon'$ *we obtain*

$$(2.39)\qquad\qquad u_\varepsilon \leq u_{\varepsilon'}.$$

Proof. Let T_ε be the operator introduced in Theorem 2.1 by (2.12). First, we are going to prove that

$$(2.40)\qquad\qquad T_\varepsilon u_{\varepsilon'}\leq u_{\varepsilon'}\quad\text{if }0<\varepsilon\leq\varepsilon'.$$

Indeed, as in Theorem 2.6, we obtain for any $u\in B(\bar{\mathcal{O}})$ which satisfies (2.35)[14] and vanishes on Γ_0,

$$(2.41)\qquad T_\varepsilon u - u = E\left\{\int_0^\tau\left[f - Au - \frac{1}{\varepsilon}(u-\psi)^+\right](y(t))\exp\left(-\alpha t - \frac{1}{\varepsilon}t\right)dt\right\}.$$

So using the equality

$$f - Au_{\varepsilon'} - \frac{1}{\varepsilon}(u_{\varepsilon'}-\psi)^+ = \left(\frac{1}{\varepsilon'} - \frac{1}{\varepsilon}\right)(u_{\varepsilon'}-\psi)^+,$$

and taking $u=u_\varepsilon$ in (2.41), we deduce (2.40).

Further, knowing that T_ε has the monotone property (if $u\leq u'$ then $T_\varepsilon u\leq T_\varepsilon u'$), from (2.40) we obtain

$$(2.42)\qquad\qquad T_\varepsilon^k u_{\varepsilon'}\leq u_{\varepsilon'}.$$

Hence, taking the limit in (2.42) as $k\to\infty$, we prove (2.39).   □

Remark 2.8. As for Theorem 2.7, an analogous property is obtained for the optimal cost u_ε' in the open case.

Remark 2.9. Approximating u_ε by regular functions (cf. D. W. Stroock and S. R. S. Varadhan [19, Coroll. 8.1]), we have

$$(2.43)\qquad\qquad t\to u_\varepsilon(y(t\wedge\tau))\text{ is a.s. continuous.}$$

The same argument holds for functions ψ satisfying (2.35).

---

[14] Clearly, with u instead of ψ.

Also, using the semigroup associated with the process $y(t)$ stopped at the exit from the open set $\mathcal{O}$, we prove

$$(2.44) \qquad t \to u'_\varepsilon(y(t \wedge \tau')) \text{ is a.s. right continuous,}$$

where u'_ε denote the optimal cost in the open case.

3. Integral formulation. Recall that Γ_0 denotes the set of regular points given by (0.4) and that if $u, v \in B(\bar{\mathcal{O}})$ we set

$$Au \leqq v \text{ in } \bar{\mathcal{O}} \backslash \Gamma_0 \text{ if the process}$$

$$(3.1) \qquad X_t = \int_0^{\tau \wedge \tau} v(y(s)) e^{-\alpha s} \, ds + u(y(\tau \wedge \tau)) e^{-\alpha(\tau \wedge \tau)}$$

is a strong submartingale[15] for each $x \in \bar{\mathcal{O}} \backslash \Gamma_0$.

The following problem is considered: Find $u(x)$ such that

$$(3.2) \qquad u \in B(\bar{\mathcal{O}}), \qquad u(x) = 0 \quad \forall x \in \Gamma_0,$$

$$(3.3) \qquad Au \leqq f \quad \text{in } \bar{\mathcal{O}} \backslash \Gamma_0 \text{ [in the martingale sense (3.1)]},$$

$$(3.4) \qquad u \leqq \psi \quad \text{in } \bar{\mathcal{O}} \backslash \Gamma_0.$$

In order to find solutions of problem (3.2), (3.3), (3.4) which have some continuity property, it is necessary to assume that

$$(3.5) \qquad \psi(x) \geqq 0 \quad \forall x \in \Gamma_0.$$

This section is divided into three parts. First, we consider the case where ψ is regular. Next, we extend the results for ψ continuous or upper semicontinuous. Finally, we give some complementary results.

3.1. Regular case. We have

THEOREM 3.1. *Let the conditions* (1.1), (2.1), (3.5) *hold. We also assume that*

$$(3.6) \quad \text{there exists } w \in B(\bar{\mathcal{O}}) \text{ such that } A\psi = w \text{ in } \bar{\mathcal{O}} \backslash \Gamma_0 \text{ [martingale sense].}$$

Then the problem (3.2), (3.3), (3.4) *admits a maximum solution u which is given by the decreasing limit*

$$(3.7) \qquad u(x) = \lim_{\varepsilon \downarrow 0} u_\varepsilon(x) \quad \forall x \in \bar{\mathcal{O}},$$

where u_ε is the solution of problem (2.5), (2.6).

