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ON THE OPTIMALITY OF SPRING BALANCE WEIGHING DESIGNS

BY MIKE JACROUX[1] AND WILLIAM NOTZ[2]

Washington State University and Purdue University

This paper deals with techniques for finding Φ-optimal designs for weighing v objects in b weighings using a spring balance. The optimality functions considered encompass a large class of functions. Results are applied to find A-, D- and E-optimal designs and the optimal designs obtained are seen to be related to certain types of well known block designs.

1. Introduction. This paper deals with the problem of optimally weighing v objects in b weighings on a spring balance, $b \geq v$. An experimental design d in this setting is a plan for deciding which of the v objects should be weighed on the ith weighing, $1 \leq i \leq b$. Such a design d can be represented by a $b \times v$ matrix $X_d = (x_{dij})$ having entries $x_{dij} = 1$ or 0 depending upon whether the jth object is included or excluded on the ith weighing. Thus if we let $D(v, b)$ denote the entire class of $b \times v$ matrices whose entries are 0 and 1, then we can think of $D(v, b)$ as the entire class of available weighing designs. We shall henceforth use d and X_d interchangeably when referring to some specific design. Assuming that the observations are uncorrelated, have constant variance σ^2, and that the weight of the jth object is β_j, then the total weight of the objects measured on the ith weighing is $\sum_{j=1}^{v} x_{dij}\beta_j$. If X_d has rank v, all of the β_j are estimable, and the covariance matrix of their best linear unbiased estimators is $\sigma^2(X'_dX_d)^{-1}$. The matrix $X'_dX_d = (\lambda_{dij})$ is called the information matrix of X_d. Here we consider the determination of optimal designs in $D(v, b)$.

A design is said to be optimal within a given class $D(v, b)$ provided it is determined to be "best" by some optimality function Φ. Most optimality functions Φ are real valued functions of the covariance matrices corresponding to $X_d \in D(v, b)$ and X_d is optimal provided $\Phi(X'_dX_d)$ is minimal over $D(v, b)$. Some typical Φ are the maximum eigenvalue of $(X'_dX_d)^{-1}$ (an optimal design is called E-optimal), the trace of $(X'_dX_d)^{-1}$ (an optimal design is called A-optimal), and the determinant of $(X'_dX_d)^{-1}$ (an optimal design is called D-optimal).

Before proceeding, we note that each $X_d \in D(v, b)$ can be viewed as the incidence matrix of a binary block design where the rows of X_d correspond to blocks. The row and column sums of X_d give the size and number of replications of the corresponding blocks and treatments. From time to time in the sequel we will refer to X_d as the transpose of the incidence matrix of a given type of block design. For definitions and a discussion of the properties of any block designs referenced, the reader is referred to Raghavarao (1971), Chapters 5 and 8.

The main purpose of this paper is to obtain results which can be used to establish the optimality of some previously unknown spring balance designs according to various Φ. In Section 2 we consider a general class of optimality functions Φ, obtain some preliminary results, and indicate how to obtain Φ-optimal designs. The remaining sections of this paper apply the results obtained in Section 2 to finding A, D and E-optimal designs.

2. Preliminary results. Suppose Φ is a convex real valued function on the set of all $v \times v$ positive definite matrices, such that if M is a $v \times v$ positive definite matrix with

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eigenvalues $\mu_1 \le \cdots \le \mu_v$ then $\Phi(M) = \phi(\mu_1, \cdots, \mu_v)$ where ϕ is a real valued convex function which is descreasing in each of its coordinates when the others are held constant and has the property that if $(v-1)x_1 + x_2 = (v-1)y_1 + y_2$, $0 \le x_1 \le x_2$, $0 \le y_1 \le y_2$, $x_1 \le y_1$, and $y_2 \le x_2$ then $\phi(x_1, \cdots, x_1, x_2) \ge \phi(y_1, \cdots, y_1, y_2)$. Following Cheng (1978), call such a function Φ a type 1 function on the set of all $v \times v$ positive definite matrices.

Additionally, let Π be the set of all $v \times v$ permutation matrices. We shall denote by $\bar{M}_d$ the average of $M_d = X_d'X_d$ over all elements of Π, i.e.

$$\bar{M}_d = (\textstyle\sum_{P\in\Pi} P'X_d'X_dP)/v!. \tag{2.1}$$

It is not difficult to see that $\bar{M}_d = \alpha_d I_v + \beta_d J_{v,v}$ from some α_d, $\beta_d \ge 0$. We now prove a series of lemmas.

