# On the order of the coefficients of Bessel series of a differentiable function. (*) 

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Summary. - In this paper the author has estimated the orders of the coefficients of Bessel series of a function which is differentiable $2 S(S \geq 1)$ times. From his results he also derived the uniformity of the convergence of the Bessel series in the interval $(a, b)$.

1.     - Introduction: The Fourier Bessel Series for an arbitrary function $f(x) \in \mathcal{L}(0,1)$ is given by

$$
\begin{equation*}
f(x) \sim \sum_{m=1}^{\infty} a_{m} J_{v}\left(j_{m} x\right) \tag{1.1}
\end{equation*}
$$

where

$$
\begin{equation*}
a_{m}=\frac{2}{J_{v+1}^{2}\left(j_{m}\right)} \int_{0}^{1} t f(t) J_{v}\left(j_{m} t\right) d t \tag{1.2}
\end{equation*}
$$

and $j_{1}<j_{2}<\ldots$ denote the positive zeros of the Bessel function $J_{\nu}(x)$ of the first kind of order $v>-\frac{1}{2}$.

An arbitrary function $f(x) \in \mathscr{L}(a, b)$ can be expanded in a series of the form

$$
\begin{equation*}
f(x) \sim \sum_{m=1}^{\infty} a_{m} C_{v}\left(\gamma_{m} x, \gamma_{m} b\right) \tag{1.3}
\end{equation*}
$$

where

$$
\begin{equation*}
C_{\nu}(\alpha, \beta) \equiv J_{\nu}(\alpha) Y_{\nu}(\beta)-J_{v}(\beta) Y_{\nu}(\alpha) \tag{1.4}
\end{equation*}
$$

and $\gamma_{1}<\gamma_{2}<\ldots$ denote the positive roots of the equation

$$
C_{\nu}(a z, b z)=0
$$

$Y_{\nu}(z)$ denoting the Bessel function of the second kind of order $v \geq-\frac{1}{2}$ and $a_{m}$ is defined as

$$
\begin{equation*}
a_{m}=\frac{\int_{a}^{b} t f(t) C_{v}\left(\gamma_{m} t, \gamma_{m} b\right) d t}{\int_{a}^{b} t C_{v}^{2}\left(\gamma_{m} t, \gamma_{m} b\right) d t} \tag{1.5}
\end{equation*}
$$

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$$
=\frac{\pi^{2} \gamma_{m}^{2} J_{\nu}^{2}\left(\gamma_{m} a\right)}{2\left[J_{v}^{2}\left(\gamma_{m} a\right)-J_{v}^{2}\left(\gamma_{m} b\right)\right]} \cdot \int_{a}^{b} t f(t) C_{\nu}\left(\gamma_{m} t, \gamma_{m} b\right) d t .
$$

Tifchmarsh [1] has studied the convergence of the series (1.3) when the generating fnnction $f(x)$ is of bounded variation in the neighbourhood of a point in the interval ( $a, b$ ).

The object of the present note is to prove certain results regarding the order of the co-efficients of the series (1.3) of a function which is differentiable several times. The results correspond to the well known theorems on Fourier-Bessel series (1.1) [3, pp. 228-233].
2. - We shall prove the following theorems.

Theorem 1. - If $f(x)$ be bounded and twice differentiable function defined on the interval $(a, b)$ such that

$$
\begin{aligned}
& f(a)=f^{\prime}(a)=0, \\
& f(b)=0
\end{aligned}
$$

and $f^{\prime \prime}(x)$ is bounded almost everywhere then the co-efficients $\alpha_{m}$ of the Bessel series (1.3) of the function $f(x)$ are given by

$$
\begin{equation*}
\left|a_{m}\right|=0\left(\frac{1}{\gamma_{m}}\right), \quad \text { as } \quad m \rightarrow \infty \tag{2.1}
\end{equation*}
$$

almost everywhere.
Theorem 2. - If $f(x)$ satisfy the conditions of theorem 1 then the series (1.3) converges uniformly and absolutely on the whole interval ( $a, b$ ), provided $a>0$, for $v \geq-\frac{1}{2}$.