Proof. Using Theorem 2.7 we can define a function $u(x)$ by the limit (3.7).

First we are going to prove that u, given by (3.7), is a solution or problem (3.2), (3.3), (3.4). Indeed, assertion (3.2) is trival from (2.5) and Remark 2.1. Condition (3.3) is obtained taking the limit in the martingale expression of (2.6), and (3.4) follows from the estimate (2.38).

Next, in order to show that u is the maximum solution, it is only necessary to prove that each solution v of problem (3.2), (3.3), (3.4) satisfies

$$(3.8) \qquad v \leqq u_\varepsilon \quad \text{in } \bar{\mathcal{O}} \quad \forall \varepsilon > 0.$$

But, as in Theorem 2.7, (3.8) follows from

$$(3.9) \qquad v \leqq T_\varepsilon v \quad \text{in } \bar{\mathcal{O}}.$$

---

[15] That is, X_t satisfies the Doob optional sampling theorem.

Thus, using Remark 1.3 as in Theorem 2.7, we obtain (3.9), and so the theorem is proved. □

Now, the optimal stopping time problem is considered.

THEOREM 3.2. *Under assumptions* (1.1), (2.1), (3.5), *and* (3.6) *the maximum solution $\hat{u}$ of problem* (3.2), (3.3), (3.4) *is also given as the optimal cost* (0.2), *and the estimate*

$$(3.10) \qquad \|u_\varepsilon - \hat{u}\| \leqq \varepsilon \|(f - A\psi)^+\| \quad \forall \varepsilon > 0,$$

holds. Moreover, the stopping time $\hat{\theta} = \hat{\theta}_x$ defined by

$$(3.11)^{16} \qquad \hat{\theta} = \inf \{t \in [0, \tau]/\hat{u}(y(t)) = \psi(y(t))\}$$

is optimal; i.e.,

$$(3.12) \qquad \hat{u}(x) = J_x(\hat{\theta}).$$

Proof. Denote by $\hat{u}$ the optimal cost (0.2), and by u_ε the solution of the penalized problem (2.5), (2.6).

First we are going to show that

$$(3.13) \qquad u_\varepsilon \geqq \hat{u} \quad \forall \varepsilon > 0,$$

$$(3.14)^{16} \qquad \hat{\theta}^\varepsilon = \inf \{t \in [0, \tau]/u_\varepsilon(y(t)) \geqq \psi(y(t))\}$$

satisfies

$$(3.15)^{17} \qquad \begin{aligned} & 1_{\hat{\theta}^\varepsilon \wedge \tau < \infty} \, u_\varepsilon(y(\hat{\theta}^\varepsilon \wedge \tau)) = 1_{\hat{\theta}^\varepsilon < \tau} \psi(y(\hat{\theta}^\varepsilon)), \\ & u_\varepsilon(y(t)) < \psi(y(t)) \quad \text{if } t \in [0, \hat{\theta}^\varepsilon]. \end{aligned}$$

Note that (2.6) implies

$$(3.16) \quad u_\varepsilon = E\left\{ \int_0^{\theta \wedge \tau} \left[f - \frac{1}{\varepsilon}(u_\varepsilon - \psi)^+ \right](y(t)) \, e^{-\alpha t} \, dt + 1_{\theta \wedge \tau < \infty} u_\varepsilon(y(\theta \wedge \tau)) \, e^{-\alpha(\theta \wedge \tau)} \right\},$$

for any stopping time θ. Thus, taking $\theta = \hat{\theta}_\varepsilon$ in (3.16) and regarding (3.15), we deduce

$$(3.17) \qquad u_\varepsilon(x) = J_x(\hat{\theta}^\varepsilon),$$

and so (3.13) follows.

Next we are going to prove

$$(3.18) \qquad u_\varepsilon - \hat{u} \leqq \varepsilon \|(f - A\psi)^+\|.$$

Indeed, starting at

$$(3.19) \qquad u_\varepsilon(x) - \hat{u}(x) = \sup_\theta \inf_\nu \left[J_x^\varepsilon(\nu) - J_x(\theta) \right],$$

and setting

$$\nu_\theta(s) = \begin{cases} 1 & \text{if } s > \theta, \\ 0 & \text{if } s \leqq \theta, \end{cases}$$

---

[16] With $\hat{\theta} = \tau$ or $\hat{\theta}^\varepsilon = \tau$ if the corresponding set is empty.

