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On the Origin of the BV Operator on Odd Symplectic Supermanifolds

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Abstract. Differential forms on an odd symplectic manifold form a bicomplex: one differential is the wedge product with the symplectic form and the other is de Rham differential. In the corresponding spectral sequence the next differential turns out to be the Batalin–Vilkoviski operator.

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1. Introduction

The Batalin–Vilkovisky (BV) operator [1] plays an important role in quantization of gauge theories. Its invariant meaning was discovered by Khudaverdian [2,3]. He noticed that if x^i, p_i are local Darboux coordinates on a supermanifold M with an odd symplectic structure, the operator

$$\Delta = \frac{\partial^2}{\partial x^i \partial p_i},$$

acting on *semidensities* on M, is independent of the choice of cooordinates. He also noticed that semidensities on the symplectic supermanifold restrict to densities on Lagrangian submanifolds, which then can be integrated (this explained invariantly the gauge fixing in the BV formalism).

The aim of this note is to find a natural interpretation of this somewhat miraculous operator. The basic problem is to find an intrinsic (i.e. coordinate-free) interpretation of (semi)densities, which are on supermanifolds usually defined using coordinate transformations. Fortunately, there is a (little known) intrinsic interpretation of Berezinian due tu Manin [4], which gives rise to a simple intrinsic interpretation of semidensities on M, makes the fact that they restrict to densities on Lagrangian submanifolds to a tautology, and leads to a natural construction of the BV operator, which is the subject of this note.

2. Two Differentials on Odd Symplectic Supermanifolds

In what follows, let M be a supermanifold with an odd symplectic form ω. On $\Omega(M)$ we have two anticommuting differentials: one is de Rham's d and the other is the wedge product with ω. We shall find that the Khudaverdian BV operator is (rougly speaking) the third differential of the spectral sequence of this bicomplex.

THEOREM. *Let* (M, ω) *be an odd symplectic supermanifold. In the spectral sequence of the bicomplex* $(\Omega(M), \omega\wedge, \mathrm{d})$ *we have:*

(1) *the cohomology of the complex* $(\Omega(M), \omega\wedge)$ *is naturally isomorphic to the semi-densities on* M
(2) *the next differential in the spectral sequence, de Rham's* d*, vanishes on the cohomology of* $(\Omega(M), \omega\wedge)$
(3) *the next differential* $(\mathrm{d}\circ(\omega\wedge)^{-1}\circ\mathrm{d})$ *coincides with the BV operator*
(4) *all higher differentials are zero.*

The proof of this theorem is completely straightforward; we shall do it leisurely in the rest of this note. Let us remark that the theorem remains true when one replaces $\Omega(M)$ (differential forms on M) with pseudodifferential forms on M; our choice is basically a matter of taste.

3. Cohomology of $\omega\wedge$

It is fairly simple to describe the cohomology of the complex $(\Omega(M), \omega\wedge)$ in local Darboux coordinates x^i, p_i $(i = 1, \ldots, n,\ \omega = \mathrm{d}p_i \wedge \mathrm{d}x^i)$, where x^i are the even coordinates and p_i the odd coordinates. Let $U \subset M$ be the open subset covered by the coordinates. Then any cohomology class of $(\Omega(U), \omega\wedge)$ has unique representative of the form

$$f(x, p)\,\mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \cdots \wedge \mathrm{d}x^n. \tag{1}$$

In other words, using the coordinates, we can locally identify $H(\Omega(M), \omega\wedge)$ with functions on M (this identification *does* depend on the choice of coordinates; the cohomology classes are rather sections of a line bundle over M).