Theorem 3. - If $f(x)$ be defiined on $(a, b)$ and differentiable $2 S$ times $(S>1)$, such that $f(a)=f^{\prime}(a)=f^{\prime \prime}(a)=\ldots=f^{2 s-1}(a)=0, f(b)=f^{\prime}(b)=\ldots=f^{2 s-2}(b)=0$; and $f^{2 r}(x)$ is bounded almost everywhere then the co-efficients $a_{m i}$ of the series (1,3) of the function $f(x)$ are given by

$$
\begin{equation*}
\left|\boldsymbol{a}_{m}\right|=0\left(\frac{1}{\gamma_{m}^{2 s-1}}\right) \quad \text { as } \quad m \rightarrow \infty \tag{2.2}
\end{equation*}
$$

almost everywhere.
Theorem 4. - If $f(x)$ satisfy the conditions of theorem 3 for $S \geq 1$, then for $\nu \geq-\frac{1}{2}$, the series (1.3) converges uniformly on the whole interval ( $a, b$ ) provided $a>0$.
3. - Daring the proof of our theorems we shall establish the following Lemmas.

Lemma 1. - For $v \geq-\frac{1}{2}$, and sufficiently large $\lambda>0$

$$
\left|C_{v}(\lambda x, \lambda b)\right|<\frac{k^{\prime}}{\lambda \sqrt{x b}}
$$

where $k^{\prime}$ is a constant and $0<a<\infty<b$.
Proof. - It is known that [3, p. 608] for $-\frac{1}{2} \leq \nu \leq \frac{1}{2}$ and for all values of $x$ and $\lambda$ under considerations

$$
\begin{align*}
& \left|H_{v}^{(1)}(\lambda x)\right|<\frac{k_{2}}{\sqrt{\lambda x}}  \tag{3.1}\\
& \left|H_{v}^{(2)}(\lambda x)\right|<\frac{k_{2}}{\sqrt{\lambda x}}
\end{align*}
$$

( $k_{2}$ being a constant)
where $H_{v}^{(1)}(z)$ and $H_{v}^{(2)}(z)$ are Bessel functions of third kind of order $v$ and [see, (3) p. 74].

$$
\begin{align*}
& J_{\nu}(z)=\frac{H_{v}^{(1)}(z)+H_{v}^{(2)}(z)}{2}  \tag{3.3}\\
& Y_{\nu}(z)=\frac{H_{v}^{(1)}(z)-H_{\nu}^{(2)}(z)}{2 i}
\end{align*}
$$

For $\vee \geq \frac{1}{2}$, and sufficiently large $\lambda>0$ and $0<a<x<b$ [see, (3) p. 608]

$$
\begin{align*}
& \left.\left|H_{\nu}^{(1)}(\lambda x)\right|<k_{4} \left\lvert\,(\lambda x)^{-\frac{1}{2}}+(\lambda x)^{-v}\right.\right\}  \tag{3.5}\\
& \left.\left|H_{\nu}^{(2)}(\lambda x)\right|<k_{4} \left\lvert\,(\lambda x)^{-\frac{1}{2}}+(\lambda x)^{-v}\right.\right\} \tag{3.6}
\end{align*}
$$ ( $k_{4}$ being a constant).

By straight application of (3.1), (3.2) and using (3.3), (3.4) the result of the Lemma follows immediately for $-\frac{1}{2} \leq v \leq \frac{1}{2}$.

On the otherhand for $v \geq \frac{1}{2}$, in view of (3.5), (3.6) and using (3.3), (3.4) we at once deduce that

$$
\left|C_{v}(\lambda x, \lambda b)\right|<\frac{k^{\prime}}{\lambda \sqrt{x b}}
$$

Lemma 2. - For $v \geq-\frac{1}{2}$, and sufficiently large $\lambda>0$ and fixed $x>0$

$$
\lambda \sqrt{x} C_{\nu}(\lambda x, \lambda b)=A \operatorname{Sin}(\lambda b-\lambda x)+\frac{\gamma^{\prime}}{\lambda x}
$$

Where $A$ is a constant and $\gamma^{\prime}$ remains bounded as $\lambda \rightarrow \infty$.
Proof. - The result of the lemma is evident from the asymptotic formulae for $J_{\nu}(x)$ and $Y_{\nu}(x)$ given by Tolstov [2, p. 213]

$$
\begin{align*}
& J_{v}(x)=\left(\frac{2}{\pi x}\right)^{\frac{1}{2}}\left[\operatorname{Cos}\left(x-\frac{1}{2} v \pi-\frac{1}{4} \pi\right)+\frac{\gamma_{\nu}(x)}{x}\right]  \tag{3.7}\\
& Y_{\nu}(x)=\left(\frac{2}{\pi x}\right)^{\frac{1}{2}}\left[\operatorname{Sin}\left(x-\frac{1}{2} v \pi-\frac{1}{4} \pi\right)+\frac{\rho_{v}(x)}{x}\right] \tag{3.8}
\end{align*}
$$

