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ON THE ORDER-THEORETIC CANTOR THEOREM

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Dedicated to Fon-Che Liu

Abstract. In this report[1], we present an order-theoretic version of the Cantor theorem. This result, which is based on the interplay of the notions of partial order and of completeness, permits to give a unified and simplified account to a long list of results related to the Bishop–Phelps theorem. We survey briefly only its simplest applications and refer the reader to [10] for a complete presentation of the results.

1. Cantor Spaces

Let $(X, \preceq)$ be a partially ordered set. For any $z \in X$, denote the *terminal tail* $\{y \in X \mid z \preceq y\}$ by Tz; if $y \in Tz$, the set $Ty \subset Tz$ is called a *subtail* of Tz. Clearly an element y is maximal in $(X, \preceq)$ provided $\{y\} = Ty$. A map $F : X \to X$ is said to be *expanding* if $x \preceq F(x)$ for each $x \in X$. We observe that if $F : X \to X$ is expanding then: (i) any tail in $(X, \preceq)$ is invariant under F, (ii) any maximal element of $(X, \preceq)$ is a fixed point of F.

Let $(X; d, \preceq)$ be a metric space in which a partial order $\preceq$ is defined. We say that $(X; d, \preceq)$ *admits arbitrarily small tails* if for each tail Tz and any $\varepsilon > 0$ there exists a subtail $Ty \subset Tz$ with $\operatorname{diam}(Ty) \leq \varepsilon$.

Proposition 1. *Let $(X; d, \preceq)$ be a partially ordered complete metric space which admits arbitrarily small tails. Then for any $x_0 \in X$ there exists an*

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ascending and convergent sequence $x_0 \preceq x_1 \preceq \cdots \preceq x_n \preceq \cdots$ *such that* $\lim_{n\to\infty} x_n \in \cap_{n\in N}\overline{Tx_n}$.

Proof. The point x_0 being given, we first choose $x_1 \in Tx_0$ such that $\operatorname{diam}(Tx_1) \leq 1$. Assume that we have an ascending finite sequence $x_0 \preceq x_1 \preceq \cdots \preceq x_n$ such that $\operatorname{diam}(Tx_k) \leq 1/k$ for $0 < k \leq n$. Choose $x_{n+1} \in Tx_n$ such that $\operatorname{diam}(Tx_{n+1}) \leq 1/(n+1)$. By induction, we have an increasing sequence $\{x_n\}_{n\in N}$ with $\operatorname{diam}(Tx_n) \leq 1/n$ for each $n > 0$. The sequence of sets $\{\overline{Tx_n}\}_{n\in N}$ is clearly decreasing, so by the Cantor Theorem there exists a point $\hat{x} \in X$ such that $\{\hat{x}\} = \cap_{n\in N}\overline{Tx_n}$. Obviously, $\hat{x} = \lim_{n\to\infty} x_n$. ■

Proposition 2. *Let* $(X; d, \preceq)$ *be a partially ordered complete metric space which admits arbitrarily small tails and* $f : X \to X$ *an expanding continuous map. Then for each* $x_0 \in X$ *there exists a fixed point* $\hat{x} = f(\hat{x})$ *of* f *with* $\hat{x} \in \overline{Tx_0}$.

Proof. Given $x_0 \in X$, take a convergent ascending sequence $x_0 \preceq x_1 \preceq \cdots \preceq x_n \preceq \cdots$ with $\lim_{n\to\infty} x_n = \hat{x} \in \cap_{n\in N}\overline{Tx_n}$ and $\operatorname{diam}(Tx_n) \leq 1/n$ for each $n > 0$. We have $x_n \preceq f(x_n)$ for each $n \in N$ and therefore $f(x_n) \in Tx_n$. It follows that the sequence $\{f(x_n)\}_{n\in N}$ converges to $\hat{x}$ and by continuity that $\hat{x} = f(\hat{x})$. ■

We now come to our main concept.

Definition 1. We say that $(X; d, \preceq)$ is a *partially ordered Cantor space* (or simply a *Cantor space*), provided (i) tails are closed, (ii) $(X; d, \preceq)$ admits arbitrarily small tails and (iii) d is complete.

The main property of Cantor spaces is given in

Theorem 1 (Order-theoretic Cantor theorem). *Let* $X = (X; d, \preceq)$ *be a Cantor space. Then*:

- (i) *Any tail* Tx *in* X *is also a Cantor space.*
- (ii) X *contains at least one maximal element.*
- (iii) *Any tail* Tx *in* X *contains at least one maximal element* x^* *in* X.
- (iv) *If* $F : X \to X$ *is expanding, then each tail* Tx *contains a fixed point of* F.