[17] $1_{a<b}$ denotes the function $= 1$ if $a < b$ and $= 0$ otherwise.

we deduce, as in Theorem 2.6,

$$J_x^\varepsilon(\nu_\theta) - J_x(\theta) = -E\left\{1_{\tau<\infty}1_{\theta<\tau}\psi(y(\tau)) \exp\left(-\alpha\tau - \frac{\tau-\theta}{\varepsilon}\right)\right\}$$

(3.20)

$$+E\left\{\int_{\theta\wedge\tau}^\tau (f-A\psi)(y(t)) \exp\left(-\alpha t - \frac{t-\theta\wedge\tau}{\varepsilon}\right) dt\right\}.$$

Hence, using (3.5) from (3.20) and (3.19), we have (3.18).

Clearly, (3.18) and (3.13) imply (3.10). So we obtain from (2.43),

(3.21) $\qquad\qquad t \to \hat{u}(y(t \wedge \tau))$ is a.s. continuous.

Further, from Theorem 2.7, (3.21), and estimate (3.10), we have

(3.22) $$\lim_{\varepsilon\downarrow 0} \hat{\theta}^\varepsilon = \hat{\theta} \quad \text{a.s.,}$$

where the limit is increasing.

Finally, choosing $\theta = \hat{\theta}_{\varepsilon'}$, $\varepsilon' > \varepsilon > 0$ in (3.16), and letting $\varepsilon \to 0$ and then $\varepsilon' \to 0$, and using the convergence (3.10), (3.22) we establish (3.12). $\quad\square$

3.2. Nonregular case. Now, we relax the regularity assumptions on ψ. Without assuming (3.6), ψ will be only continuous or upper semicontinuous. We have

THEOREM 3.3. *Under assumptions* (1.1), (2.1), (3.5), *and*

(3.23) $\qquad\qquad \psi$ *is uniformly continuous on* $\bar{\mathcal{O}}$,

the problem (3.2), (3.3), *and* (3.4) *admits a maximum solution* $\hat{u}$ *which is given as the optimal cost* (0.2). *Moreover,*

(3.24) $$\lim_{\varepsilon\downarrow 0} \|u_\varepsilon - \hat{u}\| = 0,$$

and the relation (3.12) *is true.*

Proof. First we remark that if u_i denotes the optimal cost (0.2) corresponding to f_i, ψ_i for $i = 1, 2$, we immediately obtain the estimate,

(3.25) $$\|\hat{u}_1 - \hat{u}_2\| \le \frac{1}{\alpha}\|f_1 - f_2\| + \|\psi_1 - \psi_2\|.$$

Next, notice that in Theorem 3.1 the assumption (3.6) was used only in order to prove (3.4). Also, the same arguments as in Theorem 3.2 show that provided (3.25) and (3.24) hold, we can deduce (3.12). So, using the fact that $\hat{u}$ defined by (0.2) satisfies (3.4), we just need to prove the convergence (3.24). Then, approximating ψ by a sequence of smooth functions and using the estimates (3.25) and (2.15) the convergence (3.24) is established. $\quad\square$

Remark 3.1. If the obstacle ψ is only continuous, the assertions of Theorem 3.3 remain true but the convergence (3.24) holds only on compact sets of $\bar{\mathcal{O}}$,

THEOREM 3.4. *Let the conditions* (1.1), (2.1), (3.5), *and*

(3.26) $\qquad\qquad \psi$ *upper semicontinuous on* $\bar{\mathcal{O}}$

hold. The problem (3.2), (3.3), (3.4) *admits a maximum solution* $\hat{u}$ *which is given as the optimal cost* (0.2). *Moreover, given any constant* $\varepsilon > 0$ *there exists a function* $\hat{\theta}^\varepsilon = \hat{\theta}^\varepsilon(x)$ *such that*

(3.27)
$$\hat{\theta}^\varepsilon : \bar{\mathcal{O}} \times \Omega \to [0, \infty] \text{ is measurable,}$$

$$\forall x \in \bar{\mathcal{O}}, \quad \tilde{\theta}^\varepsilon(x) \text{ is a stopping time,}$$

and

(3.28) $\hat{u}(x) + \varepsilon \geqq J_x(\tilde{\theta}^{\varepsilon}(x) \quad \forall x \in \bar{\mathcal{O}}.$

Proof. Since ψ is bounded and upper semicontinuous on $\bar{\mathcal{O}}$, there exists a sequence $\{\psi_k\}_{k=1}^{\infty}$ of bounded and continuous functions on $\bar{\mathcal{O}}$ decreasing to ψ (cf. Bourbaki [5, p. 30]). Let $\hat{u}$ and $\hat{u}_k$ be the optimal costs according to ψ and ψ_k respectively; then clearly, $\hat{u}_k$ is decreasing to $\hat{u}$.