where the functions $\gamma_{\nu}(x)$ and $\rho_{\nu}(x)$ remain bounded as $x \rightarrow \infty$.
Lemma. $3-$ For $v \geq-\frac{1}{2}$ and sufficiently large $\lambda>0$

$$
\begin{equation*}
\frac{M}{\lambda^{2}} \leq \int_{a}^{b} x C_{y}^{2}(\lambda x, \lambda b) d x \leq \frac{K}{\lambda^{2}} \tag{3.9}
\end{equation*}
$$

where $M>0$ and $K>0$ are constants (which can depend on $\nu$ ).
Proof. - Obviously, from Lemma 1 we have

$$
\begin{align*}
\int_{a}^{b} x C_{i}^{2}(\lambda x, \lambda b) d x & \leq \frac{k^{\prime 2}}{b \lambda^{2}} \int_{a}^{b} d x  \tag{3.10}\\
& =0\left(\frac{1}{\lambda^{2}}\right)
\end{align*}
$$

which implies the right hand in-equality of (3.9). Again

$$
\begin{equation*}
\int_{a}^{b} x C_{\nu}^{2}(\lambda x, \lambda b) d x=\frac{1}{\lambda^{2}} \int_{\lambda, a}^{\lambda b} t C_{\nu}^{2}(t, \lambda b) d t \tag{3.11}
\end{equation*}
$$

but, from Lemma 2

$$
\begin{aligned}
\lambda t G_{v}^{2}(t, \lambda b) & =A^{2} \operatorname{Sin}^{2}(\lambda b-t)+\frac{\gamma^{\prime 2}}{b^{2}}+\frac{2 A \gamma^{\prime} \operatorname{Sin}(\lambda b-t)}{t} \\
& \geq A^{2} \operatorname{Sin}^{2}(\lambda b-t)-\frac{\mathcal{Q}}{t}
\end{aligned}
$$

( $£$ being a constant).
Hence

$$
\begin{align*}
\int_{\lambda a}^{\lambda b} t C_{\nu}^{2}(t, \lambda b) d t & \geq \frac{1}{\lambda} \int_{\lambda a}^{\lambda b}\left\{A^{2} \operatorname{Sin}^{2}(\lambda b-t)-\frac{\mathcal{E}}{t}\right\} d t \\
& =\frac{A^{2}}{\lambda} \int_{\lambda,}^{\lambda b} \sin ^{2}(\lambda b-t) d t-\frac{\mathcal{L}}{\lambda}(\log \lambda b-\log \lambda a) \\
& \geq M \tag{3.12}
\end{align*}
$$

where $M(\neq 0)$ is a constant, and in view of (3.11) this implies the left hand in-equality of (3.9).

Lemma 4. - Naylor [4] has shown that for sufficiently large $m$ and $v>-1$

$$
\gamma_{m}=\frac{m \pi}{b-a}+\frac{\left(4 v^{2}-1\right)(b-a)}{8 m \pi a b}+0\left(m^{-3}\right)
$$

where $\gamma_{m}$ is the $m$-th positive root of the equation $C_{v}(a z, b z)=0$.
The result of the lemma is true when the order is small compared with the arguments $\gamma_{m} a, \gamma_{m} b$.
4. - Proof of Theorem 1. - Let $F(x) \equiv x^{\frac{1}{2}} f(x)$.

It is then evident that $F(x)$ also satisfies the conditions of the theorem.
By substituting $Y=\frac{z}{\sqrt{x}}$, [see, (2), p. 208] the transformed Bessel equation is

$$
\begin{equation*}
z^{\prime \prime}+\left(1-\frac{v^{2}-\frac{1}{4}}{x^{2}}\right) z=0 \tag{4.1}
\end{equation*}
$$

We write

$$
z=(\lambda x)^{\frac{1}{2}} C_{v}(\lambda x, \lambda b),
$$

and

$$
v^{2}-\frac{1}{4}=\mu
$$

thus, (4.1) is

$$
\begin{equation*}
\frac{d^{2} z}{d x^{2}}+\left(\lambda^{2}-\frac{\mu^{2}}{x^{2}}\right) z=0 \tag{4.2}
\end{equation*}
$$

It is easily seen that the function $\Phi \equiv x^{\frac{1}{2}} O_{y}(\lambda x, \lambda b)$ also satisfies the equation (4.2).

