Journal of Computer Assisted Tomography
4(3):361-363, June

On the Optimality of the Filtered Backprojection Algorithm

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Abstract: It is shown that under certain conditions the filtered backprojection algorithm produces a computed tomographic reconstruction for which the statistical accuracy attainable in the amplitude estimation of large-area objects meets the general lower bound derived by Tretiak. In this sense, filtered backprojection is an optimum algorithm. **Index Terms:** Computed tomography----Noise-Image reconstruction-Image quality-Data processing.

I. INTRODUCTION

Tretiak (1) has obtained a lower bound on the statistical accuracy achievable in X-ray computed tomography (CT) through the use of the Cramer-Rao (2) method. His result is attractive in that it is essentially independent of the reconstruction algorithm. Tretiak calculated that the filtered backprojection (convolutional) algorithm yields a statistical variance which is 27% greater than the lower bound and hence concluded that this algorithm is nearly optimal. In this paper, we use Tretiak's result to obtain a lower bound on the accuracy of the estimates of the amplitude of low-contrast, large-area objects that takes a particularly simple form under the assumption that approximately the same density of X-rays are detected in each projection. We will show that the accuracy set by this lower bound is attained when an optimum estimation procedure is applied to a reconstruction obtained by the filtered backprojection algorithm, provided the amplitude estimator is unbiased. In this sense, filtered backprojection is an optimum algorithm. The relationship between amplitude estimation and the problem of the detection of objects is discussed.

II. ESTIMATION OF OBJECT AMPLITUDE

In ref. 1 the quantity to be estimated is

$$(\mu,a) = \iint dx\ dy\ \mu(x,y)a(x,y) \tag{1}$$

where $\mu(x,y)$ is the x-ray attenuation coefficient of the material in the plane being scanned and $a(x,y)$ is the aperture function, normalized to have unity integral,

$$\iint dx\ dy\ a(x,y) = 1. \tag{2}$$

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We observe that (μ,a) is the weighted average of μ, where a is the weighting function. Let us consider an object of known shape and position to be placed in a known background medium. This object may be represented as

$$\mu(x,y)a(x,y) = \mu_0 a(x,y), \tag{3}$$

where μ_0 is the effective amplitude (contrast) of the object and $a(x,y)$ is its shape function. Then the quantity to be estimated is $(\mu,a) = \mu_0$, the object's effective amplitude. Since the background medium is assumed to be known, it may be trivially removed from the estimate.

Through the use of the Cramer-Rao method, Tretiak showed that the variance in an estimate of (μ,a) has a lower bound of

$$\mathrm{var}(\mu,a) \geqslant (a,a)^2/I_1 \tag{4}$$

in his nomenclature. This lower bound holds when the functional derivative of the bias with respect to the aperture function is negligible. It is the opinion of the author that it is possible to construct an estimate of the amplitude of a large-area object based on a CT reconstruction for which the effect of the bias is negligible. This will be assumed here to be the case. (See ref. 1 for further discussion of this point.)

Tretiak's evaluation of the Fisher Information Function is

$$I_1 = \sum_{i=1}^{m} \int dr\ W_i(r)\ \exp\left[-p(r,\phi_i)\right]\tilde{a}^2(r,\phi_i) \tag{5}$$

for a set of m measurements at different projection angles ϕ_i, where $W_i(r)$ is the average incident X-ray density in each projection and $\tilde{a}(r,\phi_i)$ is the projection of $a(x,y)$. Equation 5 may be simplified somewhat under the condition that $W_i(r)$ and the projections $p(r,\phi_i)$ are nearly constant over the width of the object. This condition will be met if the object of interest has low contrast relative to the surrounding, slowly varying medium. Then

$$I_1 = \sum_{i=1}^{m} W_i \exp(-p_i) \int dr\ \tilde{a}^2(r,\phi_i)$$

$$= \sum^{m} v_i \int dr\, \tilde{a}^2(r,\phi_i)\,, \quad (6)$$

where V_i are the average density of X-rays *detected* in each projection. Also, the numerator in Eq. 4 may be expressed in terms of an effective area (3) of the object:

