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On the Orlicz Minkowski Problem for Polytopes

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Abstract Quite recently, an Orlicz Minkowski problem has been posed and the existence part of this problem for even measures has been presented. In this paper, the existence part of the Orlicz Minkowski problem for polytopes is demonstrated. Furthermore, we obtain a solution of the Orlicz Minkowski problem for general (not necessarily even) measures.

Keywords Convex polytope · Orlicz norm · Minkowski problem

1 Introduction

The solution of the classical Minkowski problem for convex bodies is a central topic in convexity with many applications. Although the solution of the Minkowski problem has been known in the mathematical literature since the work of Minkowski [44, 45], Alexandrov [1–3], Fenchel and Jessen [10], analytic versions or algorithmic issues of the problem are still subject of current research and highly relevant (see, e.g., Chou and Wang [7], Jerison [21], Klain [23], Lamberg [24], Lamberg and Kaasalainen [25], and the reference therein).

The L_p Minkowski problem extends the classical Minkowski problem, which was first defined by Lutwak in [30] as part of the L_p Brunn-Minkowski theory (see, e.g., [4, 5, 8, 13–16, 20, 26–40, 43, 46–48, 50, 55, 56]). It requires necessary and sufficient conditions on a Borel measure μ on S^{n-1} to be the L_p surface area measure of a convex body; i.e., is there a convex body K such that

$$h_K^{1-p}\,dS_K = d\mu?$$

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Here, h_K is the support function of K and S_K is the surface area measure of K. The even L_p Minkowski problem asks which even measures are L_p surface area measures of convex bodies. For $p = 1$, this problem reduces to the classical Minkowski problem. For $p > 1$ but $p \neq n$, the even L_p Minkowski problem was solved by Lutwak [30]. However, in [36] an equivalent volume-normalized version of the L_p Minkowski problem was proposed, and the even volume-normalized L_p Minkowski problem was solved for all $p > 1$. The solution of the L_p Minkowski problem for polytopes for all $p > 1$ was given by Chou and Wang [8], while an alternate approach to this problem was presented by Hug et al. [20]. Other approaches towards the L_p Minkowski problem have also been extensively studied over the last years (see, e.g., [6, 9, 17, 19, 22, 51–54]). Despite impressive success in this direction, not all problems concerning the L_p Minkowski problem are completely solved.

Quite recently, an embryonic Orlicz–Brunn–Minkowski theory emerged in a series of papers [18, 41, 42]. The Orlicz Minkowski problem is a natural and important task to be considered. Haberl, Lutwak, Yang and Zhang [18] first proposed the following Orlicz Minkowski problem: what are necessary and sufficient conditions for a Borel measure μ on S^{n-1} to be the Orlicz surface area of a convex body; i.e., given a suitable continuous function $\varphi : (0, +\infty) \to (0, +\infty)$, is there a convex body K such that for some c

$$c\varphi(h_K)\, dS_K = d\mu?$$

In the case that $\varphi(t) = t^{1-p}$ $(p \neq n)$, this problem reduces to L_p Minkowski problem.

Under some suitable conditions on φ, the even Orlicz Minkowski problem was solved by Haberl, Lutwak, Yang and Zhang in [18]. One of their results [18, Theorem 2] is the following: Suppose $\varphi : (0, \infty) \to (0, \infty)$ is a continuous function such that $\phi(t) = \int_o^t \frac{1}{\varphi(s)}\, ds$ exists for every positive t and is unbounded as $t \to \infty$, and μ is an even finite Borel measure on S^{n-1} which is not concentrated on a great subsphere of S^{n-1}, then there exists an origin symmetric convex body $K \subset \mathbb{R}^n$ and $c > 0$ such that $c\varphi(h_K)\, dS_K = d\mu$ and $\|h_K\|_{\phi,\mu} = 1$ simultaneously, where $\|h_K\|_{\phi,\mu}$ is the Orlicz norm of the support function h_K with respect to μ.

The main purpose of this paper is to provide a solution of the general Orlicz Minkowski problem without assuming that μ is an even measure. But besides the assumptions on φ in [18], we have to assume that $\varphi(s)$ tends to infinity as $s \to 0^+$. In order to serve this purpose, we first solve the Orlicz Minkowski problem for discrete measures, that is, if μ is a discrete measure then the solution body K is a polytope (Theorem 1.1). The proof relies on techniques developed by Haberl, Hug, Lutwak, Yang, and Zhang in [18] and [20], but it is not just an immediate generalization. Next, we give a solution of the Orlicz Minkowski problem for general measures (Theorem 1.2), by means of an approximation argument.