Next, from Theorem 3.4 and Remark 3.1, the functions $\hat{u}_k$ verify (3.2), (3.3), and

(3.29) $\hat{u}_k \leqq \psi_k.$

So, if we let $k \to \infty$, the function $\hat{u}$ satisfies (3.4). Moreover, from monotonicity, $\hat{u}$ is the maximum solution of (3.2), (3.3), (3.4).

Finally, we set

(3.30) $k_{\varepsilon}(x) = \inf\{k \geqq 1 / \hat{u}_k(x) \leqq \hat{u}(x) + \varepsilon\},$

and

(3.31) $\tilde{\theta}^{\varepsilon} = \inf\{t \in [0, \tau] / \hat{u}_{k_{\varepsilon}}(y(t)) = \psi_{k_{\varepsilon}}(y(t))\}.$

It is easy to check that $\tilde{\theta}^{\varepsilon}$ satisfies (3.27), (3.28), and the proof is completed. □

Now, using Theorem 3.4, Theorem 2.7 and Theorem 2.2, we obtain

COROLLARY 3.1. *Let the conditions* (1.1), (2.1), *and* (3.5) *hold. Then if f and ψ are nonnegative upper semicontinuous on $\bar{\mathcal{O}}$, so is the optimal cost $\hat{u}$ defined by* (0.2).

Next, using Remark 3.1, Theorem 3.4, and Theorem 2.3, we obtain

COROLLARY 3.2. *Assume* (1.1), (2.1), (2.20), *and* (3.5). *Then if f and ψ are upper semicontinuous or (continuous) on $\bar{\mathcal{O}}$, so is the optimal cost $\hat{u}$ defined by* (0.2).

Remark 3.2. With suitable modification in the proofs, results similar to Theorem 3.1, Theorem 3.2, Theorem 3.3 and Corollary 3.1 are obtained for the optimal cost u' in the case of the open set $\mathcal{O}$.

3.3. Complementary results. A relation between the two problems, in the closed set $\bar{\mathcal{O}}$ and in the open set $\bar{\mathcal{O}}$, is given by

THEOREM 3.5. *Let the conditions* (1.1) *and* (2.1) *hold. Then the following estimates hold,*

(3.32) $\|(\hat{u}' - \hat{u})^+\| \leqq \dfrac{1}{\alpha}\|f^-\| + \|\psi^-\|,$

(3.33) $\|(\hat{u}' - \hat{u})^-\| \leqq \|1_{\Gamma \backslash \Gamma_0}\psi^+\|,$

where $\hat{u}'$ and $\hat{u}$ denote the optimal cost corresponding to the problem in the open subset $\mathcal{O}$ and closed set $\bar{\mathcal{O}}$ respectively.

Proof. Recall that τ' denotes the first exit time of process $y(t)$ from the open subset $\mathcal{O}$, and $J'_x(\theta)$ the functional cost given by (0.1) with τ' instead of τ.

Starting at

(3.34) $\hat{u}(x) - \hat{u}(x) = \sup_{\theta} \inf_{\theta'} [J'_x(\theta') - J_x(\theta)],$

and choosing for the infimum $\theta' = \theta$ in (3.34), we deduce

(3.35 $\hat{u}'(x) - \hat{u}(x) \leqq E\left\{\int_{\tau'}^{\tau} f^-(y(t) e^{-\alpha t} dt\right\} + \sup_{\theta} E\{1_{\tau' \leqq \theta < \tau}\psi^-(y(\theta)) e^{-\alpha\theta}\},$

and (3.32) follows.

Further, taking from the supremum $\theta = \theta' \wedge \tau'$ in (3.34), we have

$$(3.36) \qquad \hat{u}'(x) - \hat{u}(x) \geqq -E\{1_{\tau' < \tau} \psi^+(y(\tau')) e^{-\alpha \tau'}\}.$$

Hence (3.33) is proved. $\square$

Next, combining Theorem 3.5, Corollary 3.1 and Remark 3.2, we obtain

COROLLARY 3.3. *Assume* (1.1), (2.1), (3.5), *and*

$$(3.37) \qquad \psi(x) = 0 \quad \forall x \in \Gamma \backslash \Gamma_0.$$

Then if f and ψ are nonnegative continuous on $\bar{\mathcal{O}}$, the two optimal costs $\hat{u}'$ and $\hat{u}$ coincide. It follows from Theorem 2.2 and Remark 2.4 that the optimal cost $\hat{u}$ given by (0.2) *is continuous on $\bar{\mathcal{O}}$.*