LEMMA 2.1. *Suppose Φ is a type 1 function on the set of all $v \times v$ positive definite matrices. Let $X_{d1} \in D(v, b)$ be such that it has exactly $N = mb + p$ $(0 \le m < v, 0 \le p < b)$ $+1$ entries. Furthermore suppose the first m columns of X_d consist entirely of $+1$ entries, the $m + 1$st column has $+1$ as its first p entries and 0 as its last $b - p$ entries, and the remaining columns consist entirely of zeroes. Then for any other $X_d \in D(v, b)$ having exactly N entries being $+1$, $\Phi(\bar{M}_{d1}) \le \Phi(\bar{M}_d)$, assuming $\bar{M}_{d1}$ and $\bar{M}_d$ are nonsingular.*

PROOF. First notice that if $A_{d1}, A_{d2} \in D(v, b)$ have exactly N entries being $+1$, if $\bar{M}_{di} = \alpha_{di}I_v + \beta_{di}J_{v,v}$ is the average of $A_{di}'A_{di}$ over Π for $i = 1, 2$, and if $\alpha_{d1} \ge \alpha_{d2}$ and $\alpha_{d1} + v\beta_{d1} \le \alpha_{d2} + v\beta_{d2}$, then $\Phi(\bar{M}_{d1}) \le \Phi(\bar{M}_{d2})$ assuming $\bar{M}_{d1}$ and $\bar{M}_{d2}$ are nonsingular. This follows easily from the fact that the eigenvalues of $\alpha I + \beta J_{v,v}$ are well known to be α with multiplicity $v - 1$ and $\alpha + \beta v$ with multiplicity 1 and the fact that $N = \mathrm{tr}(\bar{M}_{di}) = (v-1)\alpha_{di} + (\alpha_{di} + \beta_{di}v)$ for $i = 1, 2$.

Now for any $X_d \in D(v, b)$ as stated in the lemma, $N = \mathrm{tr}\, X_d'X_d = \mathrm{tr}\, \bar{M}_d$. Thus all the diagonal entries of $\bar{M}_d$ must be N/v. Suppose Q is the sum of the off diagonal entries of $X_d'X_d$. Then the sum of the off diagonal entries of $\bar{M}_d$ is also Q and the off diagonal entries of $\bar{M}_d$ are $Q/v(v-1)$. The lemma will now follow from the first paragraph of this proof if we can show Q is minimized over all $X_d \in D(v, b)$ having exactly N entries equal to $+1$, for $X_d = X_{d1}$,

Let T_k be the kth row sum of X_d. It is straightforward to verify that $Q = \sum_{k=1}^{b} T_k^2 - N$. Since Q is convex in T_k, it is minimized over T_k, subject to $\sum T_k = N$ and the T_k being nonnegative integers, when the T_k are as nearly equal as possible. This occurs when p of the T_k have value $m + 1$ and $b - p$ of the T_k have value m. Since X_{d1} has precisely these values for its row sums, it attains the minimum value for Q over $X_d \in D(v, b)$.

LEMMA 2.2. *Suppose Φ is a type 1 function on the set of all $v \times v$ positive definite matrices. Suppose $X_{d1} \in D(v, b)$ is as in Lemma 2.1 and X_{d1} has exactly N entries of $+1$. Then for any $X_d \in D(v, b)$ of rank v having exactly N entries of $+1$, $\Phi(\bar{M}_{d1}) \le \Phi(X_d'X_d)$.*

PROOF. This follows from Lemma 2.1 and the convexity of Φ.

Lemma 2.2 suggests that when Φ is a type 1 function, a Φ-optimal spring balance design may be found by comparing the values of $\Phi(\bar{M}_{d1})$ for various N, where $\bar{M}_{d1}$ is the average over Π of $M_{d1} = X_{d1}'X_{d1}$ and X_{d1} is as in Lemma 2.1 If the minimizing $\bar{M}_{d1}$ satisfies the property that there exists $X_d \in D(v, b)$ with $X_d'X_d = \bar{M}_{d1}$, then X_d is a Φ-optimal spring balance design over $D(v, b)$. In general, which $\bar{M}_{d1}$ (considered as a function of N) minimizes Φ depends on Φ. However we can prove the following.