Proof. (i) is obvious from the definitions involved; (iii) and (iv) follow clearly from (i) and (ii). It thus remains to verify that (ii) is true. The existence in X of a maximal element follows from Proposition 1. Indeed, let $x_0 \preceq x_1 \preceq \cdots \preceq x_n \preceq \cdots$ be an ascending sequence which converges to a point

$\hat{x}$ such that $\hat{x} \in \cap_{n\in N} Tx_n$. We claim that $\hat{x}$ is maximal in X: for, if $z \succeq \hat{x}$, then $z \succeq \hat{x} \succeq x_n$ for each $n \geq 0$, so $z \in \cap_{n\in N} Tx_n$ and therefore $z = \hat{x}$. This completes the proof. ∎

2. Bishop–Phelps Theorem

Following Bishop–Phelps, we introduce the following:

Definition 2. Let (X, d) be a metric space, $\varphi : X \to R$ be a real-valued function and λ a positive number. Following Bishop–Phelps, we define a relation $\preceq_{\varphi,\lambda}$ on X by

(BP) $$x \preceq_{\varphi,\lambda} y \quad \text{if and only if} \quad \varphi(y) + \lambda d(x, y) \leq \varphi(x).$$

This is in fact a partial ordering on X: clearly, $x \preceq_{\varphi,\lambda} x$ for each $x \in X$; if $x \preceq_{\varphi,\lambda} y$ and $y \preceq_{\varphi,\lambda} x$, then $2\lambda d(x, y) = \lambda d(x, y) + \lambda d(y, x) \leq \varphi(x) - \varphi(y) + \varphi(y) - \varphi(x) = 0$ and $x = y$; finally, if $x \preceq_{\varphi,\lambda} y$ and $y \preceq_{\varphi,\lambda} z$, then from the triangle inequality, we find $x \preceq_{\varphi,\lambda} z$. The space (X, d) together with this partial ordering is denoted by $X_{\varphi,\lambda}$. In the special case $\lambda = 1$, we shall write $\preceq_\varphi$ for $\preceq_{\varphi,\lambda}$ and X_φ for $X_{\varphi,1}$. Observe that if x, y are known to be related then the condition $\varphi(y) \leq \varphi(x)$ alone assures that both $x \preceq_{\varphi,\lambda} y$ and $\varphi(y) + \lambda d(x, y) \leq \varphi(x)$.

Proposition 3. *Let $\varphi : X \to R$ be a function and $\lambda > 0$. Then* (i) *if $\varphi : X \to R$ is bounded below, then $X_{\varphi,\lambda}$ admits arbitrarily small tails*; (ii) *if φ is lower semicontinuous, then each tail in $X_{\varphi,\lambda}$ is closed.*

Proof. Clearly, for the proof, we may assume that $\lambda = 1$. (i) Letting $x \in X$ and $\varepsilon > 0$ be given, we choose an element $y \in Tx$ so that

$$\varphi(y) - \inf_{t\in Tx} \varphi(t) \leq \varepsilon/2.$$

From $Ty \subset Tx$ we have, for any z_1, $z_2 \in Ty$,

$$d(z_1, z_2) \leq d(z_1, y) + d(z_2, y) \leq 2\varphi(y) - 2 \inf_{t\in Ty} \varphi(t) \leq 2\varphi(y) - 2 \inf_{t\in Tx} \varphi(t) \leq \varepsilon.$$

From this we get $\operatorname{diam}(Ty) \leq \varepsilon$ as asserted.

(ii) Indeed, given a tail $Tx = \{y \mid \varphi(y) + d(x, y) \leq \varphi(x)\}$, because the map $y \mapsto \varphi(y) + d(x, y)$ is lower semicontinuous, the conclusion follows. ∎

Theorem 1 leads immediately to the following fundamental result:

Theorem 2 (Bishop–Phelps). *Let (X, d) be complete, $\varphi : X \to R$ a l.s.c. function on X with a finite lower bound and λ a positive number. Then given any x_0 there exists at least one maximal element x^* in $X_{\varphi,\lambda}$ with $x^* \in Tx_0$. Precisely, for any x_0 there is at least one $x^* \in X$ such that*

$$\varphi(x^*) + \lambda d(x_0, x^*) \le \varphi(x_0)$$

and

$$\varphi(x^*) < \varphi(x) + \lambda d(x, x^*)$$

for any $x \neq x^*$.