To prove this claim we split $(\Omega(U), \omega\wedge)$ into subcomplexes: we assign an auxiliary degree 1 to each $\mathrm{d}x$ and -1 to each $\mathrm{d}p$, and denote this degree auxdeg (x's and p's will have auxdeg$=0$); since $\mathrm{auxdeg}\,\omega = 0$, the subspaces of $\Omega(U)$ of fixed auxdeg are subcomplexes. We shall see that each of them has zero cohomology, except for the one with auxdeg$=n$, where the differential vanishes. We shall prove it using an explicit homotopy. Let us consider the operator $L\colon \Omega(U) \to \Omega(U)$ given by

$$L\colon \alpha \mapsto \partial_{x^i} \lrcorner\, \partial_{p_i} \lrcorner\, \alpha.$$

A direct computation shows that

$$L\circ(\omega\wedge)+(\omega\wedge)\circ L:\alpha\mapsto(n-\operatorname{auxdeg}\alpha)\,\alpha.$$

This concludes the proof.

Now we also see that d is 0 on $H(\Omega(M),\,\omega\wedge)$, since

$$\mathrm{d}(f(x,p)\,\mathrm{d}x^1\wedge\mathrm{d}x^2\wedge\cdots\wedge\mathrm{d}x^n)=\frac{\partial f}{\partial p_k}\,\mathrm{d}p_k\wedge\mathrm{d}x^1\wedge\mathrm{d}x^2\wedge\cdots\wedge\mathrm{d}x^n,$$

which is $\omega\wedge$-exact (having auxdeg $=n-1$).

4. The Third Differential

Let us now compute the third differential $\mathrm{d}\circ(\omega\wedge)^{-1}\circ\mathrm{d}$ in local Darboux coordinates. We have

$$\mathrm{d}(f(x,p)\,\mathrm{d}x^1\wedge\mathrm{d}x^2\wedge\cdots\wedge\mathrm{d}x^n)=\omega\wedge\alpha,$$

where

$$\alpha=L(\mathrm{d}(f(x,p)\,\mathrm{d}x^1\wedge\mathrm{d}x^2\wedge\cdots\wedge\mathrm{d}x^n))=\frac{\partial f}{\partial p_k}\,\partial_{x^k}\lrcorner\,\mathrm{d}x^1\wedge\mathrm{d}x^2\wedge\cdots\wedge\mathrm{d}x^n$$

and $\mathrm{d}\alpha$ is (up to a $\omega\wedge$-exact term)

$$\frac{\partial^2 f}{\partial x^k\partial p_k}\,\mathrm{d}x^1\wedge\mathrm{d}x^2\wedge\cdots\wedge\mathrm{d}x^n.$$

The third differential in the spectral sequence is thus equal to the BV operator

$$\Delta=\frac{\partial^2}{\partial x^k\partial p_k}.$$

Now if M is contractible and admits global Darboux coordinates, the cohomology of Δ is isomorphic to $\mathbb{R}$ (since Δ can be identified with de Rham's d on a contractible subset of $\mathbb{R}^n$), and any Δ-cohomology class has a representative in $\Omega(M)$ which is a constant multiple of $\mathrm{d}x^1\wedge\mathrm{d}x^2\wedge\cdots\wedge\mathrm{d}x^n$. This representative is d-closed and thus is annuled by all higher differentials in the spectral sequence. Since any M can be covered by such patches, these higher differentials vanish for any M. This concludes the proof of the theorem, except for the part (1).

5. Semidensities

Now we'll prove the part (1) of the theorem. We locally identified the cohomology of $H(\Omega(M),\,\omega\wedge)$ with functions on M by choosing the representant (1); it is fairly easy to see that when we pass to another system of local Darboux coordinates, the function f gets multiplied by the square root of the corresponding Berezinian.

We shall however give a different proof, using Manin's cohomological interpretation of Berezinian [4]. The claim we are proving here, together with the proof, is taken from [5].

Let us recall Manin's idea. Let V be a vector supespace. Let us choose a vector superspace W with an odd symplectic form $\omega \in \bigwedge^2 W^*$, such that V is its Lagrangian subspace (for example $W = V \oplus \Pi V^*$). Then $\mathrm{Ber}(V^*)$ (the one dimensional vector space of constant densities on V) is defined as the cohomology $H\left(\bigwedge W^*, \omega\wedge\right)$ (this definition is easily seen to be independent of the choice of W).

