On the order three Brauer classes for cubic surfaces

Andreas-Stephan Elsenhans* and Jörg Jahnel‡

Abstract

We describe a method to compute the Brauer-Manin obstruction for smooth cubic surfaces over $\mathbb{Q}$ such that $\mathrm{Br}(S)/\mathrm{Br}(\mathbb{Q})$ is a 3-group. Our approach is to associate a Brauer class with every ordered triplet of Galois invariant pairs of Steiner trihedra. We show that all order three Brauer classes may be obtained in this way. To show the effect of the obstruction, we give explicit examples.

1 Introduction

1.1. —— For cubic surfaces, weak approximation and even the Hasse principle are not always fulfilled. The first example of a cubic surface violating the Hasse principle was constructed by Sir Peter Swinnerton-Dyer [SD1] in 1962. A series of examples generalizing that of Swinnerton-Dyer is due to L. J. Mordell [Mo]. An example of a different sort was given by J. W. S. Cassels and M. J. T. Guy [CG].

Around 1970, Yu. I. Manin [Ma] presented a way to explain these examples in a unified manner. This is what today is called the Brauer-Manin obstruction. Manin's idea is that a non-trivial Brauer class may be responsible for the failure of weak approximation.

1.2. —— Let S be a projective variety over $\mathbb{Q}$ and π be the structural morphism. Then, for the Brauer-Manin obstruction, only the factor group $\mathrm{Br}(S)/\pi^*\mathrm{Br}(\mathbb{Q})$ of the Grothendieck-Brauer group is relevant. When $\mathrm{Br}(S_{\overline{\mathbb{Q}}}) = 0$, that group is isomorphic to the Galois cohomology group $H^1(\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}), \mathrm{Pic}(S_{\overline{\mathbb{Q}}}))$.

*Mathematisches Institut, Universität Bayreuth, Univ'straße 30, D-95440 Bayreuth, Germany, Stephan.Elsenhans@uni-bayreuth.de, Website: http://www.staff.uni-bayreuth.de/~btm216

‡Département Mathematik, Universität Siegen, Walter-Flex-Str. 3, D-57068 Siegen, Germany, jahnel@mathematik.uni-siegen.de, Website: http://www.uni-math.gwdg.de/jahnel

*The first author was supported in part by the Deutsche Forschungsgemeinschaft (DFG) through a funded research project.

Key words and phrases. Cubic surface, Steiner trihedron, Triplet, Twisted cubic curve, Weak approximation, Explicit Brauer-Manin obstruction

2000 Mathematics Subject Classification. Primary 11D25; Secondary 11D85, 14J26, 14J20

For S a smooth cubic surface, a theorem of Sir Peter Swinnerton-Dyer [SD3] states that, for $H^1(\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}), \mathrm{Pic}(S_{\overline{\mathbb{Q}}}))$, there are only five possibilities. It may be isomorphic to 0, $\mathbb{Z}/2\mathbb{Z}$, $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, $\mathbb{Z}/3\mathbb{Z}$, or $\mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$.

1.3. — The effect of the Brauer-Manin obstruction has been studied by several authors. For instance, the examples of Mordell and Cassels-Guy were explained by the Brauer-Manin obstruction in [Ma]. For diagonal cubic surfaces, the computations were carried out by J.-L. Colliot-Thélène and his coworkers in [CKS]. More recently, we treated the situations that $H^1(\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}), \mathrm{Pic}(S_{\overline{\mathbb{Q}}})) \cong \mathbb{Z}/2\mathbb{Z}$ and $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ [EJ2]. It seems, however, that for only a few of the cases when

$$H^1(\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}), \mathrm{Pic}(S_{\overline{\mathbb{Q}}})) \cong \mathbb{Z}/3\mathbb{Z}\,,$$

computations have been done up to now. The goal of the present article is to fill this gap.

1.4. Remark (On the combinatorics of the 27 lines)**.** — On a smooth cubic surface S over an algebraically closed field, there are exactly 27 lines. The configuration of these was intensively studied by the geometers of the 19th century. We will be able to recall only the most important facts from this fascinating part of classical geometry. The reader may find more information in [Do, Chapter 9].

Up to permutation, the intersection matrix is always the same, independently of the surface considered. Further, there are exactly 45 planes cutting out three lines of the surface. These are called the *tritangent planes.*

The combinatorics of the tritangent planes is of interest in itself. For us, special sets of three tritangent planes, which are called *Steiner trihedra,* will be important. We will recall them in section 3.

On the other hand, there are 72 so-called *sixers,* sets of six lines that are mutually skew. Further, the sixers come in pairs. Every sixer has a partner such that each of its lines meets exacly five of the other. Together with its partner, a sixer forms a *double-six.* There are 36 double-sixes on a smooth cubic surface.

Let us finally mention one more recent result. The automorphism group of the configuration of the 27 lines is isomorphic to the Weyl group $W(E_6)$ [Ma, Theorem 23.9].

1.5. — If S is defined over $\mathbb{Q}$ then $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ operates on the 27 lines. The operation always takes place via a subgroup of $W(E_6)$.

The starting point of our investigations is now a somewhat surprising observation. It turns out that $H^1(\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}), \mathrm{Pic}(S_{\overline{\mathbb{Q}}}))$ is of order three or nine only in cases when, on S, there are three Galois invariant pairs of Steiner trihedra that are complementary in the sense that together they contain all the 27 lines. Three pairs of this sort are classically said to form a triplet.

This observation reduces the possibilities for the action of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on the 27 lines. Among the 350 conjugacy classes of subgroups of $W(E_6)$, exactly 140 stabilize a pair of Steiner trihedra, while only 54 stabilize a triplet. Exactly 17 of these conjugacy classes lead to a $H^1(\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}), \mathrm{Pic}(S_{\overline{\mathbb{Q}}}))$ of order three or nine.

1.6. Remark (Overview over the 350 conjugacy classes**).** — It is known [EJ2] that $H^1(\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}), \mathrm{Pic}(S_{\overline{\mathbb{Q}}}))$ is of order two or four only when there is a Galois-invariant double-six. There are the following cases.

i) Exactly 175 of the 350 classes neither stabilize a double-six nor a triplet. Then, clearly, $H^1(\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}), \mathrm{Pic}(S_{\overline{\mathbb{Q}}})) = 0$.

ii) Among the other conjugacy classes, 158 stabilize a double-six and 54 stabilize a triplet, whereas 37 do both. For the latter, one has $H^1(\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}), \mathrm{Pic}(S_{\overline{\mathbb{Q}}})) = 0$, too.

iii) For the remaining 17 classes stabilizing a triplet, $H^1(\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}), \mathrm{Pic}(S_{\overline{\mathbb{Q}}}))$ is a non-trivial 3-group.

iv) Among the 121 remaining conjugacy classes that stabilize a double-six, 37 even stabilize a sixer. Those may be constructed by blowing up a Galois-invariant set in $\mathbf{P}^2$ and, thus, certainly fulfill weak approximation.

There are eight further conjugacy classes such that $H^1(\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}), \mathrm{Pic}(S_{\overline{\mathbb{Q}}})) = 0$, each leading to an orbit structure of $[2, 5, 10, 10]$ or $[2, 5, 5, 5, 10]$. For the remaining 76 classes, $H^1(\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}), \mathrm{Pic}(S_{\overline{\mathbb{Q}}}))$ is a non-trivial 2-group.

1.7. — In this article, we will compute a non-trivial Brauer class for each of the 16 cases such that $H^1(\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}), \mathrm{Pic}(S_{\overline{\mathbb{Q}}})) \cong \mathbb{Z}/3\mathbb{Z}$. We will start with a "model case". This is the maximal subgroup $U_t \subset W(E_6)$ stabilizing a triplet. We will show that, for every subgroup $G \subset W(E_6)$ leading to $H^1(G, \mathrm{Pic}(S_{\overline{\mathbb{Q}}}))$ of order three, the restriction map from $H^1(U_t, \mathrm{Pic}(S_{\overline{\mathbb{Q}}}))$ is bijective.

1.8. — We then present a method to explicitly compute the local evaluation maps induced by an element $c \in \mathrm{Br}(S)/\pi^*\mathrm{Br}(\mathbb{Q})$ of order three. An advantage of our approach is that it requires, at most, a quadratic extension of the base field.

To evaluate, we test whether the value of a certain rational function is a norm from a cyclic cubic extension. The construction of the rational function involves beautiful classical geometry, to wit the determination of a twisted cubic curve on S.

We show the effect of the Brauer-Manin obstruction in explicit examples. It turns out that, unlike the situation described in [CKS], where a Brauer class of order three typically excludes two thirds of the adelic points, various fractions are possible.

1.9. — Up to conjugation, there is only one subgroup $U_{tt} \subset W(E_6)$ such that $H^1(U_{tt}, \mathrm{Pic}(S_{\overline{\mathbb{Q}}})) \cong \mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$. This group of order three was known to Yu. I. Manin, in 1969 already.

In a final section, which is purely theoretic in nature, we show that U_{tt} actually fixes four triplets. The corresponding four restriction maps are injections $\mathbb{Z}/3\mathbb{Z} \hookrightarrow \mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$. Together, each of the eight non-zero elements of the right hand side is met exactly once.

2 The Brauer-Manin obstruction – Generalities

2.1. —— For cubic surfaces, all known counterexamples to the Hasse principle or weak approximation are explained by the following observation.

2.2. Definition. —— Let X be a projective variety over $\mathbb{Q}$ and $\mathrm{Br}(X)$ its Grothendieck-Brauer group. Then, we will call

$$\mathrm{ev}_\nu\colon \mathrm{Br}(X) \times X(\mathbb{Q}_\nu) \longrightarrow \mathbb{Q}/\mathbb{Z}\,, \quad (\alpha, \xi) \mapsto \mathrm{inv}_\nu(\alpha|_\xi)$$

the *local evaluation map*. Here, $\mathrm{inv}_\nu\colon \mathrm{Br}(\mathbb{Q}_\nu) \to \mathbb{Q}/\mathbb{Z}$ (and $\mathrm{inv}_\infty\colon \mathrm{Br}(\mathbb{R}) \to \frac{1}{2}\mathbb{Z}/\mathbb{Z}$) denote the canonical isomorphisms.

2.3. Observation (Manin)**.** —— *Let $\pi\colon X \to \mathrm{Spec}\,\mathbb{Q}$ be a projective variety over $\mathbb{Q}$. Choose an element $\alpha \in \mathrm{Br}(X)$. Then, every $\mathbb{Q}$-rational point $x \in X(\mathbb{Q})$ gives rise to an adelic point $(x_\nu)_\nu \in X(\mathbf{A}_\mathbb{Q})$ satisfying the condition*

$$\sum_{\nu \in \mathrm{Val}(\mathbb{Q})} \mathrm{ev}_\nu(\alpha, x_\nu) = 0\,.$$

2.4. Remarks. —— i) It is obvious that altering $\alpha \in \mathrm{Br}(X)$ by some Brauer class $\pi^*\rho$ for $\rho \in \mathrm{Br}(\mathbb{Q})$ does not change the obstruction defined by α. Consequently, it is only the factor group $\mathrm{Br}(X)/\pi^*\mathrm{Br}(\mathbb{Q})$ that is relevant for the Brauer-Manin obstruction.

ii) The local evaluation map $\mathrm{ev}_\nu\colon \mathrm{Br}(X) \times X(\mathbb{Q}_\nu) \to \mathbb{Q}/\mathbb{Z}$ is continuous in the second variable.

iii) Further, for every projective variety X over $\mathbb{Q}$ and every $\alpha \in \mathrm{Br}(X)$, there exists a finite set $S \subset \mathrm{Val}(\mathbb{Q})$ such that $\mathrm{ev}(\alpha, \xi) = 0$ for every $\nu \notin S$ and $\xi \in X(\mathbb{Q}_\nu)$.