$$(a,a) = \iint dx\, dy\, a^2(x,y) = 1/A. \quad (7)$$

The above may be further simplified if it is assumed that the projection measurements are taken at equally spaced angles between 0 and π and that m is large enough that the sum over i can be well approximated by an integral over ϕ. This assumption is implicit in the use of the conventional filtered backprojection algorithm, since if it were not fulfilled, serious artifacts would result in the reconstruction. It is fulfilled by all commercial CT scanners. Let us also assume that the density of detected X-rays V_i is approximately the same in each projection. While this assumption is not critical to the comparison to be made in Section III, it leads to a greatly simplified result. Under these assumptions, Eq. 6 becomes

$$I_i \approx mV_i \int_0^{\pi} \frac{d\phi}{\pi} \int_{-\infty}^{\infty} dr\, \tilde{a}^2(r,\phi)$$

$$= \frac{\text{NEQ}}{\pi} \int_0^{\pi} d\phi \int_{-\infty}^{\infty} df\, |\mathcal{A}(f,\phi)|^2 \quad (8)$$

by Parseval's theorem, where $\mathcal{A}(f,\phi)$ is the Fourier transform of the projection $\tilde{a}$ and, by the Projection-Slice Theorem (4), is also the Fourier transform of the object shape function $a(x,y)$. The total number of detected X-rays per unit length in the projections has been represented by NEQ, the number (density) of noise-equivalent quanta. The final result is that the lower bound on the variance in the estimated object amplitude is

$$\text{var}(\mu_0) \geq \frac{\pi}{A^2\text{NEQ}} \left[\int_0^{2\pi} d\phi \int_0^{\infty} df\, |\mathcal{A}(f,\phi)| \right]^{-1}. \quad (9)$$

It should be noted that while this lower bound was derived for specific conditions of the projection measurements (typical of most CT applications), it does not depend on any specific reconstruction algorithm. As pointed out by Tretiak at the close of his article, this lower bound is only accurate when the object's dimension is somewhat larger than the spatial resolution of the projection measurements; that is, it holds for large objects.

For a Gaussian-shaped object considered by Tretiak, the double integral in Eq. 9 is simply $A^{-1/2}$, where A is the effective area of the Gaussian ($A = 4\pi\sigma^2$), yielding

$$\text{var}(\mu_0) = \pi/(NEQ\ A^{3/2}). \quad (10)$$

It has been found (3) that this is a good approximation for other reasonably compact objects (e.g., for circles and squares, but not for long, thin rectangles).

III. COMPARISON WITH FILTERED BACKPROJECTION

There is a close connection between the estimation of an object's amplitude and the problem of the detection of the object for a fixed amplitude. In the binary decision case, it is to be determined whether an object of known amplitude and shape is present at a known location. The amplitude estimate of that object may be used as the decision function. The estimate will yield a value approximately equal to the known amplitude if the object is present and a value near zero if the object is not present. Then the signal-to-noise ratio (SNR) for the detection of an object with amplitude μ_0 is simply related to the variance in the amplitude estimate as

$$\text{SNR}^2 = \mu_0^2/\text{var}(\mu_0). \quad (11)$$

This relation may be turned around to determine the variance in the amplitude estimate if the detection SNR is known. The minimum amplitude variance possible will be achieved for the maximum detection SNR.

The maximum SNR achievable in the binary decision case can be expressed in Fourier space as (5)

$$\text{SNR}^2_{\text{max}} = \iint df_x\, df_y\, \frac{|R(f_x,f_y)|^2}{S(f_x,f_y)}\,, \quad (12)$$

where R is the Fourier transform of the object and S is the power spectral density of the image noise. Equation 12 is the "matched filter" result for the detection of signals in the presence of nonwhite noise. In this formalism, the object is considered to be the attenuation coefficient as a function of x and y, which is related to the shape function $a(x,y)$ as

$$r(x,y) = \mu_0 A\ a(x,y). \quad (13)$$

This relationship maintains the effective contrast, defined as

$$\mu_{\text{eff}} = \frac{1}{A}\iint dx\, dy\, r(x,y) = \mu_0\,. \quad (14)$$

Under the same restrictions imposed on the projections in Section II, S for the filtered backprojection CT reconstruction algorithm has the low-frequency limit (3,6-9)