Now, $\hat{u}$ is regarded as a distribution in $\mathcal{O}$. Recalling that A represents the differential operator given by (2.30), we have

THEOREM 3.6. *Suppose that the boundary Γ is smooth and the conditions* (1.1), (2.1), (2.31), (3.5), *and*

$$(3.38) \qquad \psi \text{ continuous on } \bar{\mathcal{O}}$$

hold. Then the optimal cost $\hat{u}$ satisfies

$$(3.39) \qquad A\hat{u} \leqq f \quad \text{in } \mathscr{D}'(\mathcal{O}),$$

$$(3.40)^{18} \qquad A\hat{u} = f \quad \text{in } \mathscr{D}'([\hat{u} < \psi]).$$

Furthermore, if ψ verifies (3.6), *the following estimate is true*

$$(3.41) \qquad \|A\hat{u}\| \leqq \|f\| + \|(f - A\psi)^+\|.$$

So $A\hat{u} \in L^\infty(\mathcal{O})$.

Proof. First we recall that the condition (3.40) has meaning if the subset $[\hat{u} < \psi]$ is open. Using Corollary 3.1 and Corollary 3.2 this fact can be deduced.

Next the conditions (3.39) and (3.41) are immediate from Theorems 3.4 and 2.6.

Finally, if $\phi \in \mathscr{D}([\hat{u} \in \psi])$, using the uniform convergence (3.24) we obtain

$$(3.42) \qquad (\hat{u}_\varepsilon - \psi)^+ \phi = 0 \quad \text{if } \varepsilon \text{ is small enough.}$$

Therefore, from (3.42) and (2.34) the equality (3.40) is proved. $\square$

Remark 3.3. Let U be the subset of $\mathcal{O}$ where $\sigma(x)$ is nondegenerate. Suppose that $\hat{u}$ is continuous (see Corollary 3.2). Then, from (3.41), $\hat{u}$ can be regarded as the unique solution of a Dirichlet problem on U. This fact leads to a $W^{2,p}_{\text{loc}}(U)$, $1 < p < \infty$, regularity for the optimal cost $\hat{u}$ given by (0.2).

Remark 3.4. All these results can be extended for f and ψ with polynomial growth.

Remark 3.5. It is possible to consider a more general case of a cost functional $J_x(\theta)$, exchanging the term $\exp(-\alpha t)$ with

$$\exp\left(-\int_0^\tau c(y(s)) \, ds\right),$$

and adding a final cost

$$1_{\tau < \infty} 1_{\theta \geqq \tau} h(y(\tau)) \exp\left(-\int_0^\tau c(y(t)) \, dt\right),$$

provided $c(y) \geqq \alpha_0 > 0$.

---

[18] $[\hat{u} < \psi]$ denotes the subset of points $x \in \bar{\mathcal{O}}$ such that $\hat{u}(x) < \psi(x)$.

Remark 3.6. A result analogous to Theorem 3.6 is given for the problem in the open set $\mathcal{O}$.

Remark 3.7. All these results can be extended to the parabolic case.

4. Variational inequality. Let $a_{ij}(x)$, $a_i(x)$ be functions for $i, j = 1, \cdots, N$, such that

$(a_{ij})_{ij}$ is a nonnegative symmetric matrix and

$$(4.1) \qquad a_{ij} \in C^1(\mathbb{R}^N), \qquad \frac{\partial^2 a_{ij}}{\partial x_k \, \partial x_l} \in L^\infty(\mathbb{R}^N) \quad \forall i, j, k, l = 1, \cdots, N,$$

$$(4.2) \qquad a_i \in C(\mathbb{R}^N), \qquad \frac{\partial a_i}{\partial x_k} \in L^\infty(\mathbb{R}^N) \quad \forall i, k = 1, \cdots, N.$$

Define the following differential operator A,

$$(4.3) \qquad A = -\sum_{i,j=1}^N \frac{\partial}{\partial x_i} a_{ij} \frac{\partial}{\partial x_j} + \sum_{i=1}^N a_i \frac{\partial}{\partial x_i} + \alpha,$$

where α is a positive constant.