Consequently, we have

$$
\begin{equation*}
\frac{d^{2} \Phi}{d x^{2}}+\left(\lambda^{2}-\frac{\mu^{2}}{x^{2}}\right) \Phi=0 \tag{4.3}
\end{equation*}
$$

hence

$$
\begin{equation*}
\Phi=\frac{1}{\lambda_{2}}\left(\frac{\mu}{x_{2}} \Phi-\Phi^{\prime \prime}\right) . \tag{4.4}
\end{equation*}
$$

Consider now

$$
\begin{aligned}
I & =\int_{a}^{b} x f(x) C_{v}(\lambda x, \lambda b) d x \\
& =\int_{a}^{b} \sqrt{x} F(x) C_{v}(\lambda x, \lambda b) d x \\
& =\int_{a}^{b} F(x) \Phi d x \\
& =\frac{1}{\lambda^{2}} \int_{a}^{b} F(x)\left(\frac{\mu}{x^{2}} \Phi-\Phi^{\prime \prime}\right) d x
\end{aligned}
$$

Since

$$
\left(F^{\prime} \Phi-F \Phi^{\prime}\right)^{\prime}=F^{\prime \prime} \Phi-F \Phi^{\prime \prime}
$$

we see that

$$
\begin{align*}
I & =\frac{1}{\lambda^{2}} \int_{a}^{b}\left[\left\{F(x) \frac{\mu}{x^{2}}-F^{\prime \prime \prime}(x)\right\} \Phi+\left\{F^{\prime}(x) \Phi-F(x) \Phi^{\prime}\right\}^{\prime}\right] d x  \tag{4.6}\\
& =\frac{1}{\lambda^{2}} \int_{a}^{b}\left[F(x) \frac{\mu}{x_{2}}-F^{\prime \prime}(x)\right] \Phi d x+\left[F^{\prime}(x) \Phi-F(x) \Phi^{\prime}\right]_{x=a}^{x=b}
\end{align*}
$$

Here under our hypothesis
$F^{\prime}(b)$ is finite,

$$
F(b)=0, \quad F(\alpha)=F^{\prime}(a)=0 ;
$$

and

$$
\begin{gathered}
\Phi(b)=b^{\frac{1}{2}} C_{y}(\lambda b, \lambda b)=0 . \\
\Phi(a)=a^{\frac{1}{2}} C_{y}(\lambda a, \lambda b)=0 ;
\end{gathered}
$$

if $\lambda$ is a root of the equation $C_{,}(a z, b z)=0$.
Also

$$
\Phi^{\prime}(b)=-\frac{2}{\pi b^{\frac{1}{2}}}
$$

is finite and

$$
\Phi^{\prime}(a)=-\frac{2}{\pi \beta} a^{\frac{1}{2}}
$$

where

$$
\beta=\frac{J_{v}(\lambda a)}{J_{v}(\lambda b)} .
$$

Since, from the asymptotic expansion (3.7) of $J_{\nu}(x)$

$$
\begin{aligned}
\Phi^{\prime}(a) & =-\frac{2}{\pi} a^{\frac{1}{2}} \frac{J_{v}(\lambda b)}{J_{v}(\lambda a)} \\
& =-\frac{2}{\pi} a^{\frac{1}{2}}\left[\frac{(2 / \pi \lambda b)^{\frac{1}{2}}\left\{\cos \left(\lambda b-\frac{\nu \pi}{2}-\frac{1}{4} \pi\right)+\frac{\gamma_{\nu}(\lambda b)}{\lambda b}\right\}}{(2 / \pi \lambda a)^{\frac{2}{2}}\left\{\cos \left(\lambda a-\frac{\nu \pi}{2}-\frac{1}{4} \pi\right)+\frac{\gamma_{v}(\lambda a)}{\lambda a}\right\}}\right] \\
& =-\frac{2}{\pi} \frac{a}{\sqrt{b}} \frac{\cos \left(\lambda b-\frac{v \pi}{2}-\frac{1}{4} \pi\right)+\frac{\gamma_{v}(\lambda b)}{\lambda b}}{\cos \left(\lambda a-\frac{v \pi}{2}-\frac{1}{4} \pi\right)+\frac{\gamma_{v}(\lambda a)}{\lambda a}} .
\end{aligned}
$$

we see that $\Phi^{\prime}(a)$ is finite for sufficiently large values of $\lambda$. Collecting these results we obtain that

$$
\begin{aligned}
{\left[F^{\prime} \Phi-F \Phi^{\prime}\right]_{x=a}^{x=b} } & =\left[F^{\prime}(b) \Phi(b)-F^{\prime}(b) \Phi^{\prime}(b)\right]-\left[F^{\prime}(a) \Phi(a)-F(a) \Phi^{\prime}(a)\right] \\
& =0 .
\end{aligned}
$$