If now V' is a Lagrangian complement of V in W, then again $\mathrm{Ber}(V'^*) = H\left(\bigwedge W^*, \omega\wedge\right)$; on the other hand,

$$\mathrm{Ber}(W^*) = \mathrm{Ber}(V^*) \otimes \mathrm{Ber}(V'^*) = H\left(\bigwedge W^*, \omega\wedge\right)^{\otimes 2},$$

and thus $H\left(\bigwedge W^*, \omega\wedge\right) = \mathrm{Ber}(W^*)^{1/2}$ (we should write everywhere "naturally isomorphic" instead of "equal", but hopefully it's not a big crime). This identity is valid for any vector superspace with an odd symplectic form. We apply it to the bundle of symplectic vector spaces TM, which concludes the proof.

6. Final Remarks

1. We should say a few remarks about the spectral sequence of the bicomplex $(\Omega(M), \omega\wedge, \mathrm{d})$, since $\Omega(M)$ is *not* bigraded. The spectral sequence is constructed in this way: we take $\Omega(M)[\hbar]$ (differential forms on M depending polynomially on an indeterminate $\hbar$; we could just as well take $\Omega(M)[[\hbar]]$), with the differential $\hbar\mathrm{d}+\omega\wedge$. Then multiplication by $\hbar$ is an endomorphism of the complex $(\Omega(M)[\hbar], \hbar\mathrm{d}+\omega\wedge)$, and our spectral sequence is the Bochstein spectral sequence of this endomorphism. That is, we start with the short exact sequence of complexes

$$0 \to (\Omega(M)[\hbar], \hbar\mathrm{d}+\omega\wedge) \xrightarrow{\hbar\cdot} (\Omega(M)[\hbar], \hbar\mathrm{d}+\omega\wedge) \to (\Omega(M), \omega\wedge) \to 0,$$

out of which we get the exact couple

$$\begin{array}{ccc} H(\Omega(M)[\hbar], \hbar\mathrm{d}+\omega\wedge) & \xrightarrow{(\hbar\cdot)_*} & H(\Omega(M)[\hbar], \hbar\mathrm{d}+\omega\wedge) \\ & \nwarrow \quad \swarrow & \\ & H(\Omega(M), \omega\wedge) & \end{array}$$

which generates the spectral sequence. If we denote E_∞ its ultimate term, we have

$$H\left(\Omega(M)[\hbar, \hbar^{-1}], \hbar\mathrm{d}+\omega\wedge\right) \cong E_\infty[\hbar, \hbar^{-1}]$$

(and also $H\left(\Omega(M)[[\hbar]][\hbar^{-1}], \hbar\mathrm{d}+\omega\wedge\right) \cong E_\infty[[\hbar]][\hbar^{-1}]$).

2. The odd symplectic form ω on M gives us an isomorphism between $\Omega(M)$ and $\Gamma(STM)$, i.e. the space of polynomial functions on T^*M. This isomorphism

transfers $\omega\wedge$ to multiplication by the odd Poisson structure π corresponding to ω (recall that since π is an odd Poisson structure, it is a function on T^*M), and d to $\{\pi, \cdot\}$ (where $\{,\}$ is the Poisson bracket on T^*M); the differential $\hbar d + \omega\wedge$ becomes $\pi + \hbar\{\pi, \cdot\}$. This suggests some generalizations, e.g. we can take an odd Poisson structure which is not symplectic, or more generally, instead of T^*M we can take an aritrary supermanifold with an even symplectic form, and consider on it an odd function π such that $\{\pi, \pi\} = 0$. The spectral sequences would still be defined, but it is not clear to me if they are good for anything.
3. The result in this note is extremely simple; it is written with the hope that it might be helpfull in situations where BV-like operators are much less trivial.

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