Proof. Let $Tx_0 \subset X_{\varphi,\lambda}$ be the tail containing a given element x_0 in X; by Proposition 3, Tx_0 is a Cantor space, and thus our assertion is an immediate consequence of Theorem 1. ∎

3. Applications to Fixed Points

Let us say that a function $F : X \to X$ defined on a metric space (X, d) fulfils *Caristi's condition* with respect to a given function $\varphi : X \to R_+$ if

$$(*) \qquad d(x, Fx) \le \varphi(x) - \varphi(Fx) \quad \text{for each } x \in X.$$

If it is clear from the context which function φ is involved, we will simply say that Caristi's condition holds. Notice that if $(*)$ holds with respect to a function $\varphi : X \to R$ which is only assumed to be bounded below, then it obviously holds with respect to $\varphi - \inf_{x\in X} \varphi(x)$. So, there is no loss of generality if φ is assumed to be positive instead of bounded below. We establish now a version of the Caristi–Brøndsted theorem (cf. [3] and [5]).

Theorem 3. *Let (X, d) be complete and $\varphi : X \to R_+$ be a l.s.c. function on X. Then given any x_0, there exists an $x^* \in X$ such that $x_0 \preceq_\varphi x^*$ and x^* is a common fixed point for the family of functions (not necessarily continuous) $F : X \to X$ for which Caristi's condition holds.*

Proof. Consider the Cantor space X_φ and note that the estimate $(*)$ means that $F : X_\varphi \to X_\varphi$ is expanding with respect to the partial order $\preceq_\varphi$. Now, by the introductory remarks on expanding maps, and because a tail $Tx_0 \subset X_\varphi$ is a Cantor space, the conclusion follows. ∎

Theorem 4 (Brøndsted). *Let (X, d) be complete and $\varphi : X \to R_+$ an arbitrary function. Then given any $x_0 \in X$, there exists an ascending convergent sequence $x_0 \preceq_\varphi x_1 \preceq_\varphi \dots \preceq x_n \preceq_\varphi \dots$ such that $\lim_{n\to\infty} x_n$ is a* *common fixed point for the family of all continuous functions* $F : X \to X$ *for which Caristi's condition holds.*

Proof. We have seen that X_φ admits arbitrarily small tails and that the estimate $(*)$ means that $F : X_\varphi \to X_\varphi$ is expanding with respect to the partial order $\preceq_\varphi$. Now, the conclusion follows from Propositions 1 and 2. ∎

The order-theoretic Cantor theorem is equally useful for dealing with multivalued maps. Following W. Takahashi [15], we give the multivalued extension of Caristi's theorem and then establish Nadler's fixed point theorem for set-valued contractions.

Theorem 5 (W. Takahashi [15]). *Let* (X, d) *be complete and* $\varphi : X \to R$ *be a l.s.c. function bounded below on* X. *Let* $\mathcal{F} : X \to X$ *be a multivalued map such that for each* $x \in X$ *there is* $y \in \mathcal{F}x$ *satisfying*

$$d(x, y) \leq \varphi(x) - \varphi(y).$$

Then given any x_0 *there exists at least one fixed point* x^* *of* $\mathcal{F}$ *with* $x_0 \preceq_\varphi x^*$.

Proof. For each $x \in X$, choose $Fx \in \mathcal{F}x$ such that $d(x, y) \leq \varphi(x) - \varphi(Fx)$. Obviously, F is a single-valued selector of $\mathcal{F}$. By Caristi's theorem, there is a point x_0 such that $x_0 \preceq_\varphi x^*$ and $Fx_0 = x_0$. Obviously, $x_0 \in \mathcal{F}x_0$. ∎

Given a metric space (X, d), let us denote by $\mathcal{CB}(X)$ the family of closed nonempty bounded subsets of X. The Hausdorff metric on $\mathcal{CB}(X)$ is denoted by d_H. A map $\mathcal{F} : X \to \mathcal{CB}(X)$ is *α-contractive*, where $0 \leq \alpha < 1$, if

$$d_H(\mathcal{F}x, \mathcal{F}y) \leq \alpha d(x, y) \quad \text{for all } x, y \in X.$$