LEMMA 2.3. *If Φ is a type 1 function on the set of all $v \times v$ positive definite matrices, then an $X_{d1} \in D(v, b)$ of the form in Lemma 2.1 yielding $\bar{M}_{d1}$ which minimizes Φ can be found among the X_{d1} with $N = mb + p$ $(0 \le m \le v, 0 \le p < b)$ $+1$ entries and $m \ge v/2$.*

PROOF. Suppose $X_{d1} \in D(v, b)$ is as in Lemma 2.1 and X_{d1} as $N = mb + p(0 \le m \le v, 0 \le p < b)$ +1 entries and $m \le (v-2)/2$. Notice

$$\bar{M}_{d1} = \{(mb+p)/v - (bm(m-1) + 2mp)/v(v-1)\}I_v + \{(bm(m-1)+2mp)/v(v-1)\}J_{v,v} \tag{2.2}$$

and the eigenvalues of $\bar{M}_{d1}$ are

$$\mu_{d1} = \{mb(v-m) + p(v-1-2m)\}/v(v-1) \tag{2.3}$$

with multiplicity $v-1$ and

$$\lambda_{d1} = \{bm^2 + p(2m+1)\}/v \tag{2.4}$$

with multiplicity 1.

Let $X_{d2} \in D(v, b)$ be as in Lemma 2.1 with $bv - N = (v-1-m)b + (b-p)$ entries of +1. The eigenvalues of $\bar{M}_{d2}$, the average of $X'_{d2}X_{d2}$ over Π, by reasoning similar to that used to get those of $\bar{M}_{d1}$ are

$$\begin{aligned}\mu_{d2} &= \{(v-1-m)b(v-\{v-1-m\}) + (b-p)(v-1-2\{v-1-m\})\}/v(v-1)\\ &= \{mb(v-m) + p(v-1-2m)\}/v(v-1) = \mu_{d1}\end{aligned}$$

with multiplicity $v-1$ and

$$\begin{aligned}\lambda_{d2} &= \{b(v-1-m)^2 + (b-p)(2\{v-1-m\}+1)\}/v\\ &= \{bm^2 + p(2m+1) + v(bv - 2mb - 2p)\}/v = \lambda_{d1} + bv - 2mb - 2p\end{aligned}$$

with multiplicity 1. Since $m \le (v-2)/2$ we see $bv - 2mb - 2p \ge 2(b-p) \ge 0$ so $\lambda_{d2} \ge \lambda_{d1}$. Since Φ is a type 1 function the above implies $\Phi(\bar{M}_{d2}) \le \Phi(\bar{M}_{d1})$. Thus if X_{d1} has $m \le (v-2)/2$ there exists an X_{d2} as in Lemma 2.1 with $m > (v-2)/2$ (since m must be an integer we must in fact have $m \ge (v-1)/2$ if v is odd and $m \ge v/2$ if v is even) which is at least as good as X_{d1}.

To complete the proof, suppose v is odd and X_{d1} is such that $m = (v-1)/2$. Examination of equation (2.3) shows that μ_{d1} is then independent of p and has the same value it would have if m was $(v+1)/2$ and p was 0. Further examination of (2.4) indicates that λ_{d1} is increasing in both p and m. We therefore conclude that since Φ is a type 1 function, hence decreasing in the value of μ_{d1} and λ_{d1}, there is a design $X_{d2} \in D(v, b)$ as in Lemma 2.1 with $m \ge (v+1)/2$ (recall v is odd now) which has $\Phi(\bar{M}_{d2}) \le \Phi(\bar{M}_{d1})$ where $\bar{M}_{di}$ is the average over Π of $X'_{di}X_{di}$, $i = 1, 2$. The lemma now follows.

These results will now be applied to some special type 1 functions Φ of interest.

3. *A*-optimality.

Suppose $\Phi(X'_dX_d) = \text{tr}((X'_dX_d)^{-1}) = \sum_{i=1}^{v} 1/\mu_{di}$, where $\mu_{d1} \le \mu_{d2} \le \cdots \le \mu_{dv}$ are the eigenvalues of X'_dX_d. Since $\sum_{i=1}^{v} 1/\mu_{di}$ is a type 1 function, we can apply the results of Section 2 to find Φ-optimal designs. Here Φ corresponds to A-optimality. Suppose $b \ge v \ge 3$ is fixed. If $\bar{M}_{d1}$ is the average of $X'_{d1}X_{d1}$ over Π where $X_{d1} \in D(v, b)$ is as in Lemma 2.1 with $N = mb + p$ entries of +1,

$$\Phi(\bar{M}_{d1}) = v(v-1)^2/\{mb(v-m) + p(v-1-2m)\} + v/\{bm^2 + p(2m+1)\} \equiv f(m, p). \tag{3.1}$$

From Lemma 2.3 we know that the minimum of $f(m, p)$ is to be found among the integers $v/2 \le m \le v$, $0 \le p < b$, and $mb + p \le bv$.