We always identify g and σ given by (1.1) as

$$(4.4) \qquad \begin{aligned} (a_{ij})_{ij} &= \tfrac{1}{2}\sigma\sigma^*, \\ a_i &= \sum_{j=1}^N \frac{\partial a_{ij}}{\partial x_j} - g_i. \end{aligned}$$

Let $\beta_0(x)$ and $\beta_1(x)$ be the weight functions $(1+|x|^2)^{-(\lambda+1)/2}$ and $(1+|x|^2)^{-\lambda/2}$, $\lambda > N/2$, respectively. Introduce the following Hilbert spaces:

$$(4.5) \qquad H = \{v / \beta_0 v \in L^2(\mathcal{O})\},$$

with the inner product

$$(4.6) \qquad (u, v) = \int_{\mathcal{O}} (\beta_0 u)(\beta_0 v) \, dx$$

and the norm $|\cdot|$;

$$(4.7) \qquad V = \left\{ v \in H / \beta_1 \frac{\partial v}{\partial x_k} \in L^2(\mathcal{O}), \forall k = 1, \cdots, N \right\},$$

with the norm

$$(4.8) \qquad \|v\| = \left(|v|^2 + \sum_{k=1}^N \int_{\mathcal{O}} \left| \beta_1 \frac{\partial v}{\partial x_k} \right|^2 dx \right)^{1/2}.$$

V' denotes the dual space of V, and $\langle \cdot, \cdot \rangle$ the duality between V' and V.

We have

$$(4.9) \qquad \begin{aligned} &V \subset H \subset V'; \ L^\infty(\mathcal{O}) \subset H; \\ &\left\{ v / \frac{\partial v}{\partial x_i} \in L^\infty(\mathcal{O}) \ \forall i = 1, \cdots, N \right\} \subset V. \end{aligned}$$

Let $a(\cdot, \cdot)$ be the bilinear form associated to the operator A,

$$(4.10) \quad a(u, v) = \sum_{i,j=1}^N \int_{\mathcal{O}} \tilde{a}_{ij} \left(\beta_1 \frac{\partial u}{\partial x_j} \right) \left(\beta_1 \frac{\partial v}{\partial x_j} \right) dx + \sum_{i=1}^N \int_{\mathcal{O}} \tilde{a}_i \left(\beta_1 \frac{\partial u}{\partial x_i} \right) (\beta_0 v) \, dx + \alpha(u, v),$$

where

$$
(4.11) \quad
\begin{aligned}
& \bar{a}_{ij}(x) = (1 + |x|^2)^{-1} a_{ij}(x), \\
& \tilde{a}_i(x) = (1 + |x|^2)^{-1/2} a_i(x) - 2(\lambda + 1)(1 + |x|^2)^{-3/2} \sum_{j=1}^{N} a_{ij}(x) x_j.
\end{aligned}
$$

Notice that a_{ij}, a_i are not supposed to be bounded, but a_{ij} is at most of quadratic growth, and a_i of linear growth. Then $\tilde{a}_{ij}$, $\tilde{a}_i$ in (4.11) are bounded.

This section is divided into three parts. First, we consider the case where $\mathcal{O} = \mathbb{R}^N$. Next, we give a weak formulation. Finally, we study the general case.

4.1. Case $\mathcal{O} = \mathbb{R}^N$. Assume $\mathcal{O} = \mathbb{R}^N$. After some computation we deduce

$$(4.12) \qquad a(u, v) = (Au, v) \quad \forall u, v \in V, \, Au \in H,$$

$$(4.13)[19] \qquad |a(u, v)| \leq C \|u\| \|v\| \quad \forall u, v \in V,$$

and if α is large enough there exists $\alpha_0 > 0$ such that

$$(4.14) \qquad a(u, u) \geq \alpha_0(u, u) \quad \forall u \in V.$$

Next, from (4.12) and (4.13) it follows that

$$(4.15) \qquad a(u, v) = \langle Au, v \rangle, \qquad u, v \in V.$$

Now, let K be the following closed cone in V:

$$(4.16) \qquad K = \{ v \in V / v(x) \leq \psi(x) \text{ a.e. in } \mathbb{R}^N \},$$

and let us consider the variational inequality

$$(4.17) \qquad \text{Find } u \in K \text{ such that } a(u, v - u) \geq (f, v - u) \quad \forall v \in K.$$