Therefore, (4.6) reduces to
(4.8)

$$
I=\frac{1}{\lambda^{2}} \int_{a}^{b}\left[F(x) \frac{\mu}{x^{2}}-F^{\prime \prime}(x)\right] \Phi d x
$$

Now

$$
\begin{equation*}
\left|\int_{a}^{b}\left[F(x) \frac{\mu}{\boldsymbol{x}^{2}}-F^{\prime \prime}(x)\right] \Phi d x\right| \leq \mathfrak{\Sigma}^{\prime} \int_{a}^{b}|\Phi(x)| d x . \quad\left(\mathfrak{L}^{\prime} \text { being a constant }\right) \tag{4.9}
\end{equation*}
$$

Applying Schwarz in-equality, we have

$$
\begin{align*}
\left(\int_{a}^{b}|\Phi(x)| d x\right)^{2} & \leq(b-a) \int_{a}^{b} \Phi^{2}(x) d x \\
& =(b-a) \int_{a}^{b} x C_{v}^{2}(\lambda x, \lambda b) d x \\
& =\frac{K}{\lambda^{2}} \cdot(b-a) \tag{4.10}
\end{align*}
$$

$$
\int_{a}^{b}|\Phi(x)| d x=0\left(\frac{1}{\lambda}\right)
$$

It follows that

$$
\begin{equation*}
|I|=0\left(\frac{1}{\lambda^{3}}\right) \tag{4.12}
\end{equation*}
$$

And, in view of (4.12) we at once obtain that

$$
\begin{equation*}
\left|\int_{a}^{b} x f(x) C_{\nu}\left(\gamma_{m} x, \quad \gamma_{m} b\right) d x\right|=0\left(\frac{1}{\gamma_{m}^{3}}\right) . \tag{4.13}
\end{equation*}
$$

Also

$$
\begin{equation*}
\int_{a}^{b} x C_{\nu}^{2}\left(\gamma_{m} \boldsymbol{x}, \gamma_{m} b\right) d x \geq \frac{M}{\gamma_{m}^{2}} \tag{4.14}
\end{equation*}
$$

by lemma 3 ,

Therefore, from (4.13) and (4.14)

$$
\begin{align*}
\left|a_{m}\right| & =\frac{\left|\int_{a}^{b} x f(x) C_{\nu}\left(\gamma_{m} x, \gamma_{m} b\right) d x\right|}{\left|\int_{a}^{b} x C_{\nu}^{2}\left(\gamma_{m} x, \gamma_{m} b\right) d x\right|} \\
& =0\left(\frac{1}{\gamma_{m}}\right), \quad \text { as } m \rightarrow \infty
\end{align*}
$$

almost everywhere.
This completes the proof of Theorem 1.
5. Proof of Theorem 2. - Since, by hypothesis, $f(x)$ satisfies all the conditions of Theorem 1, hence from (4.15) and Lemma 1 we have

$$
\begin{equation*}
\left|a_{m} C_{\chi}\left(\gamma_{m} x, \quad \gamma_{m} b\right)\right| \leq \frac{\mathfrak{S}^{\prime \prime}}{\gamma_{m}^{2}} \quad \text { where } \mathcal{L}^{\prime \prime} \text { is a constant. } \tag{5.1}
\end{equation*}
$$

Applying Lemma 2, we have

$$
\begin{equation*}
\gamma_{m+1}-\gamma_{m} \rightarrow \frac{\pi}{b-a}, \quad \text { as } m \rightarrow \infty \tag{5.2}
\end{equation*}
$$

which follows that for $m>n$ (where $n$ is some fixed number)

$$
\gamma_{m}>\gamma_{n}+(m-n)=m+h \quad(h \text { being a constant }) .
$$

If $m$ is large enough, then

$$
\gamma_{m} \geq \frac{1}{2} m
$$

or

$$
\frac{1}{\gamma_{m}} \leq \frac{2}{m}
$$

Consequently, for sufficiently large $m$

$$
\left|a_{m} O_{\chi}\left(\gamma_{m} x, \quad \gamma_{m} b\right)\right|=0\left(\frac{1}{m^{2}}\right) .
$$

Hence, the series (1.3) converges absolutely and uniformly for $0<a<$ $<x<b$,

This completes the proof of Theorem 2.
6. Proof of Theorem 3.-We set

$$
F(x)=\sqrt{\bar{x}} f(x)
$$

By hypothesis, it is easily seen that $F(x)$ satisfies the conditions of Theo. rem 1.