Now

$$\partial f(m,p)/\partial p = -v(v-1)^2(v-1-2m)/\{mb(v-m) + p(v-1-2m)\}^2 - v(2m+1)/\{bm^2 + p(2m+1)\}^2. \tag{3.2}$$

Since $\{bm^2 + p(2m+1)\}^2 \geq \{mb(v-m) + p(v-1-2m)\}^2$ and $v(v-1)^2|v-1-2m| > v(2m+1)$ for $m \geq v/2$ we conclude $\partial f(m,p)/\partial p > 0$ for $m \geq v/2$. Thus the minimum of $f(m,p)$ is to be found among the integers $v/2 \leq m \leq v$, $p = 0$.

Next we notice

$$df(m,0)/dm = (\{-(v-1)^2(v-2m)/m^2(v-m)^2\} - \{2/m^3\})v/b. \tag{3.3}$$

Direct calculation shows that this has positive root

$$m_0 = \{(v-3)(v+1) + \sqrt{(v-3)^2(v+1)^2 + 16v(v-2)}\}/4(v-2)$$

and $df(m,0)/dm < 0$ for $m < m_0$ and $df(m,0)/dm > 0$ for $m > m_0$. A little additional calculation shows that for $v \geq 3$

$$v/2 < m_0 < (v+1)/2. \tag{3.4}$$

From (3.4) and the behavior of $df(m,0)/dm$ we conclude $f(m,p)$ is minimized for $m = (v+1)/2$, $p = 0$ if v is odd and $m = v/2$ or $(v+2)/2$ and $p = 0$ if v is even. Direct calculation shows $f(v/2, 0) < f((v+2)/2, 0))$ and hence we have the following.

LEMMA 3.1. $\operatorname{tr}(\bar{M}_{d1}^{-1})$ *is minimized for the following values m and p.*
(i) *If v is even,* $m = v/2, p = 0$.
(ii) *If v is odd,* $m = (v+1)/2, p = 0$.

Application of Lemmas 2.2, 2.3, and 3.1 yields the following.

THEOREM 3.1. *Any* $X_d \in D(v,b)$ *for which*
(i) $X'_dX_d = bv/4(v-1)I_v + b(v-2)/4(v-1)J_{v,v}$ *if v is even*
(ii) $X'_dX_d = b(v+1)/4vI_v + b(v+1)/4vJ_{v,v}$ *if v is odd*
is A-optimal over $D(v,b)$.

A direct consequence of this theorem is:

COROLLARY 3.1. *Suppose* $X_d \in D(v,b)$ *is the incidence matrix of a* B.I.B. *design with parameters*
(i) b, v, *and* $r = b/2$ *if v is even*
(ii) b, v, *and* $r = b(v+1)/2v$ *if v is odd*
then X_d *is A-optimal over* $D(v,b)$.

4. *D*-optimality. Suppose $\Phi(X'_dX_d) = \det(X'_dX_d)^{-1} = \prod_{i=1}^{v} 1/\mu_{di}$ where $\mu_{d1} \leq \cdots \leq \mu_{dv}$ are the eigenvalues of X'_dX_d. Since $\prod_{i=1}^{v} 1/\mu_{di}$ is a type 1 function, we can apply the results of Section 2 to find Φ-optimal designs. Here Φ-optimality corresponds to D-optimality. Suppose $b \geq v \geq 3$ is fixed. If $\bar{M}_{d1}$ is the average of $X'_{d1}X_{d1}$ over Π where $X_{d1} \in D(v, {}_1b)$ is as in Lemma 2.1 with $N = mb + p + 1$ entries,

$$\det(\bar{M}_{d1})^{-1} = \Phi(\bar{M}_{d1}) = (v(v-1)/\{mb(v-m) + p(v-1-2m)\})^{v-1}(v/\{bm^2 + p(2m+1)\}). \tag{4.1}$$

Let

$$g(m,p) = (\{mb(v-m) + p(v-1-2m)\}/v(v-1))^{v-1}(\{bm^2 + p(2m+1)\}/v) = 1/\Phi(\bar{M}_{d1}). \tag{4.2}$$

We seek values of m and p with $0 \leq m \leq v$, $0 \leq p < b$, and $0 \leq mb + p \leq bv$ which will minimize, or equivalently, maximize $g(m,p)$. Proceeding in a manner analogous to that used in Section 3 to find the minima of $f(m,p)$ one can here determine the values of m and p maximizing $g(m,p)$. The results are stated in the following lemma.