Theorem 6 (Nadler [12]). *If* (X, d) *is a complete metric space, then every α-contractive map* $\mathcal{F} : X \to \mathcal{CB}(X)$ *has a fixed point.*

Proof. First, notice that for any $x \in X$ and any $y \in \mathcal{F}x$, we have $d(y, \mathcal{F}y) \leq \alpha d(x, y)$. Indeed, for any $\delta > 0$ we have

$$\mathcal{F}x \subset \bigcup_{z \in \mathcal{F}y} B(z, \alpha d(x, y) + \delta).$$

Therefore, if $y \in \mathcal{F}x$ there is a point $z \in \mathcal{F}y$ such that $d(y, z) < \alpha d(x, y) + \delta$. Taking the infimum over $z \in \mathcal{F}y$ yields $d(y, \mathcal{F}y) \leq \alpha d(x, y) + \delta$. Since $\delta > 0$ was arbitrary, the conclusion follows.

Now, fix $\epsilon > 0$. For any $x \in X$ there exists a point $y_\epsilon(x) \in \mathcal{F}x$ such that

$$d(x, y_\epsilon(x)) \leq (1+\epsilon) d(x, \mathcal{F}x).$$

From this we get

$$\left[\left(\frac{1}{1+\epsilon}\right) - \alpha\right] d(x, y_\epsilon(x)) \leq d(x, \mathcal{F}x) - \alpha d(x, y_\epsilon(x)) \leq d(x, \mathcal{F}x) - d(y_\epsilon(x), \mathcal{F}y_\epsilon(x)).$$

Let

$$\varphi_\epsilon(x) = \left[\left(\frac{1}{1+\epsilon}\right) - \alpha\right]^{-1} d(x, \mathcal{F}x).$$

If ϵ is chosen such that $(1+\epsilon)^{-1} > \alpha$, then φ_ϵ is continuous and bounded below. Furthermore, we have just shown that for any $x \in X$,

$$x \preceq_{\varphi_\epsilon} y_\epsilon(x) \quad \text{and} \quad y_\epsilon(x) \in \mathcal{F}x.$$

Let x^* be a maximal element for the partial order $\preceq_{\varphi_\epsilon}$. From $x^* \preceq_{\varphi_\epsilon} y_\epsilon(x^*)$ we get $x^* = y_\epsilon(x^*)$, and therefore $x^* \in \mathcal{F}x^*$. ∎

4. Applications to Geometry of Banach Spaces

Let $B = B(z, r)$ be a closed ball in a Banach space. For any $x \notin B$, the convex hull of x and B is called a *drop* and is denoted by $D(x, B)$; it is clear that if $y \in D(x, B)$, then $D(y, B) \subset D(x, B)$, and, if $z = 0$, that $\|y\| \leq \|x\|$.

Theorem 7 (Daneš [7]). *Let A be a closed subset of a Banach space E, let $z \in E - A$, and let $B = B(z, r)$ be a closed ball of radius $0 < r < d(z, A) = R$. Let $F : A \to A$ be any map such that $F(a) \in A \cap D(a, B)$ for each $a \in A$. Then for each $x \in A$, the map F has at least one fixed point in $A \cap D(x, B)$.*

Proof. We can assume $z = 0$. Let $\|x\| = \varrho \geq R$ and let $X = A \cap D(x, B)$; clearly, F maps X into itself and we shall develop an expression for $\|x - F(x)\|$ on X.

Given $y \in X$, there is a $b \in B$ with $F(y) = tb + (1-t)y$; since $\|F(y)\| \leq t\|b\| + (1-t)\|y\|$, we have $t[\|y\| - \|b\|] \leq \|y\| - \|F(y)\|$ so because $\|y\| - \|b\| \geq R - r$, we find

$$t \leq \frac{\|y\| - \|Fy\|}{R - r}.$$

Thus,

$$\|y - F(y)\| \leq t\|y - b\| \leq t[\|y\| + \|b\|] \leq t[\varrho + r] \leq \frac{\varrho + r}{R - r}[\|y\| - \|F(y)\|].$$