Now by similar process as in Theorem 1, we can also show that

$$
\begin{align*}
I & =\int_{a}^{b} x f(x) C_{\nu}(\lambda x, \lambda b) d x \\
& =\frac{1}{\lambda^{2}} \int_{a}^{b}\left(\frac{\mu}{x^{2}} F-F^{\prime \prime}\right) \Phi d x \\
& =\frac{1}{\lambda^{2}} \int_{a}^{b} F_{\lambda} \cdot \Phi d x \tag{6.1}
\end{align*}
$$

say, where

$$
\Phi=x^{\frac{1}{2}} C_{\nu}(\lambda x, \lambda b)
$$

and

$$
F_{1} \equiv \frac{\mu}{x^{2}} F-F^{\prime \prime}
$$

Under our hypothesis $F_{1}$ also satisfies the conditions of Theorem 1 which at once renders

$$
\begin{equation*}
I=\frac{1}{\lambda^{4}} \int_{a}^{b} F_{2} \cdot \Phi d x \tag{6.2}
\end{equation*}
$$

where

$$
F_{2} \equiv \frac{\mu}{x^{2}} F_{1}-F_{1}^{\prime \prime} .
$$

If $S>2, F_{2}$ again satisfies the conditions of Theorem 1 , and hence repeating the argument $S$ times, finally we obtain that

$$
\begin{equation*}
I=\frac{1}{\lambda^{2_{s}}} \int_{a}^{b} F_{S} \Phi d x \tag{6.3}
\end{equation*}
$$

where

$$
F_{s}=\frac{\mu}{x^{2}} F_{S-1}-F_{s-1}^{\prime \prime}
$$

Also, it follows that

$$
\begin{equation*}
\left|\int_{a}^{b} F_{S} \Phi d x\right| \leq \mathfrak{L}^{\prime \prime} \int_{a}^{b} \mid \Phi d x . \quad\left(\mathcal{L}^{\prime \prime \prime} \text { being a constant }\right) \tag{6.4}
\end{equation*}
$$

Applying Sohwarz inequality as ia Theorem 1, we have

$$
\begin{align*}
|I| & =\left|\int_{a}^{b} x f(x) C_{\nu}\left(\gamma_{m} x, \gamma_{m} b\right) d x\right| \\
& =0\left(\frac{1}{\gamma_{m}^{2 s+1}}\right) \tag{6.5}
\end{align*}
$$

Therefore, using lemma 3

$$
\begin{align*}
\left|a_{m}\right| & =\frac{\left|\int_{a} x f(x) C_{v}\left(\gamma_{m} x, \quad \gamma_{m} b\right) d x\right|}{\left|\int_{a}^{b} x C_{v}^{2}\left(\gamma_{m} x, \quad \gamma_{m} b\right) d x\right|} \\
& =0\left(\frac{1}{\gamma_{m}^{2 s}-1}\right), \quad \text { as } \quad m \rightarrow \infty \tag{6.6}
\end{align*}
$$

almost everywhere.
7. Proof of Theorem 4. - Since, by hypothesis, $f(x)$ satisfies all the conditions of Theorem 3, we see that applying Lemma 1 and in view* ${ }^{*}$ (6.6)

$$
\begin{equation*}
\left.\left|a_{m} O_{\nu}\left(\gamma_{m} x, \gamma_{m} b\right)\right| \leq \frac{\mathfrak{L}^{i v}}{\gamma_{m}^{2 s}} \quad \text { ( } \mathcal{L}^{i v} \text { being a costant }\right) \tag{7.1}
\end{equation*}
$$

Consequently, for sufficiently large $m$

$$
\begin{equation*}
\left|a_{m} C_{\nu}\left(\gamma_{m} x, \gamma_{m} b\right)\right|=0\left(\frac{1}{m^{2 s}}\right), \quad(s>1) \tag{7.2}
\end{equation*}
$$

## by Lemma 4.

Hence, the series (1.3) is absolutely and uniformly convergent for $0<a<$ $<x<b$.

This completes the proof of Theorem 4.
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