LEMMA 4.1. $\det(\bar{M}_{d1})^{-1}$ *is minimized for the following values of m and p*
(i) $m = (v + 1)/2$ *and* $p = 0$, *if v is odd*
(ii) $m = v/2$ *and* $p = bv/2(v + 1)$ *if v is even.*

REMARK. In (ii) above, if $bv/2(v + 1)$ is not an integer, $\det(\bar{M}_{d1})^{-1}$ is minimized by one of the two integers closest to $bv/2(v + 1)$.

THEOREM 4.1. *Any* $X_d \in D(v, b)$ *for which*
(i) $X'_dX_d = b(v + 1)/4vI_v + b(v + 1)/4vJ_{v,v}$, *if v is odd.*
(ii) $X'_dX_d = b(v + 2)/4(v + 1)I_v + b(v + 2)/4(v + 1)J_{v,v}$, *if v is even*
is D-optimal over $D(v, b)$.

A direct consequence of this theorem is:

COROLLARY 4.1. *Suppose* $X_d \in D(v, b)$ *is the incidence matrix of a* B.I.B. *design with parameters b, v, and* $r = b(v + 1)/2v$ *if v is odd. Then* X_d *is D-optimal over* $D(v, b)$.

Suppose $X_d \in D(v, b)$, v even, is of the form

$$X'_d = (X'_{d1}X'_{d2} \cdots X'_{dt})$$

where each X_{di} is the incidence matrix of a B.I.B. design with parameters b_i, v, and r_i satisfying $\sum_{i=1}^{t} b_i = b$, and $\sum_{i=1}^{t} r_i = b(v + 2)/2(v + 1)$. Then X_d is D-optimal over $D(v, b)$.

REMARK. The v even version of Corollary 4.1 is interesting. One can verify that there does not exist a B.I.B. design with parameters b, v, and $r = b(v + 2)/2(v + 1)$ when v is even, for this would require the block size k to be $v(v + 2)/2(v + 1)$ which is not an integer since $v + 1$ is relatively prime to both v and $v + 2$. However, one can piece together the incidence matrices of several B.I.B. designs having differing block sizes to produce an optimal design. For example, when $v = 4$ and $b = 10$, if we let X_{d1} be the incidence matrix of the B.I.B. design with parameters $v = 4$, $b = 6$, and $r = 3$ and X_{d2} be the incidence matrix of the B.I.B. design with parameters $v = 4$, $b = 4$, and $r = 3$ then $X'_d = (X'_{d1}X'_{d2})$ is D-optimal for D (4, 10).

REMARK. Hedayat and Wallis (1978) show that a design X_d corresponding to a B.I.B.D. having parameters $v = b = 4t - 1$, $r = k = 2t$ and $\lambda = t$ for $t \geq 1$ is D-optimal in $D(v, b)$. Clearly such designs also satisfy Corollary 4.1. Thus the result given by Hedayat and Wallis is a special case of Corollary 4.1.

5. *E*-optimality. Suppose $\Phi(X'_dX_d)$ = maximum eigenvalue of $(X'_dX_d)^{-1} = 1/\mu_{d1}$ where $\mu_{d1} \leq \cdots \leq \mu_{dv}$ are the eigenvalues of X'_dX_d. Since $1/\mu_{d1}$ is a type 1 function, we can apply the results of Section 2 to find Φ-optimal designs. Here Φ corresponds to E-optimality. Suppose $b \geq v \geq 3$ is fixed. If $\bar{M}_{d1}$ is the average of $X'_{d1}X_{d1}$ over Π where $X_{d1} \in D(v, b)$ is as in Lemma 2.1 with $N = mb + p + 1$ entries,

$$\Phi(\bar{M}_{d1}) = v(v - 1)/\{mb(v - m) + p(v - 1 - 2m)\}. \tag{5.1}$$

Let

$$h(m, p) = mb(v - m) + p(v - 1 - 2m). \tag{5.2}$$

We seek values of m and p with $0 \leq m \leq v$, $0 \leq p < b$, and $0 \leq mb + p \leq bv$ which will minimize Φ, or equivalently, maximize $h(m, p)$. Lemma 2.3 tells us that the maximum value of $h(m, p)$ can be found among the integers $v/2 \leq m \leq v$, $0 \leq p < b$, and $mb + p \leq bv$. In this range $v - 1 - 2m < 0$ so $h(m, p)$ is decreasing in p, implying the maximum of $h(m, p)$ is among the values $v/2 \leq m \leq v$ and $p = 0$. Examination of $h(m, 0)$ shows the maximum occurs when $m = v/2$ if v is even, and $m = (v + 1)/2$ if v is odd. Actually when v is odd $h((v - 1)/2, p) = h((v + 1)/2, 0)$ for all $0 \leq p < b$. We therefore conclude:

LEMMA 5.1. *The maximum eigenvalue of $\bar{M}_{d1}^{-1}$ is minimized for the following values of m and p.*
(i) $m = v/2$ *and* $p = 0$, *if v is even*
(ii) $m = (v - 1)/2$ *and any* $0 \leq p < b$ *or* $m = (v + 1)/2$ *and* $p = 0$, *if v is odd.*

THEOREM 5.1. *Any $X_d \in D(v, b)$ for which*
(i) $X'_dX_d = bv/4(v - 1)I_v + b(v - 2)/4(v - 1)J_{v,v}$, *if v is even*
(ii) $X'_dX_d = b(v + 1)/4vI_v + \{b(v - 3) + 4p\}/4vJ_{v,v}$ *for some* $0 \leq p < b$
or
$X'_dX_d = b(v + 1)/4vI_v + b(v + 1)/4vJ_{v,v}$, *if v is odd*
is E-optimal over $D(v, b)$.

A direct consequence of this theorem is:

COROLLARY 5.1. *Suppose $X_d \in D(v, b)$ is the incidence matrix of a* B.I.B. *design with parameters*
(i) b, v, *and* $r = b/2$, *if v is even*
(ii) b, v, *and* $r = \{b(v - 1) + 2p\}/2v$ *for some* $0 \leq p < b$ *or* $r = b(v + 1)/2v$, *if v is odd then X_d is E-optimal over $D(v, b)$.*

COROLLARY 5.2. *For any $X_d \in D(v, b)$*
(i) min. *eignevalue of* $X'_dX_d \leq bv/4(v - 1)$, *if v is even*
(ii) min. *eigenvalue of* $X'_dX_d \leq b(v + 1)/4v$, *if v is odd.*

Corollary 5.2 can be improved upon slightly by a more careful argument. Letting $[x]$ be the greatest integer $\leq x$, we can establish:

THEOREM 5.2. *Suppose $b \geq v$ are such that*
(i) $bv/4(v - 1) \leq [bv/4(v - 1)] + 1 - v/4(v - 1)$ *if v is even*
(ii) $b(v + 1)/4v \leq [b(v + 1)/4v] + 1 - (v + 1)/4v$ *if v is odd.*
If $X_\delta \in D(v, b)$ has its minimum eigenvalue $\mu_{\delta 1} \geq [bv/4(v - 1)]$, and if v is even, or $\mu_{\delta 1} \geq [b(v + 1)/4v]$, if v is odd, then X_δ is E-optimal in $D(v, b)$.

PROOF. Since the proofs for v even and v odd are similar, we shall only give the proof for the case v odd.

Let $X_d \in D(v, b)$ be arbitrary with $N = mb + p$, $0 \leq m \leq v$, $0 \leq p < b$, $0 \leq N \leq bv$, being the number of ones in X_d. Let S_{ij} denote a $v \times 1$ column vector with a $+1$ in the ith coordinate, a -1 in the jth coordinate, and zeroes elsewhere. If μ_{d1} represents the minimum eigenvalue of X'_dX_d then it follows from Rayleigh's inequality that

$$\mu_{d1} \leq \frac{1}{2} S'_{ij}X'_dX_dS_{ij} = \frac{1}{2} (\lambda_{dii} + \lambda_{djj} - 2\lambda_{dij})$$

where λ_{dij} is the i, jth entry of X'_dX_d. For X_d to have $\mu_{d1} > \mu_{\delta 1}$ it must be true that if $\lambda_{dii} + \lambda_{djj}$ is even then

$$\frac{1}{2} (\lambda_{dii} + \lambda_{djj} - 2\lambda_{dij}) \geq [b(v + 1)/4v] + 1$$

and if $\lambda_{dii} + \lambda_{djj}$ is odd then

$$\frac{1}{2} (\lambda_{dii} + \lambda_{djj} - 2\lambda_{dij}) \geq [b(v + 1)/4v] + 1/2.$$

If we let v_1 and v_2 represent the numbers of columns in X_d with even and odd numbers of ones occurring in them, then it must also be true that

$$\frac{1}{2}\sum_{i=1}^{v}\sum_{j\neq i}^{v}(\lambda_{dii}+\lambda_{djj}-2\lambda_{dij}) = (v-1)N - \sum_{i=1}^{v}\sum_{j\neq i}^{v}\lambda_{dij}$$
$$\geq \{v_1(v_1-1)+v_2(v_2-1)\}\{[b(v+1)/4v]+1\} + 2v_1v_2\{[b(v+1)/4v]+1/2\}.$$