Recalling the cost functional

$$(4.18) \qquad J_x(\theta) = E \left\{ \int_0^\theta f(y(t)) e^{-\alpha t} \, dt + 1_{\theta < \infty} \psi(y(\theta)) e^{-\alpha \theta} \right\},$$

we have

THEOREM 4.1. *Let the assumptions* (4.1), (4.2), *and*[20]

$$(4.19) \qquad \frac{\partial f}{\partial x_k}, \quad \frac{\partial \psi}{\partial x_k} \in L^\infty(\mathbb{R}^N), \qquad k = 1, \cdots, N,$$

hold. Then there exists one and only one solution u of the variational inequality (4.17). *This solution u is given as the optimal cost,*

$$(4.20) \qquad u(x) = \inf \{ J_x(\theta) / \theta \text{ is a stopping time} \}.$$

Moreover, the following estimate is true:

$$(4.21) \qquad \left\| \frac{\partial u}{\partial x} \right\|_{L_\infty} \leq \frac{1}{\alpha - \gamma_0} \left\| \frac{\partial f}{\partial x} \right\|_{L_\infty} + \left\| \frac{\partial \psi}{\partial x} \right\|_{L_\infty},$$

where $\|\partial u/\partial x\|_{L_\infty}$ denotes the smallest Lipschitz constant of the function u.[21]

---

[19] C denotes a constant.

[20] α is assumed large enough, and f, ψ are not necessarily bounded.

[21] There exists also an optimal stopping time (Theorem 3.2).

Proof. Without loss of generality, we may assume that f, ψ are bounded (Remark 3.4). From (4.14) the uniqueness of the variational inequality (4.17) is obtained by classic methods (cf. A. Bensoussan and J. L. Lions [3]).

Using Theorem 2.6, we have for the optimal penalized cost u_ε given by (2.2),

$$(4.22) \qquad Au_\varepsilon + \frac{1}{\varepsilon}(u_\varepsilon - \psi)^+ = f \quad \text{in } \mathscr{D}'(\mathbb{R}^N).$$

Thus, from the convergence (3.24) and the estimate (2.23), we can take limits when $\varepsilon \to 0$ in (4.22) for the weak convergence in V, and using the monotonicity of operator A, we obtain (4.17); so the theorem is proved. $\square$

4.2. Weak formulation. In order to give a weak formulation of the variational inequality (4.17) we introduce the Hilbert space D_A which is the closure of the set

$(4.23)^{[22]}$ $\qquad\qquad\qquad\qquad \{v \in V / Av \in H\},$

with the graph norm

$$(4.24) \qquad \|v\|_{D_A} = (|v|^2 + |Av|^2)^{1/2}.$$

Using density arguments we also have

$$(4.25) \qquad (Au, u) \geqq \alpha_0(u, u) \quad \forall u \in D_A.$$

The following problem is considered,

(4.26) $\qquad$ Find $u \in D_A$ such that $u \leqq \psi$ a.e., and

$$(Au, v - u) \geqq (f, v - u) \quad \forall v \in D_A, v \leqq \psi \quad \text{a.e.}$$

THEOREM 4.2. *Assume* (4.1), (4.2) *and*[23]

$$(4.27) \qquad f, \psi \in C(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N),$$

$$(4.28) \qquad A\psi \in L^\infty(\mathbb{R}^N).$$

Then problem (4.26) *has one and only one solution u which is given as the optimal cost* (4.20). *Moreover, the function u is bounded and continuous, and the following estimate holds*:

$$(4.29) \qquad \|Au\|_{L^\infty} \leqq \|f\|_{L^\infty} + \|(f - A\psi)^+\|_{L^\infty}.$$

Proof. Notice that (4.27) and (4.28) imply (Remark 2.7) that

$(4.30)^{[24]}$ $\quad$ There exists $w \in B(\mathbb{R}^N)$ such that $A\psi = w$ in the martingale sense.

So, using Theorem 2.6, we have

$$(4.31) \qquad \|Au_\varepsilon\|_{L^\infty} \leqq \|f\|_{L^\infty} + \|(f - A\psi)^+\|_{L^\infty},$$

and also (Remark 2.3)

$$(4.32) \qquad \|u_\varepsilon\|_{L^\infty} \leqq \frac{1}{\alpha}\|f\|_{L^\infty} + \|\psi\|_{L^\infty}.$$

Then we take limits when $\varepsilon \to 0$ in (4.22) as in Theorem 4.1, and the proof is complete. $\square$

---

[22] A denotes the differential operator (4.3).

[23] α is assumed large enough in order to have (4.25).

[24] In the sense of (2.4), $\mathcal{O} = \mathbb{R}^N$.

Remark 4.1. Under assumption (4.30), Theorem 4.2 remains true for f and ψ upper semicontinuous and bounded instead of (4.27).