Therefore, applying the Theorem of Caristi with

$$\varphi(x) = \frac{\varrho + r}{R - r}\|x\|,$$

the result follows. ∎

As a consequence, we obtain

Theorem 8 (Supporting Drops Theorem). *Let A be a closed set in a Banach space E, and $z \in E - A$ a point with $d(z, A) = R > O$. Then for any $r < R < \varrho$ there is an $x_0 \in \partial A$ with*

$$\|z - x_0\| \leq \varrho \quad and \quad A \cap D(x_0, B(z, r)) = \{x_0\}.$$

Proof. Let $\tilde{A} = A \cap B(z, \varrho)$. It is a closed and nonempty subset of E. For each point $x \in \tilde{A}$, choose a point $F(x) \in \tilde{A} \cap D(x, B)$ such that $F(x) \neq x$ if $A \cap D(x, B) \neq \{x\}$. One can easily see that a fixed point x_0 of F occurs at points of ∂A and that $\tilde{A} \cap D(x, B) = A \cap D(x, B)$. ∎

5. Applications to Critical Point Theory

Let $\varphi : X \to R$ be a real-valued function[2] on a metric space X with a finite $\eta = \inf\{\varphi(x) \mid x \in X\}$. Recall that a *minimizer* (resp. a *strict minimizer*) of φ is an element $x_0 \in X$ with $\varphi(x_0) = \eta$ (resp. such that the relation $\varphi(z) \leq \varphi(x_0)$ implies $z = x_0$). A sequence $\{x_n\}$ in X for which $\varphi(x_n) \to \eta$ is called a *minimizing sequence* for φ.

Theorem 9 (Ekeland [9]). *Let (X, d) be complete and let $\varphi : X \to R$ be a lower semicontinuous function with finite lower bound η. Let $\{x_n\}$ be a minimizing sequence for ϕ and $\lambda_n = (\varphi(x_n) - \eta)^{1/2}$. Then there exists a minimizing sequence $\{y_n\}$ for φ such that for any natural n we have:*

(i) *$\varphi(y_n) \leq \varphi(x_n)$ and $d(x_n, y_n) \leq \lambda_n$,*

(ii) *y_n is a strict minimizer of the function $\varphi_n : X \to R$ given by*

$$\varphi_n(z) = \varphi(z) + \lambda_n d(z, y_n) \quad for\ z \in X,$$

(iii) *$\varphi(y_n) = \varphi_n(y_n) \leq \varphi(z) + \lambda_n d(z, y_n)$ for $z \in X$.*

[2] For simplicity, we avoid considering the extended real functions $\varphi : X \to R \cup \{\infty\}$.

Proof. We first describe the construction of $\{y_n\}$. For a given natural n, consider the space X_{φ,λ_n}, where $\lambda_n = (\varphi(x_n) - \eta)^{1/2}$. By the Bishop–Phelps theorem applied in X_{φ,λ_n} for the point x_n, there exists an element y_n in X_{φ,λ_n} such that (a) $x_n \preceq_{\varphi,\lambda_n} y_n$ and (b) y_n is maximal in X_{φ,λ_n}. We now show that y_n and the function φ_n defined in (ii) have the properties (i)–(iii).

Indeed, the relation $x_n \preceq_{\varphi,\lambda_n} y_n$ in X_{φ,λ_n} translates into the estimate

$$\lambda_n d(x_n, y_n) \leq \varphi(x_n) - \varphi(y_n),$$

and gives

$$d(x_n, y_n) \leq \frac{1}{\lambda_n}(\varphi(x_n) - \varphi(y_n)) \leq \frac{1}{\lambda_n}(\eta + \lambda_n^2 - \eta) = \lambda_n;$$

thus (i) is satisfied.

To establish (ii), suppose that $\varphi_n(z) \leq \varphi_n(y_n)$ for some z in X; we then have

$$\varphi_n(z) = \varphi(z) + \lambda_n d(z, y_n) \leq \varphi(y_n) = \varphi_n(y_n),$$

which (by the definition of the order in X_{φ,λ_n}) gives $y_n \preceq_{\varphi,\lambda_n} z$. Since y_n is maximal in X_{φ,λ_n}, the last relation implies $y_n = z$, showing that y_n is a strict minimizer of φ_n, as asserted.

(iii) is an obvious consequence of (ii).