When v is odd the right hand side of this last inequality is minimal for $v_1 = (v+1)/2$ and $v_2 = (v-1)/2$. Thus for X_d to have $\mu_{d1} > \mu_{s1}$ we must have

$$\frac{1}{2}\sum_{i=1}^{v}\sum_{j\neq 1}^{v}(\lambda_{dii}+\lambda_{djj}-2\lambda_{dij}) \geq v(v-1)[b(v+1)/4v] + (3v-1)(v-1)/4.$$

From Lemma 2.1 and its proof we see that

$$\sum_{i=1}^{V}\sum_{i\neq j}^{V}\lambda_{dij} \geq bm(m-1)+2mp.$$

Thus

$$(v-1)N - \sum_{i=1}^{v}\sum_{j\neq i}^{v}\lambda_{dij} \leq (v-1)(mb+p) - bm(m-1) - 2mp$$
$$= mb(v-m) + p(v-1-2m).$$

We have seen in the argument following equation (5.2) that $mb(v-m)+p(v-1-2m)$ is maximized for $m = (v-1)/2$, $0 \leq p < b$, or $m = (v+1)/2$, $p = 0$, and hence has maximum value $b(v^2-1)/4$. Thus for $\mu_{d1} > \mu_{\delta 1}$ to hold we must have

$$v(v-1)[b(v+1)/4v] + (3v-1)(v-1)/4 \leq (v-1)N - \sum_{i=1}^{v}\sum_{j\neq 1}^{v}\lambda_{dij} \leq b(v^2-1)/4$$

or

$$[b(v+1)/4v] + 1 - (v+1)/4v \leq b(v+1)/4v$$

which cannot hold by assumption. Thus X_d cannot have $\mu_{d1} > \mu_{\delta 1}$ and so X_δ is E-optimal.

Corollary 5.3. *Let v and b satisfy the conditions of Theorem* 5.2. *If $X_d \in D(v, b)$ corresponds to the incidence matrix of a group divisible* (GD) *design having parameters, v, b, $r = b/2$, $k = v/2$ and $\lambda_2 = \lambda_1 - 1$, then X_d is E-optimal in $D(v, b)$.*

Proof. Suppose X_d satisfies the conditions given. Then $\mu_{d1} = r - \lambda_2 - 1 = r - \lambda_1$ and it is easy to verify that

$$\lambda_2 = [b(v/2)(v/2-1)/v(v-1)] = [b(v-2)/4(v-1)] = [(b/2) - (bv/4(v-1))]$$
$$= b/2 - [bv/4(v-1)] - 1$$

where the last equality follows from the fact that $b/2$ is an integer, $b(v-2)/4(v-1)$ is not an integer, and condition (i) of Theorem 5.2. Thus

$$\mu_{d1} = r - \lambda_1 = [bv/4(v-1)]$$

and the result follows from Theorem 5.2.

Comment. Takeuchi (1963) established the E-optimality of GD designs having $\lambda_2 = \lambda_1 + 1$ in a general block design experimental setting. Corollary 5.3 is the first general result known to the authors concerning the optimality of GD designs having $\lambda_2 = \lambda_1 - 1$ in any experimental setting.

While Corollaries 5.1 and 5.3 characterize two well known classes of block designs which can serve as optimal spring balance designs, many other do exist.

Corollary 5.4. *Let v, b, and $X_s \in D(v, b)$ satisfy the conditions of Theorem* 5.2.

Suppose $\hat{b}$ *is such that*
(i) $(b + \hat{b})v/4(v-1) < [bv/4(v-1)] + 1 - v/4(v-1)$ *if* v *is even*
(ii) $(b + \hat{b})(v+1)/4v < [b(v+1)/4v] + 1 - (v+1)/4v$ *if* v *is odd.*
Let $X_d \in D(v, \hat{b})$ *be arbitrary. Then*

$$\tilde{X}_{d^*} = \begin{pmatrix} X_s \\ X_d \end{pmatrix}$$

is E-optimal in $D(v, b + \hat{b})$

PROOF. Since $\tilde{X}'_{d^*}\tilde{X}_{d^*} = X'_sX_s + X'_dX_d$, we have $\mu_{d^*1} \geq \mu_{s1}$ and the result now follows from Theorem 5.2.