These facts imply that the Brauer-Manin obstruction, if present, is an obstruction to the principle of weak approximation.

2.5. Lemma. —— *Let $\pi\colon S \to \mathrm{Spec}\,\mathbb{Q}$ be a non-singular cubic surface. Then, there is a canonical isomorphism*

$$\delta\colon H^1(\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}), \mathrm{Pic}(S_{\overline{\mathbb{Q}}})) \longrightarrow \mathrm{Br}(S)/\pi^*\mathrm{Br}(\mathbb{Q})$$

making the diagram

$$\begin{array}{ccc}
H^1(\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}), \mathrm{Pic}(S_{\overline{\mathbb{Q}}})) & \xrightarrow{\ \delta\ } & \mathrm{Br}(S)/\pi^*\mathrm{Br}(\mathbb{Q}) \\
\downarrow{\scriptstyle d} & & \\
H^2(\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}), \overline{\mathbb{Q}}(S)^*/\overline{\mathbb{Q}}^*) & & \downarrow{\scriptstyle \mathrm{res}} \\
\| & & \\
H^2(\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}), \overline{\mathbb{Q}}(S)^*)/\pi^*\mathrm{Br}(\mathbb{Q}) & \xrightarrow{\ \mathrm{inf}\ } & \mathrm{Br}(\mathbb{Q}(S))/\pi^*\mathrm{Br}(\mathbb{Q})
\end{array}$$

commute. Here, d is induced by the short exact sequence

$$0 \to \overline{\mathbb{Q}}(S)^*/\overline{\mathbb{Q}}^* \to \mathrm{Div}(S_{\overline{\mathbb{Q}}}) \to \mathrm{Pic}(S_{\overline{\mathbb{Q}}}) \to 0$$

and the other morphisms are the canonical ones.

Proof. The equality at the lower left corner comes from the fact [Ta, section 11.4] that $H^3(\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}), \overline{\mathbb{Q}}^*) = 0$. The main assertion is [Ma, Lemma 43.1.1]. $\square$

2.6. Remark. —— The group $H^1(\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}), \mathrm{Pic}(S_{\overline{\mathbb{Q}}}))$ is always finite. Hence, by Remark 2.4.iii), we know that only finitely many primes are relevant for the Brauer-Manin obstruction.

3 Steiner trihedra and the Brauer group

Steiner trihedra. Triplets.

3.1. Definition. —— i) Three tritangent planes such that no two of them have one of the 27 lines in common are said to be a *trihedron.*

ii) For a trihedron $\{E_1, E_2, E_3\}$, a plane E is called a *conjugate plane* if each of the lines $E_1 \cap E$, $E_2 \cap E$, and $E_3 \cap E$ is contained in S.

iii) A trihedron may have either no, exactly one, or exactly three conjugate planes. Correspondingly, a trihedron is said to be of the *first kind, second kind,* or *third kind.* Trihedra of the third kind are also called *Steiner trihedra.*

3.2. Remarks. —— i) Steiner trihedra come in pairs. Actually, the three planes conjugate to a Steiner trihedron form another Steiner trihedron.

ii) All in all, there are 120 pairs of Steiner trihedra on a non-singular cubic surface. The automorphism group $W(E_6)$ transitively operates on them.

The subgroup of $W(E_6)$ stabilizing one pair of Steiner trihedra is isomorphic to $[(S_3 \times S_3) \rtimes \mathbb{Z}/2\mathbb{Z}] \times S_3$ of order 432. This group operates on the pairs of Steiner trihedra such that the orbits have lengths 1, 2, 27, 36, and 54. The orbit of size two is of particular interest.

3.3. Fact-Definition. —— For every pair of Steiner trihedra, there are exactly two other pairs having no line in common with the pair given. An ordered triple of pairs of Steiner trihedra obtained in this way will be called a *triplet*.

3.4. Remarks. —— i) Together, a triplet contains all the 27 lines.

ii) There are 240 triplets on a non-singular cubic surface, corresponding to the 40 *decompositions* of the 27 lines into three pairs of Steiner trihedra. Clearly, the operation of $W(E_6)$ is transitive on triplets and, thus, on decompositions.

The largest subgroup U_t of $W(E_6)$, stabilizing a triplet, is of order 216. It is a subgroup of index two in $[(S_3 \times S_3) \rtimes \mathbb{Z}/2\mathbb{Z}] \times S_3$. As an abstract group, $U_t \cong S_3 \times S_3 \times S_3$.

3.5. Notation. —— Let $l_1, \ldots, l_9$ be the nine lines defined by a Steiner trihedron. Then, we will denote the corresponding pair of Steiner trihedra by a rectangular symbol of the form

$$\begin{bmatrix} l_1 & l_2 & l_3 \\ l_4 & l_5 & l_6 \\ l_7 & l_8 & l_9 \end{bmatrix}.$$

The planes of the trihedra contain the lines noticed in the rows and columns.

3.6. —— To describe Steiner trihedra explicitly, one works best in the blown-up model. In Schläfli's notation [Sch, p. 116] (cf. Hartshorne's notation in [Ha, Theorem V.4.9]), there are 20 pairs of type I,

$$\begin{bmatrix} a_i & b_j & c_{ij} \\ b_k & c_{jk} & a_j \\ c_{ik} & a_k & b_i \end{bmatrix},$$

10 pairs of type II,

$$\begin{bmatrix} c_{il} & c_{jm} & c_{kn} \\ c_{jn} & c_{kl} & c_{im} \\ c_{km} & c_{in} & c_{jl} \end{bmatrix},$$

and 90 pairs of type III,

$$\begin{bmatrix} a_i & b_l & c_{il} \\ b_k & a_j & c_{jk} \\ c_{ik} & c_{jl} & c_{mn} \end{bmatrix}.$$

3.7. —— Consequently, there are two types of decompositions of the 27 lines, 10 consist of two pairs of type I and one pair of type II,

$$\left\{ \begin{bmatrix} a_i & b_j & c_{ij} \\ b_k & c_{jk} & a_j \\ c_{ik} & a_k & b_i \end{bmatrix}, \begin{bmatrix} a_l & b_m & c_{lm} \\ b_n & c_{mn} & a_m \\ c_{ln} & a_n & b_l \end{bmatrix}, \begin{bmatrix} c_{il} & c_{jm} & c_{kn} \\ c_{jn} & c_{kl} & c_{im} \\ c_{km} & c_{in} & c_{jl} \end{bmatrix} \right\},$$

30 consist of three pairs of type III,

$$\left\{ \begin{bmatrix} a_i & b_l & c_{il} \\ b_k & a_j & c_{jk} \\ c_{ik} & c_{jl} & c_{mn} \end{bmatrix}, \begin{bmatrix} a_k & b_n & c_{kn} \\ b_m & a_l & c_{lm} \\ c_{km} & c_{ln} & c_{ij} \end{bmatrix}, \begin{bmatrix} a_m & b_j & c_{jm} \\ b_i & a_n & c_{in} \\ c_{im} & c_{jn} & c_{kl} \end{bmatrix} \right\}.$$

We will denote these decompositions by $\mathrm{St}_{(ijk)(lmn)}$ and $\mathrm{St}_{(ij)(kl)(mn)}$, respectively.

3.8. Remark. —— Except possibly for some of the group-theoretic observations, all the facts on Steiner trihedra given here were well known to the geometers of the 19th century. They are due to J. Steiner [St].

The model case.

3.9. Theorem. —— *Let* $\pi\colon S \to \operatorname{Spec}\mathbb{Q}$ *be a non-singular cubic surface. Suppose that* S *has a* $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$*-invariant triplet and an orbit structure of* $[9,9,9]$ *on the 27 lines. Then,* $\mathrm{Br}(S)/\pi^*\mathrm{Br}(\mathbb{Q}) \cong \mathbb{Z}/3\mathbb{Z}$*.*

3.10. Remark. —— There is a different orbit structure of type $[9,9,9]$, in which the orbits correspond to trihedra of the first kind. In this case, $\mathrm{Br}(S)/\pi^*\mathrm{Br}(\mathbb{Q}) = 0$. The maximal subgroup $G \subset W(E_6)$ stabilizing the orbits is isomorphic to the dihedral group of order 18.

3.11. Proof of Theorem 3.9. —— *First step.* Manin's formula.
We have the isomorphism $\delta\colon H^1(\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}), \mathrm{Pic}(S_{\overline{\mathbb{Q}}})) \to \mathrm{Br}(S)/\pi^*\mathrm{Br}(\mathbb{Q})$. In order to explicitly compute $H^1(\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}), \mathrm{Pic}(S_{\overline{\mathbb{Q}}}))$ as an abstract abelian group, we will use Manin's formula [Ma, Proposition 31.3]. This means the following.

$\mathrm{Pic}(S_{\overline{\mathbb{Q}}})$ is generated by the 27 lines. The group of all permutations of the 27 lines respecting the canonical class and the intersection pairing is isomorphic to the Weyl group $W(E_6)$ of order 51 840. The group $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ operates on the 27 lines via a finite quotient $G := \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})/H$ that is isomorphic to a subgroup of $W(E_6)$. According to Shapiro's lemma, $H^1(G, \mathrm{Pic}(S_{\overline{\mathbb{Q}}})) \cong H^1(\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}), \mathrm{Pic}(S_{\overline{\mathbb{Q}}}))$.

Further, $H^1(G, \mathrm{Pic}(S_{\overline{\mathbb{Q}}})) \cong \mathrm{Hom}((ND \cap D_0)/ND_0, \mathbb{Q}/\mathbb{Z})$. Here, D is the free abelian group generated by the 27 lines and D_0 is the subgroup of all principal divisors. $N\colon D \to D$ denotes the norm map from G-modules to abelian groups.

Second step. Divisors.
The orbits of size nine define three divisors, which we call Δ_1, Δ_2, and Δ_3. By assumption, these are defined by three pairs of Steiner trihedra forming a triplet. Thus, without restriction,

$$\begin{aligned} \Delta_1 &= (a_1) + (a_2) + (a_3) + (b_1) + (b_2) + (b_3) + (c_{12}) + (c_{13}) + (c_{23}), \\ \Delta_2 &= (a_4) + (a_5) + (a_6) + (b_4) + (b_5) + (b_6) + (c_{45}) + (c_{46}) + (c_{56}), \\ \Delta_3 &= (c_{14}) + (c_{15}) + (c_{16}) + (c_{24}) + (c_{25}) + (c_{26}) + (c_{34}) + (c_{35}) + (c_{36}). \end{aligned}$$

Put $g := \#G/9$. Then, the norm of a line is either D_1, D_2, or D_3 for $D_i := g\Delta_i$. Hence,

$$ND = \{n_1D_1 + n_2D_2 + n_3D_3 \mid n_1, n_2, n_3 \in \mathbb{Z}\} .$$

Further, a direct calculation shows $LD_1 = LD_2 = LD_3 = 3g$ for every line L on $\mathrm{Pic}(S_{\overline{\mathbb{Q}}})$. Therefore, a divisor $n_1D_1 + n_2D_2 + n_3D_3$ is principal if and only if $n_1 + n_2 + n_3 = 0$.

Finally, D_0 is generated by all differences of the divisors given by two tritangent planes. The three lines on a tritangent plane are either contained in all three of the divisors D_1, D_2, and D_3 or only in one of them. Consequently,

$$ND_0 = \langle 3D_1 - 3D_2, 3D_1 - 3D_3, D_1 + D_2 - 2D_3\rangle .$$

The assertion immediately follows from this. $\square$

3.12. Remarks. —— i) A generator of $\mathrm{Br}(S)/\pi^*\mathrm{Br}(\mathbb{Q})$ is given by the homomorphism

$$(ND \cap D_0)/ND_0 \longrightarrow \tfrac{1}{3}\mathbb{Z}/\mathbb{Z}, \quad n_1D_1 + n_2D_2 + n_3D_3 \mapsto (\tfrac{n_1-n_2}{3} \bmod \mathbb{Z}) .$$

ii) Observe that $\frac{n_1-n_2}{3} \equiv \frac{n_2-n_3}{3} \equiv \frac{n_3-n_1}{3} \pmod{\mathbb{Z}}$ as $n_1 + n_2 + n_3 = 0$.