Thus we have constructed a minimizing sequence $\{y_n\}$ satisfying (i)–(iii). ∎

Corollary 1. *Let E be a Banach space, $\varphi : E \to R$ be a differentiable function on E with finite lower bound, and $\{x_n\}$ be a minimizing sequence for φ. Then there exists a minimizing sequence $\{y_n\}$ in E for φ such that $\varphi(y_n) \leq \varphi(x_n)$ for each n and $D\varphi(y_n) \to 0$ in E^*.*

Proof. By Theorem 9, there exists a minimizing sequence $\{y_n\}$ in E for φ such that for all n, $\varphi(y_n) \leq \varphi(x_n)$ and

$$(*) \qquad \varphi(y_n) \leq \varphi(z) + \lambda_n \|z - y_n\| \quad \text{for all } z \in E.$$

For a given n, letting $z = y_n + v$ we obtain from $(*)$ the estimate

$$\begin{aligned}\varphi(y_n) &\leq \varphi(y_n + v) + \lambda_n \|(y_n + v) - y_n)\| \\ &= \varphi(y_n + v) + \lambda_n \|v\| \quad \text{for all } v \in E,\end{aligned}$$

and consequently

$$\|D\varphi(y_n)\|_{E^*} = \lim_{\varrho \to 0} \sup_{\substack{\|v\| \leq \varrho \\ v \neq 0}} \frac{\varphi(y_n) - \varphi(y_n + v)}{\|v\|} \leq \lambda_n.$$

Thus, $\|D\varphi(y_n)\|_{E^*} \leq \lambda_n$ for each n and, because $\lambda_n \to 0$, our assertion follows. ∎

6. Remarks

(1) The Bishop–Phelps technique presented in Sections 2–5 originated in and evolved from the work of the above authors in the theory of support functionals in Banach spaces. Let E be a Banach space and $X \subset E$. A point $x_0 \in X$ is a *support point of* X if for some $f \in E^*$, called a *support functional* of X, we have $f(x_0) = \sup\{f(x) \mid x \in X\}$. The following theorem was established by Bishop–Phelps [1]: *Let* C be a *closed convex subset of* E. *Then* (a) *the support points* of C *are dense in the boundary* ∂C of C, *and* (b) *the support functionals of* C *are norm dense in the set* $\{f \in E^* \mid \sup_C f < \infty\}$.

In connection with the Bishop–Phelps theorem, we make the following comments:

(i) If $\mathrm{Int}(C) \neq \emptyset$, then every $x \in C$ is a support point of C; this follows at once from the Mazur separation theorem.

(ii) If C is the closed unit ball in E, then the set $\{f \in E^* \mid f(x) = \|f\|$ for some $x \in \partial C\}$ is norm dense in E^*; this is a special case of the Bishop–Phelps theorem.

(iii) If C is the closed unit ball in E, then [each $f \in E^*$ is a support functional of C] $\Leftrightarrow$ [the space E is reflexive] (theorem of James [11]).

(iv) Let $\varphi : E \to R$ be convex and lower semicontinuous. Let

$$\partial\varphi(x) = \{f \in E^* \mid f(y-x) \leq \varphi(y) - \varphi(x) \quad \text{for} \quad y \in E\}$$

be the subdifferential of φ at $x \in E$. Because the elements of $\partial\varphi(x)$ can be identified with support functionals of the closed convex epigraph $\mathrm{epi}(\varphi) \subset E \times R$ of φ at $(x, \varphi(x))$, the Bishop–Phelps theorem leads to the following theorem: *The set* $\{x \in E \mid \partial\varphi(x) \neq 0\}$ *is dense in* E. This important result (and, in fact, its "extended" version valid for functions φ possibly equal to ∞), is due to Brøndsted–Rockafellar [4].

(2) The order-theoretic Cantor theorem implies the usual Cantor theorem. Indeed, let $\{F_n\}_{n\in N}$ be a decreasing sequence of nonempty closed sets in a complete metric space (X, d) (we can always assume $F_0 = X$) such that $\inf_{n\in N} \operatorname{diam} F_n = 0$. Let $x \preceq y$ if $x = y$ or there exists $n \in N$ such that $y \in F_n$ and $x \notin F_n$. Then $\preceq$ is compatible with the metric since $Tx = \{x\}$ if $x \in \cap_{n\in N} F_n$ and $Tx = \{x\} \cup F_{n(x)+1}$ otherwise, where $n(x) = \max\{n \in N \mid x \in F_n\}$. Clearly, any maximal element belongs to $\cap_{n\in N} F_n$.

(3) The theorem of Daneš can be proved by replacing the norm by a function $\varphi : E \to R \cup \{\infty\}$ which is l.s.c., coercive, bounded below and convex.

(4) The formulation of the Bishop–Phelps theorem is taken from [8]; the result appeared in a different form in the survey by Phelps [14] written in 1971. For various interrelations between results related to the Bishop–Phelps theorem, the reader is referred to [6] and [13].

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