$$\lim_{f\to 0} S(f) = \frac{\pi}{\text{NEQ}}\, |f|, \quad (15)$$

where NEQ has the same meaning as before. It should be noted that the apodization function used in filtering the projections to suppress the high-frequency components does not appear in Eq. 15, since it must approach unity in the low-frequency limit in order to provide a proper reconstruction. For large objects, for which R^2 is concentrated at low frequencies, we find the minimum variance in the amplitude estimate to be

$$\text{var}(\mu_0) = \frac{\mu_0^2}{\text{SNR}^2} = \frac{\pi\mu_0^2}{\text{NEQ}} \left[\int_0^{2\pi} d\phi \int_0^{\infty} df\, |R(f,\phi)|^2 \right]^{-1}. \quad (16)$$

With the definition of the object, Eq. 13, this may be expressed in terms of the Fourier transform of the shape function:

$$\text{var}(\mu_0) = \frac{\pi}{A^2\text{NEQ}} \left[\int_0^{2\pi} d\phi \int_0^{\infty} df\, |\mathcal{A}(f,\phi)|^2 \right]^{-1}. \quad (17)$$

Thus the optimum estimate of the amplitude of large objects based on the filtered backprojection reconstruction yields the same variance as the Cramer-Rao lower bound (Eq. 9). We conclude that the filtered backprojection algorithm does indeed provide an optimum reconstruction of large objects under the assumption that the bias term is negligible.

IV. DISCUSSION

The optimum nature of the filtered backprojection algorithm has already been discussed in terms of the detection of large objects. Hanson (3) has shown that for large objects the optimum detection SNR in the filtered backprojection reconstruction is the same as that obtained when the decision task is based on the projection data themselves. In principle, the ability to detect a large object is the same in the reconstruction as in the projections. Interpreted in terms of information content (9), this is equivalent to the statement that the information content of the projection data is preserved in the reconstruction process. Of course, from a practical point of view, the removal of structural overlap accomplished by CT reconstruction greatly facilitates the human interpretation of the projection data. The above statements concerning detection assume that the background on which the object is superimposed is completely known beforehand.

The discussion in this paper has been restricted to large objects to avoid the complications of finite spatial resolution in the projections and discrete coordinates. For small objects the representation of the reconstruction in discrete coordinates can reduce the detection SNR relative to that in the projections if the pixel size is not much smaller than the projection resolution (10).

Tretiak (1) compared the Cramer-Rao lower bound with the variance in the filtered backprojection reconstruction calculated by Brooks and Di Chiro (11). While the latter calculation is correct (3,12), it could not compare well with Tretiak's Cramer-Rao bound because of differences in the aperture function and in the resolution parameterization. On repeating Brooks and Di Chiro's calculation using a Gaussian aperture function to match his Cramer-Rao analysis, Tretiak obtained a result which was 27% higher than the Cramer-Rao bound.

In the present work, attention was focused on the estimation of an object's amplitude rather than on the variance in the reconstruction. Agreement with the Cramer-Rao lower bound was only achieved because an *optimum amplitude estimate* was used. It should be noted that the amplitude estimation procedure can only be optimum if it takes into account the correlation properties of the noise (described above by S). Thus the weighted mean which provides an optimum estimate for white (uncorrelated) noise, will not provide an optimum estimate for the nonwhite noise found in CT reconstructions (Eq. 15). It has been shown that for CT noise the variance in the estimate based on uncorrelated noise is worse than the optimum estimate by a factor of at least $(\pi/2) = 1.57$ (Eqs. 43 and 49 of ref. 3). An interesting example is the determination of the amplitude of a uniform disk. The conventional estimate would be the average value of the reconstruction within the region of the disk. However, the optimum estimate would be based on reconstruction values outside the disk as well as inside, since the noise outside the disk contains some information about that inside through the long-range correlations present in CT noise. Thus the assumption that the background is known *a priori* plays a key role in the development of the optimum estimate. Whether the noise correlations can be incorporated into an amplitude estimate where the background is not known, as in the clinical situation, remains an unanswered question.

ACKNOWLEDGMENT

This work was supported by the U.S. Department of Energy.

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