3.13. Definition. —— Let $\pi\colon S \to \operatorname{Spec}\mathbb{Q}$ be a smooth cubic surface and $\mathscr{T} = (T_1, T_2, T_3)$ be a Galois invariant triplet. This induces a group homomorphism $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to U_t$, given by the operation on the 27 lines.

i) Then, the image of the generator $(\frac{1}{3} \bmod \mathbb{Z})$ under the natural homomorphism

$$\tfrac{1}{3}\mathbb{Z}/\mathbb{Z} \xrightarrow{\cong} H^1(U_t, \mathrm{Pic}(S_{\overline{\mathbb{Q}}})) \longrightarrow H^1(\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}), \mathrm{Pic}(S_{\overline{\mathbb{Q}}})) \xrightarrow[\delta]{\cong} \mathrm{Br}(S)/\pi^*\mathrm{Br}(\mathbb{Q})$$

is called *the Brauer class associated with* the Galois invariant triplet $\mathscr{T}$. We denote it by $\mathrm{cl}(\mathscr{T})$.

ii) This defines a map

$$\mathrm{cl}\colon \Theta_S^{\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})} \longrightarrow \mathrm{Br}(S)/\pi^*\mathrm{Br}(\mathbb{Q})$$

from the set of all Galois invariant triplets, which we will call the *class map*.

3.14. Lemma. —— *Let* $\pi\colon S \to \operatorname{Spec}\mathbb{Q}$ *be a smooth cubic surface and* (T_1, T_2, T_3) *be a Galois invariant triplet. Then,*

$$\mathrm{cl}(T_1, T_2, T_3) = \mathrm{cl}(T_2, T_3, T_1) = \mathrm{cl}(T_3, T_1, T_2) .$$

On the other hand, $\mathrm{cl}(T_1, T_2, T_3) = -\,\mathrm{cl}(T_2, T_1, T_3)$.

Proof. This is a direct consequence of the congruences observed in 3.12.ii). $\square$

The general case of a Galois group stabilizing a triplet. Using Manin's formula, we computed $H^1(G, \mathrm{Pic}(S_{\overline{\mathbb{Q}}}))$ for each of the 350 conjugacy classes of subgroups of $W(E_6)$. The computations in `gap` took only a few seconds of CPU time. We systematized the results in Remark 1.6. Note that, in particular, we recovered Sir Peter Swinnerton-Dyer's result [SD3] and made the following observation.

3.15. Proposition. — *Let S be a non-singular cubic surface over $\mathbb{Q}$. If*

$$H^1(\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}), \mathrm{Pic}(S_{\overline{\mathbb{Q}}})) = \mathbb{Z}/3\mathbb{Z} \quad \text{or} \quad \mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$$

then, on S, there is a Galois invariant triplet.

Proof. This is seen by a case-by-case study using `gap`. □

3.16. Remarks. — i) On the other hand, if there is a Galois invariant triplet on S then $H^1(\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}), \mathrm{Pic}(S_{\overline{\mathbb{Q}}}))$ is either 0, or $\mathbb{Z}/3\mathbb{Z}$ or $\mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$.

ii) Proposition 3.15 immediately provokes the question whether the cohomology classes are always "the same" as in the $[9, 9, 9]$-cases. I.e., of the type $\mathrm{cl}(\mathscr{T})$ for a certain Galois invariant triplet $\mathscr{T}$. Somewhat surprisingly, this is indeed the case.

3.17. Lemma. — *Let $\mathscr{S}$ be a non-singular cubic surface over an algebraically closed field, H a group of automorphisms of the configuration of the 27 lines, and $H' \subseteq H$ any subgroup. Each of the criteria below is sufficient for*

$$\mathrm{res}\colon H^1(H, \mathrm{Pic}(\mathscr{S})) \to H^1(H', \mathrm{Pic}(\mathscr{S}))$$

being an injection.

i) *H' is of index prime to 3 in H and $H^1(H, \mathrm{Pic}(\mathscr{S}))$ is a 3-group.*

ii) *H' is a normal subgroup in H and $\mathrm{rk}\,\mathrm{Pic}(\mathscr{S})^H = \mathrm{rk}\,\mathrm{Pic}(\mathscr{S})^{H'}$.*

Proof. i) Here, $\mathrm{cores} \circ \mathrm{res}\colon H^1(H, \mathrm{Pic}(\mathscr{S})) \to H^1(H, \mathrm{Pic}(\mathscr{S}))$ is the multiplication by a number prime to 3, hence a bijection.

ii) The assumption ensures that H/H' operates trivially on $\mathrm{Pic}(\mathscr{S})^{H'}$. Hence, $H^1\big(H/H', \mathrm{Pic}(\mathscr{S})^{H'}\big) = 0$. The inflation-restriction sequence

$$0 \to H^1\big(H/H', \mathrm{Pic}(\mathscr{S})^{H'}\big) \to H^1(H, \mathrm{Pic}(\mathscr{S})) \to H^1(H', \mathrm{Pic}(\mathscr{S}))$$

yields the assertion. □

3.18. Proposition. — *Let $\mathscr{S}$ be a non-singular cubic surface over an algebraically closed field and U_t be the group of automorphisms of the configuration of the 27 lines that stabilize a triplet.*

Further, let $H \subseteq U_t$ be such that $H^1(H, \mathrm{Pic}(\mathscr{S})) \cong \mathbb{Z}/3\mathbb{Z}$. Then, the restriction

$$\mathrm{res}\colon H^1(U_t, \mathrm{Pic}(\mathscr{S})) \longrightarrow H^1(H, \mathrm{Pic}(\mathscr{S}))$$

is bijective.

Proof. This proof has a computer part. Using `gap`, we made the observation that $\mathrm{rk}\,\mathrm{Pic}(\mathscr{S})^H = 1$ for every $H \subset W(E_6)$ such that $H^1(H, \mathrm{Pic}(\mathscr{S})) \cong \mathbb{Z}/3\mathbb{Z}$.

To verify the assertion, we will show injectivity. According to Lemma 3.17.i), it suffices to test this on the 3-Sylow subgroups of H and U_t. But the 3-Sylow subgroup $U_t^{(3)} \cong \mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$ is abelian. Thus, Lemma 3.17.ii) immediately implies the assertion. □

3.19. Corollary. — *Let $H' \subseteq H \subseteq U_t$ be arbitrary. Then, for the restriction map* $\mathrm{res}\colon H^1(H, \mathrm{Pic}(\mathscr{S})) \longrightarrow H^1(H', \mathrm{Pic}(\mathscr{S}))$, *there are the following limitations.*

i) *If $H^1(H, \mathrm{Pic}(\mathscr{S})) = 0$ then $H^1(H', \mathrm{Pic}(\mathscr{S})) = 0$.*

ii) *If $H^1(H, \mathrm{Pic}(\mathscr{S})) \cong \mathbb{Z}/3\mathbb{Z}$ and $H^1(H', \mathrm{Pic}(\mathscr{S})) \neq 0$ then* res *is an injection.*

iii) *If $H^1(H, \mathrm{Pic}(\mathscr{S})) \cong \mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$ then $H^1(H', \mathrm{Pic}(\mathscr{S})) \cong \mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$ or 0. In the former case, $H' = H$. In the latter case, $H' = 0$.*

Proof. We know from Remark 3.16.i) that both groups may be only 0, $\mathbb{Z}/3\mathbb{Z}$, or $\mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$.

i) If $H^1(H', \mathrm{Pic}(\mathscr{S}))$ were isomorphic to $\mathbb{Z}/3\mathbb{Z}$ or $\mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$ then the restriction from U_t to H' would be the zero map.

ii) is immediate from the computations above.

iii) This assertion is obvious as H is of order three. □

4 Computing the Brauer-Manin obstruction

A splitting field for the Brauer class.

4.1. — The theory developed above shows that the non-trivial Brauer classes are, in a certain sense, always the same, as long as they are of order three. This may certainly be used for explicit computations. The Brauer class remains unchanged under suitable restriction maps. These correspond to extensions of the base field.

We will, however, present a different method here. Its advantage is that it avoids large base fields.

4.2. Lemma. — *Let S be a non-singular cubic surface and let T_1 and T_2 be two pairs of Steiner trihedra that define disjoint sets of lines.*

Then, the 18 lines defined by T_1 and T_2 contain exactly three double-sixes. These form a triple of azygetic double-sixes.

Proof. We work in the blown-up model. As $W(E_6)$ operates transitively on pairs of Steiner trihedra, we may suppose without restriction that

$$T_1 = \begin{bmatrix} a_1 & b_2 & c_{12} \\ b_3 & c_{23} & a_2 \\ c_{13} & a_3 & b_1 \end{bmatrix}.$$

Further, the stabilizer of one pair of Steiner trihedra acts transitively on the two complementary ones. Thus, still without restriction,

$$T_2 = \begin{bmatrix} a_4 & b_5 & c_{45} \\ b_6 & c_{56} & a_5 \\ c_{46} & a_6 & b_4 \end{bmatrix}.$$

In this situation, we immediately see the three double-sixes

$$\begin{pmatrix} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 \\ b_1 & b_2 & b_3 & b_4 & b_5 & b_6 \end{pmatrix}, \begin{pmatrix} a_1 & a_2 & a_3 & c_{56} & c_{46} & c_{45} \\ c_{23} & c_{13} & c_{12} & b_4 & b_5 & b_6 \end{pmatrix}, \begin{pmatrix} c_{23} & c_{13} & c_{12} & a_4 & a_5 & a_6 \\ b_1 & b_2 & b_3 & c_{56} & c_{46} & c_{45} \end{pmatrix}$$

that are contained in the 18 lines.

Finally, two double-sixes contained within the 18 lines must have at least six lines in common. It is well known that, in this case, they have exactly six lines in common. Two such double-sixes are classically called azygetic [Do, Lemma 9.1.4 and the remarks before]. A double-six that is azygetic to the first of the three is necessarily of the form

$$\begin{pmatrix} a_i & a_j & a_k & c_{mn} & c_{ln} & c_{lm} \\ c_{jk} & c_{ik} & c_{ij} & b_l & b_m & b_n \end{pmatrix}.$$

Obviously, such a double-six is contained within the 18 lines considered only for $\{i, j, k\} = \{1, 2, 3\}$ or $\{i, j, k\} = \{4, 5, 6\}$. □

4.3. Proposition-Definition. —— Let S be a non-singular cubic surface over an arbitrary field K. Suppose that T_1 and T_2 are two pairs of Steiner trihedra that are $\mathrm{Gal}(\overline{K}/K)$-invariant and define disjoint sets of lines.

a) Then, there is a unique minimal Galois extension L/K such that the three double-sixes, contained within the 18 lines defined by T_1 and T_2, are $\mathrm{Gal}(\overline{K}/L)$-invariant. We will call L the *field of definition* of the double-sixes, given by T_1 and T_2.

b) Actually, each of the six sixers is $\mathrm{Gal}(\overline{K}/L)$-invariant.

c) $\mathrm{Gal}(L/K)$ is a subgroup of S_3.