Remark 4.2. The problem (4.26) can be interpreted as

$$
\begin{aligned}
& u \in D_A, \qquad u \leqq \psi \quad \text{a.e.,} \\
(4.33) \qquad & Au \leqq f \quad \text{a.e.,} \\
& (Au - f)(u - \psi) = 0 \quad \text{a.e.,}
\end{aligned}
$$

using standard methods. Clearly, under assumptions (4.28), (4.19), the weak formulation (4.26) implies the strong formulation (4.17).

4.3. General case. We come back to the general case. Now, $\mathcal{O}$ is an open subset of $\mathbb{R}^N$ with boundary Γ smooth enough. Recalling that the subset of regular point Γ_0 is given by (0.4), we have (cf. D. Stroock and S. R. S. Varadhan [19, p. 686]).

$$
(4.34) \qquad \sum_{i=1}^{N} a_i(x) n_i(x) \leqq 0 \quad \forall x \in \Gamma \backslash \Gamma_0,
$$

where $n(x) = (n_i(x))$ is the inner normal of $\mathcal{O}$.

Next, define the closed subspace of V,

$$
(4.35) \qquad V_0 = \{ v \in V / v = 0 \text{ on } \Gamma_0 \}.
$$

Then, as in the case $\mathcal{O} = \mathbb{R}^N$, if α is large enough, using (4.34) it is possible to find a constant $\alpha_0 > 0$ such that

$$
(4.36) \qquad a(u, u) \geqq \alpha_0(u, u) \quad \forall u \in V_0.
$$

Furthermore, assuming

$$
(4.37) \qquad \sum_{i=1}^{N} a_{ij}(x) n_i(x) = 0 \quad \forall x \in \Gamma \backslash \Gamma_0, \quad j = 1, \cdots, N,
$$

we deduce

$$
(4.38) \qquad a(u, v) = \langle Au, v \rangle \quad \forall u, v \in V_0.
$$

Remark 4.3. If we assume

$$
(4.39) \qquad \sum_{i,j=1}^{N} a_{ij}(x) n_i(x) n_j(x) + \left(\sum_{i=1}^{N} a_i(x) n_i(x) \right)^{+} > 0 \quad \forall x \in \Gamma,
$$

the condition (4.37) is true and $\Gamma = \Gamma_0$.

Setting K_0 the closed cone in V_0,

$$
(4.40) \qquad K_0 = \{ v \in V_0 / v(x) \leqq \psi(x) \text{ a.e. in } \mathcal{O} \},
$$

we consider the variational inequality

$$
(4.41) \qquad \text{Find } u \in K_0 \text{ such that } a(u, v - u) \geqq (f, v - u) \quad \forall v \in K_0.
$$

THEOREM 4.3. *Under assumptions* (4.1), (4.2), (2.26), *and* (2.27)[25] *the variational inequality* (4.41) *has exactly one solution* u *which is given as the optimal cost* (0.2).

---

[25] α is assumed large enough.

Moreover, the function u is Lipschitz continuous and verifies

$$(4.42) \qquad \left\| \frac{\partial u}{\partial x} \right\|_{L^\infty} \leq \frac{1}{\alpha - \gamma_0} \left\| \frac{\partial f}{\partial x} \right\|_{L^\infty} + \left\| \frac{\partial \psi}{\partial x} \right\|_{L^\infty} + \left\| \frac{\partial u}{\partial x} \right\|_{L^\infty},$$

where $\|(\partial u/\partial x)\|_{L^\infty}$ *denotes the smallest Lipschitz constant of u.*

Proof. We just need to use the estimate (2.28) and the technique of Theorem 4.1. □

Remark 4.4. Clearly, combining Lemma 1.5 and Remark 1.7, we obtain a sufficient condition in order to have a Lipschitz continuous subsolution u, i.e., assumption (2.26).

Remark 4.5. Provided (4.37) holds, a weak formulation of the variational inequality (4.41) as (4.26) also can be considered.

Remark 4.6. All these results can be extended for f and ψ with polynomial growth, and we can also consider a function $a_0(x)$ instead of the constant α for the definition of operator A. Using the same technique, we can treat the parabolic case.

Remark 4.7. An application to the optimal stopping time problem with partial information is given in [16].

Remark 4.8. In the particular case, where the operator $A = A_1(x_1) + A_2(x_2)$, $x = (x_1, x_2)$ with A_1 coercive and A_2 of first order, a weak formulation (4.41) is obtained using only analytic methods (cf. M. Langlais [10]).

Final Remark. In a separate article in this issue [15], a degenerate quasi-variational inequality corresponding to the impulse control problem is studied (cf. [13]).

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