COROLLARY 5.5. *Let* v, b, *and* $X_s \in D(v, b)$ *satisfy the conditions of Theorem* 5.2. *Suppose* $\hat{v}$ *is such that* $v - \hat{v}$ *and* b *also satisfy*
(1) $b(v-\hat{v})/4(v-\hat{v}-1) < [bv/4(v-1)] + 1 - (v-\hat{v})/4(v-\hat{v}-1)$ *if* v *and* $v - \hat{v}$ *are even*
(2) $b(v-\hat{v}+1)/4(v-\hat{v}) < [b(v+1)/4v] + 1 - (v-\hat{v}+1)/4(v-\hat{v})$ *if* v *is odd and* $v - \hat{v}$ *is odd*
(3) $b(v-\hat{v}+1)/4(v-\hat{v}) < [bv/4(v-1)] + 1 - (v-\hat{v}+1)/4(v-\hat{v})$ *if* v *is even and* $v - \hat{v}$ *is odd*
(4) $b(v-\hat{v})/4(v-\hat{v}-1) < [b(v+1)/4v] + 1 - (v-\hat{v})/4(v-\hat{v}-1)$ *if* v *is odd and* $v - \hat{v}$ *is even.*
Then the design $X_d \in D(v-\hat{v}, b)$ *obtained by deleting any* $\hat{v}$ *columns of* X_s *is E-optimal in* $D(v-\hat{v}, b)$.

PROOF. Since X'_dX_d is a principal submatrix of X'_sX_s we have $\mu_{d1} \geq \mu_{s1}$ and the result now follows from Theorem 5.2.

EXAMPLE 5.1. Consider the class of designs $D(7, 7)$ and let $X_d \in D(7, 7)$ correspond to the B.I.B.D. having parameters $r = k = 3$ and $\lambda = 1$. Then X_d satisfies Corollary 5.1 and is E-optimal in $D(7, 7)$. Also, any design obtained by adding one or two rows to X_d is E-optimal in $D(7, 8)$ and $D(7, 9)$ by Corollary 5.4 and any design obtained by deleting $\hat{v}$ columns from X_d is E-optimal in $D(7-\hat{v}, 7)$ by Corollary 5.5 for $\hat{v} = 1, 2, 3, 4$.

6. Minimizing the maximum diagonal entry of $(X'_dX_d)^{-1}$. Suppose $\Phi(X'_dX_d) =$ maximum diagonal entry of $(X'_dX_d)^{-1}$. Although Φ is not of the form given in Section 2, it is convex and permutation invariant in X'_dX_d and hence it is possible to verify that Lemmas 2.1—2.4 still hold for Φ. Since

$$(\alpha I_v + \beta J_{v,v})^{-1} = (1/\alpha)I_v - \beta/\alpha(a + \beta v)J_{v,v}$$

it follows from Lemma 2.3 that we seek values of m and p with $0 \leq m \leq v$, $0 \leq p < b$, and $0 \leq mb + p \leq bv$ which minimize

$$f(m,p) = 1/\alpha(m,p) - \beta(m,p)/\alpha(m,p)\{\alpha(m,p) + \beta(m,p)v\}$$

where

$$\alpha(m,p) = \frac{mb+p}{v} - \frac{bm(m-1) + 2mp}{v(v-1)}, \qquad \beta(m,p) = \frac{bm(m-1) + 2mp}{v(v-1)}.$$

Since $vf(m,p) = \operatorname{tr} \bar{M}_d^{-1}$, minimization of $f(m,p)$ is equivalent to finding the m and p yielding an A-optimal design. We conclude that the A-optimal designs of Section 3 are also the designs which minimize the maximum diagonal entry of $(X'_dX_d)^{-1}$ over $D(v, b)$.

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REFERENCES

[1] BANERJEE, K. S. (1975). *Weighing Designs.* Marcel Dekker, New York.

[2] CHENG, C. S. (1978). Optimality of certain asymmetrical experimental designs. *Ann. Statist.* **6** 1239–1261.

[3] HEDAYAT, A. and WALLIS, W. D. (1978). Hadamard matrices and their applications. *Ann. Statist.* **6** 1184–1238.

[4] MOOD, A. M. (1946). On Hotelling's weighing problem. *Ann. Math Statist.* **17** 432–446.

[5] RAGHAVARAO, D. (1971). *Construction and Combinatorial Problems in Design of Experiments.* Wiley, New York.

[6] TAKEUCHI, K. (1961). On the optimality of certain type of PBIB designs. *Rep. Statist. Appl. Res. JUSE* **8** 140–145.

DEPARTMENT OF PURE AND APPLIED MATHEMATICS
WASHINGTON STATE UNIVERSITY
PULLMAN, WASHINGTON 99164

DEPARTMENT OF STATISTICS
PURDUE UNIVERSITY
WEST LAFAYETTE, INDIANA 47907