Proof. a) and c) $\mathrm{Gal}(\overline{K}/K)$ permutes the three double-sixes. L corresponds to the kernel of this permutation representation.

b) The triple intersections of T_1 or T_2 with two of the double-sixes are $\mathrm{Gal}(\overline{K}/L)$-invariant. From these trios, the sixers may be combined. □

4.4. Corollary (A splitting field**).** —— *Let S be a non-singular cubic surface over a field K with a Galois-invariant triplet (T_1, T_2, T_3).*
Then, the field L of definition of the three double-sixes, given by T_1 and T_2, is a splitting field for every class in $\mathrm{Br}(S)/\pi^*\mathrm{Br}(K)$*.*

Proof. As S_L has a Galois-invariant sixer, we certainly [Ma, Theorem 42.8] have that $\mathrm{Br}(S_L)/\pi_L^*\mathrm{Br}(L) = 0$. □

4.5. Remarks. —— Let S be a non-singular cubic surface over $\mathbb{Q}$ having a Galois-invariant triplet (T_1, T_2, T_3).
i) Then, there are the three splitting fields defined by $\{T_1, T_2\}$, $\{T_2, T_3\}$, and $\{T_3, T_1\}$. It may happen that two of them or all three coincide. In general, they are cyclic cubic extensions of a quadratic number field.
ii) Among the 17 conjugacy classes of subgroups of $W(E_6)$ leading to a Brauer class of order three, there are exactly nine allowing a splitting field that is cyclic of degree three.

4.6. Remark. —— Suppose that S is a non-singular cubic surface over $\mathbb{Q}$ having a Galois-invariant triplet. Then, somewhat surprisingly, every splitting field contains one of the three, described as in Corollary 4.4. The reason for this is the following observation.

4.7. Lemma. —— *Let $G \subset U_t$ be an arbitrary subgroup. Then, either G stabilizes a double-six or $H^1(G, \mathrm{Pic}(S_{\overline{\mathbb{Q}}})) \neq 0$.*

Proof. This follows from the inspection of all subgroups summarized in 1.6. □

A rational function representing the Brauer class.

4.8. —— To compute the Brauer-Manin obstruction, we will follow the "informal" algorithm of Yu. I. Manin [Ma, Sec. 45.2]. For L a splitting field, we first have, according to Shapiro's lemma, $H^1(\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}), \mathrm{Pic}(S_{\overline{\mathbb{Q}}})) = H^1(\mathrm{Gal}(L/\mathbb{Q}), \mathrm{Pic}(S_{\overline{\mathbb{Q}}}))$. If $\mathrm{Gal}(L/\mathbb{Q}) \cong S_3$ then a class of order three is preserved under restriction to A_3. As this is a cyclic group,

$$H^1(\mathrm{Gal}(L/\mathbb{Q}(\sqrt{\Delta})), \mathrm{Pic}(S_{\overline{\mathbb{Q}}})) \cong (\mathrm{Div}_0(S_{\mathbb{Q}(\sqrt{\Delta})}) \cap N\mathrm{Div}(S_L))/N\mathrm{Div}_0(S_L)\,.$$

4.9. Definition. —— Let S be a non-singular cubic surface over $\mathbb{Q}$ having a Galois-invariant triplet. Suppose that $\mathrm{Br}(S)/\pi^*\mathrm{Br}(\mathbb{Q}) \neq 0$. Fix a splitting field $L \supset \mathbb{Q}(\sqrt{\Delta})$ of degree 6 or 3 over $\mathbb{Q}$. Let, finally, $\Psi \in \mathbb{Q}(\sqrt{\Delta})(S)$ be a rational function such that

$$\mathrm{div}\,\Psi = N_{L/\mathbb{Q}(\sqrt{\Delta})}D$$

for a divisor $D \in \mathrm{Div}(S_L)$, but not for a principal divisor. We then say that the rational function Ψ *represents* a non-trivial Brauer class.

4.10. — Since $\mathrm{Br}(S)/\pi^*\mathrm{Br}(\mathbb{Q})$ is a 3-group, the 27 lines on $S_{\mathbb{Q}(\sqrt{\Delta})}$ may have the orbit structure $[3,3,3,3,3,3,3,3,3]$, $[3,3,3,9,9]$, or $[9,9,9]$. We distinguish two cases.

A. There is an orbit of size three.

Then, each such orbit is actually formed by a triangle. If L is chosen such that some of these orbits split then one may put $\Psi := l_1/l_2$, where l_1 and l_2 are two linear forms defining distinct split orbits.

Forms of the type described have been used before to describe Brauer classes. See [Ma, Sec. 45], [CKS], or [J].

B. The orbit structure is $[9,9,9]$.

Then, the Galois-invariant pairs T_1, T_2, T_3 of Steiner trihedra immediately define the orbits of the lines. We will work with the splitting field L defined by $\{T_1, T_2\}$. Then, on S_L, there are three Galois-invariant sixers s_1, s_2, s_3. The Galois group $G := \mathrm{Gal}(L/\mathbb{Q}(\sqrt{\Delta})) \cong \mathbb{Z}/3\mathbb{Z}$ permutes them cyclically.

4.11. Caution. — The three divisors given by the orbits of size nine generate in $\mathrm{Pic}(S_{\mathbb{Q}(\sqrt{\Delta})})$ a subgroup of index three. It turns out impossible to write down a function Ψ representing a non-trivial Brauer class such that $\mathrm{div}\,\Psi$ is a linear combination of these three divisors. Thus, curves of degree >1 need to be considered.

4.12. Lemma. — *On every smooth cubic surface, there are 72 two-dimensional families of twisted cubic curves. Each such family corresponds to a sixer. Having blown down the sixer, the curves correspond to the lines in* $\mathbf{P}^2$ *through none of the six blow-up points.*

Proof. This is a classical result, due to A. Clebsch [Cl, p. 371/72]. □

4.13. Construction (of function Ψ, theoretical part)**.** — i) Let $C = C_1$ be a twisted cubic curve on S_L corresponding to s_1. Then, the G-orbit $\{C_1, C_2, C_3\}$ of C_1 consists of three twisted cubic curves corresponding to the three sixers.

ii) An elementary calculation shows that $C_1 + C_2 + C_3 \sim 3H$ for H the hyperplane section.

iii) Further, $D_1 \sim D_2 \sim D_3 \sim 3H$ for D_i the sum of the nine lines corresponding to T_i. Hence, $NC - D_1 = C_1 + C_2 + C_3 - D_1$ is a principal divisor. It is clearly the norm of a divisor.

iv) We choose Ψ such that $\mathrm{div}(\Psi) = C_1 + C_2 + C_3 - D_1$.

4.14. Remark. — It is a tedious calculation to show that $NC - D_1$ is not the norm of a principal divisor. However, $D_1 - D_2$ is. In particular, $NC - D_1$ and $NC - D_2$ define the same Brauer class.

4.15. Construction (of function Ψ, practical part). — For a concrete cubic surface, one may determine a function Ψ using the following strategy.

i) First, construct a blow-down map $S_L \to \mathbf{P}^2_L$ in two steps by first blowing down an orbit of three. There is some beautiful classical algebraic geometry involved in this step [Do, Sec. 9.3.1]. The result is a surface in $\mathbf{P}^1 \times \mathbf{P}^1 \times \mathbf{P}^1$, given by a trilinear form. The three projective lines appear as the pencils of the planes through three skew lines on S. To give the morphism down to $\mathbf{P}^2$ is then equivalent to bringing the trilinear form into standard shape.

ii) Then, choose a line through none of the blow-up points and calculate seven L-rational points on the corresponding twisted cubic curve. Generically, these determine a two-dimensional space of cubic forms over $\mathbb{Q}(\sqrt{\Delta})$ containing the form that defines the cubic surface $S_{\mathbb{Q}(\sqrt{\Delta})}$ itself. It turned out practical to work with a line that connects the preimages of two $\mathbb{Q}$-rational points.

Explicit Brauer-Manin obstruction.

4.16. Lemma. — *Let $\pi\colon S \to \operatorname{Spec}\mathbb{Q}$ be a non-singular cubic surface with a Galois invariant triplet. Further, let $c \in \operatorname{Br}(S)/\pi^*\operatorname{Br}(\mathbb{Q})$ be a non-zero class.*

Let, finally, L be a splitting field that is cyclic of degree three over a field $\mathbb{Q}(\sqrt{\Delta})$ and Ψ a rational function representing c. We consider the cyclic algebra

$$Q := L(S)\{Y\}/(Y^3 - \Psi)$$

over the function field $\mathbb{Q}(\sqrt{\Delta})(S)$. Here, $Yt = \sigma(t)Y$ for $t \in L(S)$ and a fixed generator $\sigma \in \operatorname{Gal}(L/\mathbb{Q}(\sqrt{\Delta}))$.

Then, Q is an Azumaya algebra over $\mathbb{Q}(\sqrt{\Delta})(S)$. It corresponds to the restriction to the generic point $\eta \in S_{\mathbb{Q}(\sqrt{\Delta})}$ of some lift $\overline{c} \in \operatorname{Br}(S)$ of c.

Proof. We have $H^1(\operatorname{Gal}(\overline{\mathbb{Q}}/L), \operatorname{Pic}(S_{\overline{\mathbb{Q}}})) = 0$ as the cubic surface S_L has Galois-invariant sixers. The inflation-restriction sequence yields

$$H^1(\operatorname{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}(\sqrt{\Delta})), \operatorname{Pic}(S_{\overline{\mathbb{Q}}})) \cong H^1(\operatorname{Gal}(L/\mathbb{Q}(\sqrt{\Delta})), \operatorname{Pic}(S_L))\,.$$

The assertion easily follows. □

4.17. Remark. — It is well known that a class in $\operatorname{Br}(S_{\mathbb{Q}(\sqrt{\Delta})})$ is uniquely determined by its restriction to $\operatorname{Br}(\mathbb{Q}(\sqrt{\Delta})(S))$. The corresponding Azumaya algebra over the whole of $S_{\mathbb{Q}(\sqrt{\Delta})}$ may be described as follows.

Let $x \in S_{\mathbb{Q}(\sqrt{\Delta})}$. We know that $\operatorname{div}(\Psi)$ is the norm of a divisor on S_L. That one is necessarily locally principal. I.e., we have a rational function f_x such that $\operatorname{div}(N_{L/\mathbb{Q}(\sqrt{\Delta})} f_x) = \operatorname{div}(\Psi)$ on a Zariski neighbourhood of x. Over the maximal such neighbourhood U_x, we define an Azumaya algebra by

$$Q_x := (\mathscr{O}_{U_x} \otimes_{\mathbb{Q}(\sqrt{\Delta})} L)\{Y_x\}/(Y_x^3 - \tfrac{\Psi}{N_{L/\mathbb{Q}(\sqrt{\Delta})} f_x})\,.$$

Again, we suppose $Y_x t = \sigma(t) Y_x$ for $t \in \mathscr{O}_{U_x} \otimes_{\mathbb{Q}(\sqrt{\Delta})} L$. In particular, in a neighbourhood U_η of the generic point, we have the Azumaya algebra $Q_\eta := (\mathscr{O}_{U_\eta} \otimes_{\mathbb{Q}(\sqrt{\Delta})} L)\{Y\}/(Y^3 - \Psi)$.

Over $U_\eta \cap U_x$, there is the isomorphism $\iota_{\eta,x} \colon Q_\eta|_{U_\eta \cap U_x} \to Q_x|_{U_\eta \cap U_x}$, given by

$$Y \mapsto f_x Y_x \,.$$

For two points $x, y \in S_{\mathbb{Q}(\sqrt{\Delta})}$, the isomorphism $\iota_{\eta,y} \circ \iota_{\eta,x}^{-1} \colon Q_x|_{U_\eta \cap U_x \cap U_y} \to Q_y|_{U_\eta \cap U_x \cap U_y}$ extends to $U_x \cap U_y$. Hence, the Azumaya algebras Q_x may be glued together along these isomorphisms. This yields an Azumaya algebra $\mathscr{Q}$ over $S_{\mathbb{Q}(\sqrt{\Delta})}$.

4.18. Theorem. — *Let $\pi \colon S \to \operatorname{Spec} \mathbb{Q}$ be a non-singular cubic surface having a Galois invariant triplet. Further, let $c \in \operatorname{Br}(S)/\pi^*\operatorname{Br}(\mathbb{Q})$ be a non-zero class. Let, finally, L be a splitting field that is cyclic of degree three over a field $\mathbb{Q}(\sqrt{\Delta})$ and Ψ a rational function representing c.*

We fix an isomorphism $\operatorname{Gal}(L/\mathbb{Q}(\sqrt{\Delta})) \to \frac{1}{3}\mathbb{Z}/\mathbb{Z}$. For every prime $\mathfrak{p}$ of $\mathbb{Q}(\sqrt{\Delta})$ that does not split in L, this fixes an isomorphism $\operatorname{Gal}(L_\mathfrak{p}/\mathbb{Q}(\sqrt{\Delta})_\mathfrak{p}) \to \frac{1}{3}\mathbb{Z}/\mathbb{Z}$.

Then, there is a representative $\underline{c} \in \operatorname{Br}(S)$ such that, for every prime number p, the local evaluation map $\operatorname{ev}_p(\underline{c}, \,.\,) \colon S(\mathbb{Q}_p) \to \mathbb{Q}/\mathbb{Z}$ is given as follows.

a) *If a prime $\mathfrak{p}$ above p splits in $L/\mathbb{Q}(\sqrt{\Delta})$ then $\operatorname{ev}_p(\underline{c}, x) = 0$ independently of x.*

b) *Otherwise, for $x \in S(\mathbb{Q}_p) \subseteq S(\mathbb{Q}_p(\sqrt{\Delta}))$, choose a rational function f_x on S_L such that, in a neighbourhood of x, $\operatorname{div}(Nf_x) = \operatorname{div}(\Psi)$. Then,*

$$\operatorname{ev}_p(\underline{c}, x) = \begin{cases} \theta_\mathfrak{p}(\Psi/Nf_x) & \text{if } (p) = \mathfrak{p} \text{ is inert in } L, \\ 2\theta_\mathfrak{p}(\Psi/Nf_x) & \text{if } (p) = \mathfrak{p}^2 \text{ is ramified in } L, \\ \theta_{\mathfrak{p}_1}(\Psi/Nf_x) + \theta_{\mathfrak{p}_2}(\Psi/Nf_x) & \text{if } (p) = \mathfrak{p}_1\mathfrak{p}_2 \text{ splits in } L, \end{cases}$$

where $\theta_\mathfrak{p} \colon L_\mathfrak{p}^* \to \operatorname{Gal}(L_\mathfrak{p}/\mathbb{Q}(\sqrt{\Delta})_\mathfrak{p}) \cong \frac{1}{3}\mathbb{Z}/\mathbb{Z}$ is the norm residue homomorphism for the extension $L_\mathfrak{p}/\mathbb{Q}(\sqrt{\Delta})_\mathfrak{p}$ of local fields.

Proof. The assertion immediately follows from the above. $\square$

4.19. Remarks (Diagonal cubic surfaces**).** — i) For the particular case of a diagonal cubic surface over $\mathbb{Q}$, the effect of the Brauer-Manin obstruction has been studied intensively [CKS]. Our approach covers diagonal cubic surfaces, too.

Indeed, on the surface given by $ax^3 + by^3 + cz^3 + dw^3 = 0$, a Galois invariant pair of Steiner trihedra is formed by the three planes given by $\sqrt[3]{a}x + \sqrt[3]{b}\zeta_3 y = 0$ together with the three planes given by $\sqrt[3]{c}z + \sqrt[3]{d}\zeta_3 w = 0$. The two other pairs completing the triplet are formed analogously pairing x with z or w.

ii) Diagonal cubic surfaces, are however, far from being the generic case. There are at least two reasons. First, the 27 lines are defined over $\mathbb{Q}(\zeta_3, \sqrt[3]{a/d}, \sqrt[3]{b/d}, \sqrt[3]{c/d})$, which is of degree at most 54, not 216. Furthermore, each of the six Steiner trihedra described has the property that the three planes intersect in a line while, generically, they intersect in a point.

iii) For the two pairs of Steiner trihedra defined by pairings $(xy)(zw)$ and $(xz)(yw)$, we find $L = \mathbb{Q}(\zeta_3, \sqrt[3]{ad/bc})$ for the splitting field of the corresponding three double-sixes. This is exactly the field occurring in [CKS, Lemme 1].

In fact, the corresponding 18 lines break up into the six trios

$$\begin{aligned} D_1^i\colon\ &\zeta_3^k \sqrt[3]{a/b}\, x + y = 0,\ \zeta_3^i z + \zeta_3^{-k} \sqrt[3]{d/c}\, w = 0, \quad k = 1,2,3\,, \\ D_2^j\colon\ &\zeta_3^k \sqrt[3]{a/c}\, x + z = 0,\ \zeta_3^j y + \zeta_3^{-k} \sqrt[3]{d/b}\, w = 0, \quad k = 1,2,3\,. \end{aligned}$$

each consisting of pairwise skew lines. Observe that the trios are invariant under $\mathrm{Gal}(\overline{\mathbb{Q}}/L)$ as $\sqrt[3]{a/b}\sqrt[3]{d/c} = \sqrt[3]{a/c}\sqrt[3]{d/b} \in L$.

Further, $D_1^i \cup D_2^j$ is a sixer if and only if $i \neq j$. We have the three double-sixes $(D_1^1 \cup D_2^2) \cup (D_1^2 \cup D_2^1)$, $(D_1^1 \cup D_2^3) \cup (D_1^3 \cup D_2^1)$, and $(D_1^2 \cup D_2^3) \cup (D_1^3 \cup D_2^2)$.

iv) Our method to compute the Brauer-Manin obstruction is, however, different from that of [CKS]. J.-L. Colliot-Thélène and his coworkers worked over larger extensions of the base field.

Explicit Galois descent.

4.20. —— Recall that, in [EJ3], we described a method to construct non-singular cubic surfaces over $\mathbb{Q}$ with a Galois invariant pair of Steiner trihedra. The idea was to start with cubic surfaces in Cayley-Salmon form. For these, we developed an explicit version of Galois descent.

4.21. —— More concretely, given a quadratic number field D or $D = \mathbb{Q} \oplus \mathbb{Q}$, an element $u \in D \backslash \{0\}$, and a starting polynomial $f \in D[T]$ of degree three without multiple zeroes, we constructed a cubic surface $S^{u_0,u_1}_{(a_0,\ldots,a_5)}$ over $\mathbb{Q}$ such that

$$S^{u_0,u_1}_{(a_0,\ldots,a_5)} \times_{\mathrm{Spec}\,\mathbb{Q}} \mathrm{Spec}\,\overline{\mathbb{Q}}$$

is isomorphic to the surface $S^{(a_0,\ldots,a_5)}_{u_0,u_1}$ in $\mathbf{P}^5_{\overline{\mathbb{Q}}}$ given by

$$\begin{aligned} u_0 X_0 X_1 X_2 + u_1 X_3 X_4 X_5 &= 0\,, \\ a_0 X_0 + a_1 X_1 + a_2 X_2 + a_3 X_3 + a_4 X_4 + a_5 X_5 &= 0\,, \\ X_0 + X_1 + X_2 + X_3 + X_4 + X_5 &= 0\,. \end{aligned}$$

Here, $u_0 = \iota_0(u)$ and $u_1 = \iota_1(u)$ for $\iota_0, \iota_1\colon D \to \overline{\mathbb{Q}}$ the two embeddings of D. Further, $a \in A$ and $a_i = \tau_i(a)$ for $\tau_1, \ldots, \tau_5\colon D[T]/(f) \to \overline{\mathbb{Q}}$ the six embeddings of the étale $\mathbb{Q}$-algebra $D[T]/(f)$. Thereby, τ_0, τ_1, τ_2 are supposed to be compatible with ι_0, whereas τ_3, τ_4, τ_5 are supposed to be compatible with ι_1.

4.22. Definition. —— The general cubic polynomial

$$\Phi^{(a_0,\ldots,a_5)}_{u_0,u_1}(T) := \frac{1}{u_0}(a_0+T)(a_1+T)(a_2+T) - \frac{1}{u_1}(a_3+T)(a_4+T)(a_5+T)$$

is called the *auxiliary polynomial* associated with $S^{(a_0,\ldots,a_5)}_{u_0,u_1}$.

4.23. Proposition. —— *Let A be an étale algebra of rank three over a commutative semisimple $\mathbb{Q}$-algebra D of dimension two and $u \in D$ as well as $a \in A$ as above. Further, suppose that $S^{(a_0,\ldots,a_5)}_{u_0,u_1}$ is non-singular. Then,*
a) $\Phi^{(a_0,\ldots,a_5)}_{u_0,u_1} \in \mathbb{Q}[T]$ *for* $D \cong \mathbb{Q} \oplus \mathbb{Q}$ *and* $\sqrt{d}\,\Phi^{(a_0,\ldots,a_5)}_{u_0,u_1} \in \mathbb{Q}[T]$ *for* $D \cong \mathbb{Q}(\sqrt{d})$.
b) *The operation of an element* $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ *on* $S_{(a_0,\ldots,a_5)} \times_{\mathrm{Spec}\,\mathbb{Q}} \mathrm{Spec}\,\overline{\mathbb{Q}}$ *goes over into the automorphism* $\pi_\sigma \circ t_\sigma \colon S^{(a_0,\ldots,a_5)} \to S^{(a_0,\ldots,a_5)}$.
Here, π_σ *permutes the coordinates according to the rule* $a_{\pi_\sigma(i)} = \sigma(a_i)$ *while* t_σ *is the naive operation of* σ *on* $S_{(a_0,\ldots,a_5)}$ *as a morphism of schemes twisted by* σ.
c) *Further, on the descent variety* $S^{u_0,u_1}_{(a_0,\ldots,a_5)}$ *over* $\mathbb{Q}$, *there are*
i) *nine obvious lines given by* $L_{\{i,j\}}\colon \iota^*X_i = \iota^*X_j = 0$ *for* $i = 0,1,2$ *and* $j = 3,4,5$,
ii) *18 non-obvious lines given by* $L^\lambda_\rho\colon \iota^*Z_0 + \iota^*Z_{\rho(0)} = \iota^*Z_1 + \iota^*Z_{\rho(1)} = 0$ *for* λ *a zero of* $\Phi^{(a_0,\ldots,a_5)}_{u_0,u_1}$ *and* $\rho\colon \{0,1,2\} \to \{3,4,5\}$ *a bijection. Here, the coordinates* Z_i *are given by*

$$\begin{aligned} Z_0 &:= -Y_0+Y_1+Y_2, \quad Z_1 := Y_0-Y_1+Y_2, \quad Z_2 := Y_0+Y_1-Y_2, \\ Z_3 &:= -Y_3+Y_4+Y_5, \quad Z_4 := Y_3-Y_4+Y_5, \quad Z_5 := Y_3+Y_4-Y_5 \end{aligned}$$

for $Y_i := (a_i+\lambda)X_i, \quad i = 0,\ldots,5$.
An element $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ *acts on the lines according to the rules*

$$\sigma(L_{\{i,j\}}) = L_{\{\pi_\sigma(i),\pi_\sigma(j)\}}, \qquad \sigma(L^\lambda_\rho) = L^{\lambda^\sigma}_{\rho^{\pi_\sigma}}\,.$$

d) *The nine obvious lines are formed by a pair of Steiner trihedra. The twelve non-obvious lines corresponding to two zeroes of* $\Phi^{(a_0,\ldots,a_5)}_{u_0,u_1}$ *form a double-six.*

Proof. This is a summary of results obtained in [EJ3]. a) is [EJ3, Lemma 6.1], while b) and c) are [EJ3, Proposition 6.5]. The first assertion of d) is [EJ3, Fact 4.2]. Finally, the second one is shown [EJ3, Proposition 4.6] together with [EJ3, Proposition 2.8]. □

4.24. Remarks. —— i) The construction described is actually more general than needed here. It yields cubic surfaces with a Galois invariant pair of Steiner trihedra. The two other pairs, completing the first to a triplet, may be constructed as to be Galois invariant or to be defined over a quadratic number field. Several examples of both sorts were given in [EJ3, Section 7].

ii) Fix the two Steiner trihedra corresponding to the 18 non-obvious lines. Then, parts c.ii) and d) of Proposition 4.23 together show that the field extension splitting the corresponding three double-sixes is the same as the splitting field of the auxiliary polynomial $\Phi^{(a_0,\ldots,a_5)}_{u_0,u_1}$.

Application: Manin's conjecture.

4.25. — Recall that a conjecture, due to Yu. I. Manin, asserts that the number of $\mathbb{Q}$-rational points of anticanonical height $\leq B$ on a Fano variety S is asymptotically equal to $\tau B \log^{\mathrm{rk}\,\mathrm{Pic}(S)-1} B$, for $B \to \infty$. Further, the coefficient $\tau \in \mathbb{R}$ is conjectured to be the Tamagawa-type number $\tau(S)$ introduced by E. Peyre in [Pe]. In the particular case of a cubic surface, the anticanonical height is the same as the naive height.

4.26. — E. Peyre's Tamagawa-type number is defined in [PT, Definition 2.4] as

$$\tau(S) := \alpha(S)\cdot\beta(S)\cdot\lim_{s\to 1}\,(s-1)^t L(s,\chi_{\mathrm{Pic}(S_{\overline{\mathbb{Q}}})})\cdot\tau_H\big(S(\mathbb{A}_\mathbb{Q})^{\mathrm{Br}}\big)$$

for $t = \mathrm{rk}\,\mathrm{Pic}(S)$. Here, τ_H is the Tamagawa measure on the set $S(\mathbb{A}_\mathbb{Q})$ of adelic points on S and $S(\mathbb{A}_\mathbb{Q})^{\mathrm{Br}} \subseteq S(\mathbb{A}_\mathbb{Q})$ denotes the part which is not affected by the Brauer-Manin obstruction. For details, in particular on the constant factors, we refer to the original literature. But observe that the Tamagawa measure is the product measure of measures on $S(\mathbb{Q}_\nu)$ for ν running through the places of $\mathbb{Q}$.

4.27. — Using [EJ3, Algorithm 5.8], we constructed many examples of smooth cubic surfaces over $\mathbb{Q}$ with a Galois invariant pair of Steiner trihedra. For each of them, one may apply Theorem 4.18 to compute the effect of the Brauer-Manin obstruction. Then, the method described in [EJ1] applies for the computation of Peyre's constant.

From the considerable supply, the examples below were chosen in the hope that they indicate the main phenomena. The Brauer-Manin obstruction may work at many primes simultaneously but examples where few primes are involved are more interesting. We will show that the fraction of the Tamagawa measure excluded by the obstruction can vary greatly.

4.28. Example. — Start with $g(U) := U^2-1$, i.e. $D = \mathbb{Q}\oplus\mathbb{Q}$, $u_0 := 1$, $u_1 := \frac{1}{2}$, $f_0(V) := V^3+6V^2+9V+1$, and $f_1(V) := V^3+\frac{9}{2}V^2+\frac{9}{2}V$. Then, the auxiliary polynomial is $-V^3-3V^2+1$. [EJ3, Algorithm 5.8] yields the cubic surface S given by the equation

$$T_0^3-T_0^2T_2-T_0^2T_3-2T_0T_2^2+T_0T_2T_3-T_1^3+3T_1^2T_2-3T_1T_2T_3+3T_1T_3^2-T_2^3-T_2^2T_3+T_3^3 = 0\,.$$

This example is constructed in such a way that f_1 completely splits, while f_0 and the auxiliary polynomial have the same splitting field that is an A_3-extension of $\mathbb{Q}$.

Hence, the Galois group operating on the 27 lines is of order three. We have orbit structure $[3, 3, 3, 3, 3, 3, 3, 3, 3]$ and $H^1(\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}), \mathrm{Pic}(S_{\overline{\mathbb{Q}}})) \cong \mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$.

S has bad reduction at the primes 3 and 19. It turns out that the local evaluation map at 19 is constant. Hence, the Brauer-Manin obstruction works only at the prime 3.

Here, the reduction is the cone over a cubic curve over $\mathbb{F}_3$ having a cusp. Only the nine smooth points lift to 3-adic ones. Among them, exactly one is allowed by the Brauer-Manin obstruction. Hence, from the whole of $S(\mathbb{Q}_3)$, which is of measure 1, only a subset of measure $\frac{1}{9}$ is allowed.

Using this, for Peyre's constant, we find $\tau(S) \approx 0.1567$. There are actually 599 $\mathbb{Q}$-rational points of height at most 4000 in comparison with a prediction of 627.

4.29. Remark. —— It is, may be, of interest to note that S has good reduction at 2 and the reduction has exactly one $\mathbb{F}_2$-rational point. An example of such a cubic surface over $\mathbb{F}_2$ was once much sought-after and finally found by Swinnerton-Dyer [SD2].

4.30. Example. —— Start with $g(U) := U^2 - 1$, i.e. $D = \mathbb{Q} \oplus \mathbb{Q}$, $u_0 := -\frac{1}{3}$, $u_1 := 1$, $f_0(V) := V^3 - V + \frac{1}{3}$, and $f_1(V) := V^3 + 3V^2 - 4V + 1$. Then, the auxiliary polynomial is $-4V^3 - 3V^2 + 7V - 2$. [EJ3, Algorithm 5.8] yields the cubic surface S given by the equation

$$\begin{aligned} &-3T_0^3 - 6T_0^2T_1 - 3T_0^2T_2 + 3T_0^2T_3 - 3T_0T_1^2 + 3T_0T_1T_3 + 3T_0T_2^2 + 6T_0T_3^2 + 2T_1^3 \\ &- 4T_1^2T_2 - T_1^2T_3 + 10T_1T_2^2 - 4T_1T_2T_3 - 9T_1T_3^2 + 6T_2^3 - 8T_2^2T_3 - 8T_2T_3^2 + 4T_3^3 = 0\,. \end{aligned}$$

Here, f_0, f_1, and the auxiliary polynomial have splitting fields that are linearly disjoint A_3-extensions of $\mathbb{Q}$. Hence, the Galois group operating on the 27 lines is of order 27. It is the maximal 3-group stabilizing a pair of Steiner trihedra. We have orbit structure $[9, 9, 9]$ and $H^1(\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}), \mathrm{Pic}(S_{\overline{\mathbb{Q}}})) \cong \mathbb{Z}/3\mathbb{Z}$.

The three minimal splitting fields are given by the polynomials $T^3 + T^2 - 10T - 8$, $T^3 - 21T + 28$, and $T^3 - 21T - 35$. Observe that the first one is unramified at both primes, 3 and 7, while the other two fields ramify.

S has bad reduction at the primes 3, 7, and 31. The local H^1-criterion [EJ2, Lemma 6.6] excludes the prime 31. Hence, the Brauer-Manin obstruction works only at the primes 3 and 7.

At 3, the reduction is the cone over a cubic curve over $\mathbb{F}_3$ having a cusp. The only singular point that lifts to a 3-adic one is $(1 : 0 : 1 : 1)$. The local evaluation map is constant on the points of smooth reduction and decomposes the others into two equal parts. This means, $S(\mathbb{Q}_3)$ breaks into sets of measures 1, $\frac{1}{3}$, and $\frac{1}{3}$, respectively.

At 7, the reduction is Cayley's ruled cubic surface over $\mathbb{F}_7$. Exactly one singular point lifts to a 7-adic one. This is $(1 : 5 : 1 : 0)$. Again, the local evaluation map is constant on the points of smooth reduction and decomposes the others into two equal parts. Thus, $S(\mathbb{Q}_7)$ breaks into sets of measures 1, $\frac{1}{7}$, and $\frac{1}{7}$. An easy calculation shows $\tau_H(S(\mathbb{A}_\mathbb{Q})^{\mathrm{Br}}) = \frac{11}{45}\tau_H(S(\mathbb{A}_\mathbb{Q}))$.

Using this, for Peyre's constant, we find $\tau(S) \approx 1.7311$. There are actually 6880 $\mathbb{Q}$-rational points of height at most 4000 in comparison with a prediction of 6924.

4.31. Example. —— Start with $g(U) := U^2 - 1$, i.e. $D = \mathbb{Q} \oplus \mathbb{Q}$, $u_0 := \frac{13}{7}$, $u_1 := -\frac{31}{169}$, $f_0(V) := V^3 - 16V^2 + 85V - \frac{1049}{7}$, and $f_1(V) := V^3 - \frac{337}{31}V^2 + \frac{1216}{31}V - 47$. Then, the auxiliary polynomial is $\frac{2414}{403}V^3 - \frac{848021}{12493}V^2 + \frac{595}{13}V - \frac{1537566}{12493}$. [EJ3, Algorithm 5.8] yields the cubic surface S given by the equation

$$\begin{aligned} 13T_0^3 - 8T_0^2T_1 + 9T_0^2T_2 + 44T_0^2T_3 - 9T_0T_1^2 - 5T_0T_1T_2 + T_0T_1T_3 + 4T_0T_2^2 \\ - 19T_0T_2T_3 - 61T_0T_3^2 - T_1^2T_2 - 24T_1^2T_3 + 3T_1T_2^2 + 42T_1T_2T_3 - 16T_1T_3^2 \\ + 2T_2^3 + 10T_2^2T_3 - 60T_2T_3^2 + 6T_3^3 = 0\,. \end{aligned}$$

Here, f_0, and f_1 define A_3-extensions of $\mathbb{Q}$, ramified only at 7 and 13, respectively. As the auxiliary polynomial is of type S_3, the Galois group operating on the 27 lines is of order 54. We have orbit structure $[9, 9, 9]$ and $H^1(\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}), \mathrm{Pic}(S_{\overline{\mathbb{Q}}})) \cong \mathbb{Z}/3\mathbb{Z}$.

The minimal splitting fields are given by the polynomials $T^3 - T^2 - 30T - 27$, $T^3 - T^2 - 30T + 64$, and $T^3 - 75\,986\,365T - 753\,805\,852\,436$. The first two are cyclic of degree three. Both are ramified exactly at 7 and 13. The third splitting field is of type S_3 and ramified at 7, 13, 127, and 387 512 500 241.

S has bad reduction at the primes 3, 7, 13, 127, 281, 84 629 and 387 512 500 241. The local H^1-criterion excludes all primes except 7 and 13. Hence, the Brauer-Manin obstruction works only at the primes 7 and 13.

At 7, the reduction is a normal cubic surface with three singularities of type A_2 that are defined over $\mathbb{F}_{7^3}$. The local evaluation map on $S(\mathbb{Q}_7)$ factors via $S(\mathbb{F}_7)$ and decomposes these 57 points into three equal classes. This means, $S(\mathbb{Q}_7)$ breaks into three sets, each of which is of measure $\frac{19}{49}$.

At 13, the reduction is a normal cubic surface having three singularities of type A_2 that are defined over $\mathbb{F}_{13}$. The behaviour is a bit more complicated as there are 13-adic points reducing to the singularities. On the points having good reduction, the local evaluation map factors via $S(\mathbb{F}_{13})$. It splits the 180 smooth points into three classes of 60 elements each.

Summarizing, we clearly have $\tau_H(S(\mathbb{A}_\mathbb{Q})^{\mathrm{Br}}) = \frac{1}{3}\tau_H(S(\mathbb{A}_\mathbb{Q}))$. Using this, for Peyre's constant, we find $\tau(S) \approx 0.5907$. There are actually 2370 $\mathbb{Q}$-rational points of height at most 4000 in comparison with a prediction of 2363.

4.32. Remark. —— On $S(\mathbb{F}_7)$ and on the smooth $\mathbb{F}_{13}$-rational points, we find exactly the decompositions defined by Mordell-Weil equivalence [EJ4, Corollary 3.4.3].

4.33. Example. —— Start with $g(U) := U^2 - 7$, i.e. $D = \mathbb{Q}(\sqrt{7})$, $u := \frac{1}{-1+2\sqrt{7}} = -\frac{1+2\sqrt{7}}{29}$, and

$$f(V) := (-1 + 2\sqrt{7})V^3 + (1 + 2\sqrt{7})V^2 + (-2 + \sqrt{7})V + (-1 + 2\sqrt{7})\,.$$

Then, the auxiliary polynomial is $4\sqrt{7}(V^3 + V^2 + \frac{1}{2}V + 1)$. [EJ3, Algorithm 5.8] yields the cubic surface S given by the equation

$$\begin{aligned}11T_0^3 + 32T_0^2T_1 + 31T_0^2T_2 + 52T_0^2T_3 - 33T_0T_1^2 - 93T_0T_1T_2 - 36T_0T_1T_3 + 37T_0T_2^2 \\ + 46T_0T_2T_3 - 34T_0T_3^2 + 22T_1^3 - 10T_1^2T_2 - 21T_1^2T_3 + 75T_1T_2^2 - 4T_1T_2T_3 + 30T_1T_3^2 \\ + 133T_2^3 + 34T_2^2T_3 - 8T_2T_3^2 + 2T_3^3 = 0\,.\end{aligned}$$

Here, f has Galois group S_3. Its discriminant is $(-22144 + 3806\sqrt{7})$, a number of norm 19722^2. This ensures that not only one, but three pairs of Steiner trihedra are Galois invariant and that we have orbit structure $[9, 9, 9]$. The auxiliary polynomial is of type S_3. Hence, the Galois group operating on the 27 lines is the maximal possible group U_t stabilizing a triplet. It is of order 216.

The three minimal splitting fields are given by the polynomials $T^3 - 7T - 8$, $T^3 + T^2 - 177T - 2059$, and $T^3 - T^2 + 32T + 368$.

S has bad reduction at the primes 2, 7, 19, 89, 151 and 173. The local H^1-criterion excludes 2, 7, 89, 151 and 173. Hence, the Brauer-Manin obstruction works only at the prime 19.

Here, the reduction is to three planes over $\mathbb{F}_{19}$. The intersection points do not lift to $\mathbb{Q}_{19}$-rational ones. The Brauer-Manin obstruction allows exactly one of the three planes. This means, from the whole of $S(\mathbb{Q}_{19})$, which is of measure 3, only a subset of measure 1 is allowed.

Using this, for Peyre's constant, we find $\tau(S) \approx 0.0553$. There are actually 216 $\mathbb{Q}$-rational points of height at most 4000 in comparison with a prediction of 221.

4.34. Remarks. —— i) Except for the first, these examples have orbit structures $[9, 9, 9]$. Unfortunately, the rational functions representing the Brauer class are not very well suited for a reproduction. For instance, in Example 4.30, the non-trivial Brauer class can be represented by the cubic form

$$\begin{aligned}21965T_0^3 + 55863T_0^2T_2 - 53607T_0^2T_3 - 1215T_0T_1^2 + 75402T_0T_1T_2 - 136125T_0T_1T_3 \\ + 35961T_0T_2^2 - 133200T_0T_2T_3 + 44382T_0T_3^2 + 8402T_1^3 + 149304T_1^2T_2 \\ - 272189T_1^2T_3 - 22210T_1T_2^2 - 249626T_1T_2T_3 + 526313T_1T_3^2 - 36518T_2^3 \\ - 70576T_2^2T_3 + 184098T_2T_3^2 - 26618T_3^3\end{aligned}$$

divided by the cube of a linear form.

ii) In the final example, the rational functions representing the Brauer class have coefficients in $\mathbb{Q}(\sqrt{7})$. Rational functions with coefficients in $\mathbb{Q}$ do not serve our purposes.

Indeed, the prime $19 = \mathfrak{p}_1\mathfrak{p}_2$ splits in $\mathbb{Q}(\sqrt{7})$. According to class field theory, the two norm residue homomorphisms $\theta_{\mathfrak{p}_1}, \theta_{\mathfrak{p}_2}$ are opposite to each other when applied to elements from $\mathbb{Q}_{19}$. Thus, $\Psi \in \mathbb{Q}[T_0, \ldots, T_3]$ would imply that the Brauer-Manin obstruction does not work at 19, at least not on $\mathbb{Q}$-rational points.

iii) To apply the local H^1-criterion, we compute a degree-36 resolvent the zeroes of which correspond to the double-sixes. According to Lemma 4.7, we have $H^1(\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p), \mathrm{Pic}(S_{\overline{\mathbb{Q}}})) = 0$ when the resolvent has a p-adic zero.

iv) It is noticeable from the examples that the reduction types at the relevant primes are distributed in an unusual way. This is partially explained by [EJ4, Proposition 1.12].

5 Two triplets

5.1. Lemma. —— *Let $\mathscr{T} = \{T_1, T_2, T_3\}$ and $\mathscr{T}' = \{T'_1, T'_2, T'_3\}$ be two different decompositions of the 27 lines into pairs of Steiner trihedra. Then, up to permutation of rows and columns, for the matrix J defined by $J_{ij} := \#I_{ij}$ for $I_{ij} := T_i \cap T'_j$, there are two possibilities.*

A. $\begin{pmatrix} 3 & 3 & 3 \\ 3 & 3 & 3 \\ 3 & 3 & 3 \end{pmatrix}$,

B. $\begin{pmatrix} 5 & 2 & 2 \\ 2 & 5 & 2 \\ 2 & 2 & 5 \end{pmatrix}$.

Proof. As $W(E_6)$ transitively operates on decompositions, we may suppose that $\mathscr{T} = \mathrm{St}_{(123)(456)}$ is the standard decomposition.

Then, the other nine decompositions of type $\mathrm{St}_{(ijk)(lmn)}$ yield type B. Among the 30 decompositions of type $\mathrm{St}_{(ij)(kl)(mn)}$, there are exactly 18 for which one of the sets $\{i, j\}$, $\{k, l\}$, or $\{m, n\}$ is a subset of $\{1, 2, 3\}$. These lead to type B, too. The twelve remaining triplets yield type A. $\square$

5.2. Remarks. —— i) Correspondingly, the group $U_t \subset W(E_6)$ that stabilizes a triplet, operates on the set of all decompositions such that the orbits have lengths 1, 12, and 27.

ii) The group U_t operates as well on the set of the 240 triplets. Here, the orbit lengths are 1, 1, 1, 1, 1, 1, 27, 27, 27, 27, 27, 27, and 72. In fact, in case A, U_t is able to permute the pairs T'_1, T'_2, and T'_3 although T_1, T_2, and T_3 remain fixed. The same cannot happen in case B.

5.3. Example (Type A). —— A representative case for this type is given by the decomposition $\mathscr{T} = \mathrm{St}_{(123)(456)}$, which, by definition, consists of the pairs

$$\begin{bmatrix} a_1 & b_2 & c_{12} \\ b_3 & c_{23} & a_2 \\ c_{13} & a_3 & b_1 \end{bmatrix}, \qquad \begin{bmatrix} a_4 & b_5 & c_{45} \\ b_6 & c_{56} & a_5 \\ c_{46} & a_6 & b_4 \end{bmatrix}, \quad \text{and} \quad \begin{bmatrix} c_{14} & c_{15} & c_{16} \\ c_{24} & c_{25} & c_{26} \\ c_{34} & c_{35} & c_{36} \end{bmatrix}$$

of Steiner trihedra and $\mathscr{T}' = \mathrm{St}_{(14)(25)(36)}$, consisting of

$$\begin{bmatrix} a_1 & b_5 & c_{15} \\ b_2 & a_4 & c_{24} \\ c_{12} & c_{45} & c_{36} \end{bmatrix}, \qquad \begin{bmatrix} a_2 & b_6 & c_{26} \\ b_3 & a_5 & c_{35} \\ c_{23} & c_{56} & c_{14} \end{bmatrix}, \quad \text{and} \quad \begin{bmatrix} a_3 & b_4 & c_{34} \\ b_1 & a_6 & c_{16} \\ c_{13} & c_{46} & c_{25} \end{bmatrix}.$$

For the matrix of the mutual intersections, one obtains

$$I = \begin{pmatrix} \{a_1, b_2, c_{12}\} & \{a_2, b_3, c_{23}\} & \{a_3, b_1, c_{13}\} \\ \{a_4, b_5, c_{45}\} & \{a_5, b_6, c_{56}\} & \{a_6, b_4, c_{46}\} \\ \{c_{15}, c_{24}, c_{36}\} & \{c_{14}, c_{26}, c_{35}\} & \{c_{16}, c_{25}, c_{34}\} \end{pmatrix}.$$

The maximal subgroup U_{tt} of $W(E_6)$ stabilizing both triplets is of order $216/72 = 3$. It is clearly the same as the maximal subgroup stabilizing each of the nine sets of size three obtained as intersections.

5.4. Remark. — U_{tt} actually stabilizes four decompositions of the 27 lines into three sets of nine. They are given by the rows, the columns as well as the positive and negative diagonals of the Sarrus scheme.

5.5. Remark. — Each decomposition gives rise to eight *enneahedra,* systems of nine planes containing all the 27 lines. Two decompositions meeting of type A have an enneahedron in common, as one may see at the matrix I. All in all, there are $\frac{40 \cdot 12}{2} / \binom{4}{2} = 40$ enneahedra appearing in this way.

In this form, the two types, in which two decompositions may be correlated, were known to L. Cremona [Cr] in 1870. Cremona calls these 40 the enneahedra of the first kind. There are 160 further enneahedra, which are said to be of the second kind.

5.6. Theorem. — *Let* $\pi\colon S \to \operatorname{Spec} \mathbb{Q}$ *be a non-singular cubic surface. Suppose that* S *has two* $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$*-invariant triplets that do not correspond to the same decomposition and an orbit structure of* $[3, 3, 3, 3, 3, 3, 3, 3, 3]$ *on the 27 lines.*

a) *Then,* $\mathrm{Br}(S)/\pi^*\mathrm{Br}(\mathbb{Q}) \cong \mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$.

b) *There are actually four* $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$*-invariant decompositions on* S*. The eight non-zero elements of* $\mathrm{Br}(S)/\pi^*\mathrm{Br}(\mathbb{Q})$ *are the images of these under the class map. In other words,* cl *induces a bijection*

$$\Theta_S^{\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})}/A_3 \xrightarrow{\cong} (\mathrm{Br}(S)/\pi^*\mathrm{Br}(\mathbb{Q})) \setminus \{0\}.$$

Proof. Observe that, on S, there is an enneahedron consisting of Galois invariant planes. This immediately implies the first assertion of b).

a) Once again, we will use the isomorphism

$$\delta\colon H^1(\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}), \mathrm{Pic}(S_{\overline{\mathbb{Q}}})) \to \mathrm{Br}(S)/\pi^*\mathrm{Br}(\mathbb{Q}).$$

Further, by Shapiro's lemma, $H^1(\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}), \mathrm{Pic}(S_{\overline{\mathbb{Q}}})) \cong H^1(G, \mathrm{Pic}(S_{\overline{\mathbb{Q}}}))$ for G the subgroup of $W(E_6)$, through which $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ actually operates on the 27 lines.

The assumption clearly implies that the two decompositions considered meet each other of type A. Hence, up to conjugation, $G \subseteq U_{tt}$. As U_{tt} is of order three, there are no proper subgroups, except for the trivial group. Hence, $G = U_{tt}$. It is now a direct calculation to verify $H^1(U_{tt}, \mathrm{Pic}(S_{\overline{\mathbb{Q}}})) = \mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$. This was worked out by Yu. I. Manin in [Ma, Lemma 45.4.1].

b) We will show this assertion in several steps.

First step. Divisors.

We will use Manin's formula. The orbits of size three define nine divisors, which we call Δ_{ij} for $1 \leq i, j \leq 3$. Thereby, Δ_{ij} shall correspond to the set I_{ij} for I the matrix of intersections given above.

The norm of a line is always one of the Δ_{ij}. Hence, $ND = \{\sum_{ij} n_{ij}\Delta_{ij} \mid n_{ij} \in \mathbb{Z}\}$. Further, a direct calculation shows that $L\Delta_{ij} = 1$ for every line L on $\mathrm{Pic}(S_{\overline{\mathbb{Q}}})$ and every pair (i, j). Therefore, a divisor $\sum_{ij} n_{ij}\Delta_{ij}$ is principal if and only if $\sum_{ij} n_{ij} = 0$.

Now, consider a divisor of the form

$$\Delta_{i_1 j_1} + \Delta_{i_2 j_2} + \Delta_{i_3 j_3}$$

such that

$$i_1 + i_2 + i_3 \equiv j_1 + j_2 + j_3 \equiv 0 \pmod 3 .$$

It is not hard to check that such a divisor is always the norm of the sum of three lines in a plane. Consequently, the difference of two such divisors is in ND_0.

Second step. The nine residue classes.

We therefore have the following parallelogram rule.

$$\Delta_{ij} + \Delta_{i'j'} - \Delta_{i+i',j+j'} - \Delta_{33} = 0 \in (ND \cap D_0)/ND_0 ,$$

where the arithmetic of the indices is done modulo 3. Indeed, $\Delta_{ij}+\Delta_{i'j'}+\Delta_{-i-i',-j-j'}$ and $\Delta_{i+i',j+j'} + \Delta_{33} + \Delta_{-i-i',-j-j'}$ are norms of sums of three lines in a plane.

The parallelogram rule implies, in particular, that every element of $(ND \cap D_0)/ND_0$ is equal to some $\Delta_{ij} - \Delta_{33}$. Clearly, these nine elements are mutually distinct. Further, $(\Delta_{ij} - \Delta_{33}) + (\Delta_{i'j'} - \Delta_{33}) = \Delta_{i+i',j+j'} - \Delta_{33}$.

Third step. A non-zero Brauer class.

Let now a non-zero homomorphism $g\colon (ND \cap D_0)/ND_0 \to \frac{1}{3}\mathbb{Z}/\mathbb{Z}$ be given. Then, $\ker g = \{0, \Delta_{ij} - \Delta_{33}, \Delta_{2i,2j} - \Delta_{33}\}$ for a suitable pair (i, j). This shows that

$$g^{-1}(\tfrac{1}{3}) = \{\Delta_{i_1,j_1} - \Delta_{33}, \Delta_{i_2,j_2} - \Delta_{33}, \Delta_{i_3,j_3} - \Delta_{33}\}$$

for $i_1 + i_2 + i_3 \equiv j_1 + j_2 + j_3 \equiv 0 \pmod 3$.

The divisor $\Delta_{i_1,j_1} + \Delta_{i_2,j_2} + \Delta_{i_3,j_3}$ consists exactly of the nine lines defined by a pair T_1 of Steiner trihedra that is fixed by U_{tt}. Further, $\Delta_{33} + \Delta_{ij} + \Delta_{2i,2j}$ consists of the lines defined by another pair T_2. Let $\mathscr{T} = (T_1, T_2, T_3)$ be the corresponding triplet. We claim that the restriction of $\mathrm{cl}(\mathscr{T})$ is exactly the cohomology class represented by g.

For this, observe that the restriction map in group cohomology is dual to the norm map $N_G\colon (ND \cap D_0)/ND_0 \to (N_G D \cap D_0)/N_G D_0$ for G any group causing the orbit structure $[9, 9, 9]$ given by the triplet $\mathscr{T}$. This norm map sends Δ_{i_1,j_1}, Δ_{i_2,j_2}, and Δ_{i_3,j_3} to D_2 and Δ_{33} to D_1. I.e., $(\Delta_{i_l,j_l} - \Delta_{33})$ is mapped to $(D_2 - D_1)$ in the notation of 3.12. As $\mathrm{cl}(\mathscr{T})$ sends $(D_2 - D_1)$ to $\frac{1}{3}$, this is enough to imply the claim.

Fourth step. Conclusion.
Consequently, the mapping $\Psi_S^{\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})}/A_3 \to (\mathrm{Br}(S)/\pi^*\mathrm{Br}(\mathbb{Q})) \setminus \{0\}$ induced by cl is a surjection. As, on both sides, the sets are of size eight, bijectivity follows. □

5.7. Remark. — Up to conjugation, $G = U_{tt}$ is the only subgroup of $W(E_6)$ such that $H^1(G, \mathrm{Pic}(S_{\overline{\mathbb{Q}}})) \cong \mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$. An example of a cubic surface over $\mathbb{Q}$, for which U_{tt} appears as the Galois group operating on the 27 lines, was given in [EJ3, Example 7.6]. Over $\mathbb{Q}(\zeta_3)$, one may simply consider the surface given by $x^3 + y^3 + z^3 + aw^3 = 0$ for $a \in \mathbb{Q}^*$ a non-cube.

5.8. Example (Type B). — A representative case for this type is given by the decompositions $\mathscr{T} = \mathrm{St}_{(123)(456)}$, consisting of the pairs

$$\begin{bmatrix} a_1 & b_2 & c_{12} \\ b_3 & c_{23} & a_2 \\ c_{13} & a_3 & b_1 \end{bmatrix}, \qquad \begin{bmatrix} a_4 & b_5 & c_{45} \\ b_6 & c_{56} & a_5 \\ c_{46} & a_6 & b_4 \end{bmatrix}, \qquad \begin{bmatrix} c_{14} & c_{15} & c_{16} \\ c_{24} & c_{25} & c_{26} \\ c_{34} & c_{35} & c_{36} \end{bmatrix}$$

of Steiner trihedra, and $\mathscr{T}' = \mathrm{St}_{(12)(34)(56)}$ consisting of

$$\begin{bmatrix} a_1 & b_4 & c_{14} \\ b_3 & a_2 & c_{23} \\ c_{13} & c_{24} & c_{56} \end{bmatrix}, \qquad \begin{bmatrix} a_3 & b_6 & c_{36} \\ b_5 & a_4 & c_{45} \\ c_{35} & c_{46} & c_{12} \end{bmatrix}, \qquad \begin{bmatrix} a_5 & b_2 & c_{25} \\ b_1 & a_6 & c_{16} \\ c_{15} & c_{26} & c_{34} \end{bmatrix}.$$

For the matrix of the mutual intersections, we obtain

$$\begin{pmatrix} \{a_1, a_2, b_3, c_{13}, c_{23}\} & \{a_3, c_{12}\} & \{b_1, b_2\} \\ \{b_4, c_{56}\} & \{a_4, b_5, b_6, c_{45}, c_{46}\} & \{a_5, a_6\} \\ \{c_{14}, c_{24}\} & \{c_{35}, c_{36}\} & \{c_{15}, c_{16}, c_{25}, c_{26}, c_{34}\} \end{pmatrix}.$$

The maximal subgroup $U'_{tt} \subset W(E_6)$ stabilizing both triplets is of order $216/27 = 8$. The Brauer classes defined by both triplets are annihilated when restricted to a surface on which these six pairs of trihedra are Galois invariant.

5.9. Remark. — Observe that U'_{tt} does not stabilize any pair of Steiner trihedra different from the six given above.

References

[CG] Cassels, J. W. S. and Guy, M. J. T.: *On the Hasse principle for cubic surfaces,* Mathematika **13** (1966), 111–120

[Cl] Clebsch, A: *Die Geometrie auf den Flächen dritter Ordnung,* J. für die reine und angew. Math. **65** (1865), 359–380

[CKS] Colliot-Thélène, J.-L., Kanevsky, D., and Sansuc, J.-J.: *Arithmétique des surfaces cubiques diagonales,* in: Diophantine approximation and transcendence theory (Bonn 1985), Lecture Notes in Math. 1290, Springer, Berlin 1987, 1–108

[Cr] Cremona, L.: *Sulle ventisette rette di una superficie del terzo ordine,* Rendiconti del R. Istituto Lombardo, serie II, **3** (1870), 209–219

[Do] Dolgachev, I. V.: *Topics in classical algebraic geometry,* version of September 27, 2010, `http://www.math.lsa.umich.edu/~idolga/topics1.pdf`

[EJ1] Elsenhans, A.-S. and Jahnel, J.: *Experiments with general cubic surfaces,* in: Algebra, Arithmetic, and Geometry, In Honor of Yu. I. Manin, Edited by Y. Tschinkel and Y. Zarhin, Volume I, Progress in Mathematics 269, Birkhäuser, Boston 2007, 637–654

[EJ2] Elsenhans, A.-S. and Jahnel, J.: *On the Brauer–Manin obstruction for cubic surfaces,* Journal of Combinatorics and Number Theory **2** (2010), 107–128

[EJ3] Elsenhans, A.-S. and Jahnel, J.: *Cubic surfaces with a Galois invariant pair of Steiner trihedra,* International Journal of Number Theory **7** (2011), 947–970

[EJ4] Elsenhans, A.-S. and Jahnel, J.: *On the quasi group of a cubic surface over a finite field,* Preprint

[Ha] Hartshorne, R.: *Algebraic Geometry,* Graduate Texts in Mathematics 52, Springer, New York 1977

[J] Jahnel, J.: *More cubic surfaces violating the Hasse principle,* Journal de Théorie des Nombres de Bordeaux **23** (2011), 471–477

[Ma] Manin, Yu. I.: *Cubic forms, algebra, geometry, arithmetic,* North-Holland Publishing Co. and American Elsevier Publishing Co., Amsterdam, London, and New York 1974

[Mo] Mordell, L. J.: *On the conjecture for the rational points on a cubic surface,* J. London Math. Soc. **40** (1965), 149–158

[Pe] Peyre, E.: *Hauteurs et mesures de Tamagawa sur les variétés de Fano,* Duke Math. J. **79** (1995), 101–218

[PT] Peyre, E. and Tschinkel, Yu.: *Tamagawa numbers of diagonal cubic surfaces, numerical evidence,* Math. Comp. **70** (2001), 367–387

[Sch] Schläfli, L.: *An attempt to determine the twenty-seven lines upon a surface of the third order, and to divide such surfaces into species in reference to the reality of the lines upon the surface,* Quart. J. Math. **2** (1858), 110–120

[St] Steiner, J.: *Ueber die Flächen dritten Grades,* J. für die reine und angew. Math. **53** (1857), 133–141

[SD1] Swinnerton-Dyer, Sir Peter: *Two special cubic surfaces,* Mathematika **9** (1962), 54–56

[SD2] Swinnerton-Dyer, Sir Peter: *Universal equivalence for cubic surfaces over finite and local fields,* in: Symposia Mathematica XXIV (Sympos. INDAM, Rome 1979), Academic Press, London-New York 1981, 111–143

[SD3] Swinnerton-Dyer, Sir Peter: *The Brauer group of cubic surfaces,* Math. Proc. Cambridge Philos. Soc. **113** (1993), 449–460

[Ta] Tate, J.: *Global class field theory,* in: Algebraic number theory, Edited by J. W. S. Cassels and A. Fröhlich, Academic Press and Thompson Book Co., London and Washington 1967, 162–203