We denote by $\mathcal{K}^n$ the set of convex bodies in $\mathbb{R}^n$, $\mathcal{P}_0^n$ the set of convex polytopes in $\mathbb{R}^n$ which contain the origin in their interiors, $h(P, \cdot)$ the support function of P, $S(P, \cdot)$ the surface area measure of P, δ_{u_i} the probability measure with unit point mass at u_i. One can consult Sects. 2 and 3 for more details.

Our main result may be formulated as follows:

Theorem 1.1 (Discrete measures) *Suppose* $\varphi : (0,\infty) \to (0,\infty)$ *is a continuous function such that* $\phi(t) = \int_o^t \frac{1}{\varphi(s)}\,ds$ *exists for every positive* t *and is unbounded as* $t \to \infty$, *and* $\varphi(s)$ *tends to infinity as* $s \to 0^+$. *Let vectors* $u_1, \dots, u_m \in S^{n-1}$ *that are not contained in a closed hemisphere and real numbers* $\alpha_1, \dots, \alpha_m > 0$ *be given. Then, there exist a polytope* $P \in \mathcal{P}_0^n$ *and* $c > 0$ *such that*

$$\sum_{i=1}^{m} \alpha_i \delta_{u_i} = c\varphi\big(h(P,\cdot)\big) S(P,\cdot), \tag{1}$$

$$\|h_P\|_{\phi,\mu} = 1. \tag{2}$$

Theorem 1.2 (General measures) *Suppose* $\varphi : (0,\infty) \to (0,\infty)$ *is a continuous function such that* $\phi(t) = \int_o^t \frac{1}{\varphi(s)}\,ds$ *exists for every positive* t *and is unbounded as* $t \to \infty$, *and* $\varphi(s)$ *tends to infinity as* $s \to 0^+$. *Let* μ *be a finite Borel measure on* S^{n-1} *whose support is not contained in a closed hemisphere. Then there exist a convex body* $K \in \mathcal{K}^n$ *with* $0 \in K$ *and* $c > 0$ *such that*

$$\frac{d\mu}{\varphi(h(K,\cdot))} = c\,dS(K,\cdot), \tag{3}$$

$$\|h_K\|_{\phi,\mu} = 1. \tag{4}$$

However, just as the case of even measures, the uniqueness part of the Orlicz Minkowski problem for general measures cannot also be solved. Hence, this problem is still open and seems to be very difficult.

This paper is organized as follows: In Sect. 2 we list for quick reference some basic facts regarding convex bodies. In Sect. 3 we study some properties of the Orlicz norm on the basis of [18]. The proofs of Theorem 1.1 and Theorem 1.2 are presented in Sects. 4 and 5, respectively.

2 Background and Notation

In this section we present the terminology and notation we shall use throughout. For general reference the reader may wish to consult the books of Gardner [11], Gruber [12], and Schneider [49].

For $x, y \in \mathbb{R}^n$, we denote their inner product by $x \cdot y$ and the Euclidean norm of x by $|x| = \sqrt{x \cdot x}$. The unit sphere $\{x \in \mathbb{R}^n : |x| = 1\}$ is denoted by S^{n-1} and the unit ball $\{x \in \mathbb{R}^n : |x| \le 1\}$ by B_2^n. For $u \in S^{n-1}$, let $H_{u,t}^- := \{y \in \mathbb{R}^n : u \cdot y \le t\}$ denote the halfspace with exterior normal vector u and the distance $t \ge 0$ from the origin.

Let V stand for n-dimensional Lebesgue measure, and $|\mu| := \mu(S^{n-1})$ for a finite Borel measure μ on S^{n-1}. We write $C(S^{n-1})$ for the set of continuous functions on S^{n-1} which will always be viewed as equipped with the max-norm metric:

$$\|f - g\|_\infty = \max_{u \in S^{n-1}} \big|f(u) - g(u)\big|, \tag{5}$$

for $f, g \in C(S^{n-1})$.

A convex body is a compact convex set of $\mathbb{R}^n$ with nonempty interior. Let $\mathcal{K}^n$ denote the set of convex bodies in $\mathbb{R}^n$ endowed with the Hausdorff metric, and $\mathcal{P}^n$ the subset of convex polytopes. We write $\mathcal{K}_0^n$ for the set of convex bodies containing the origin in their interiors, and $\mathcal{P}_0^n := \mathcal{P}^n \cap \mathcal{K}_0^n$.

For $K \in \mathcal{K}^n$, let $h_K = h(K, \cdot) : \mathbb{R}^n \to \mathbb{R}$ denote the support function of K; i.e., for any $x \in \mathbb{R}^n$, $h_K(x) = h(K, x) = \max\{x \cdot y : y \in K\}$. It is easy to show that the support function of the line segment $\hat{v}$ joining the points $0, v \in \mathbb{R}^n$ is given by

$$h_{\hat{v}}(u) = (u \cdot v)_+ = \max\{u \cdot v, 0\} = \frac{|u \cdot v| + u \cdot v}{2}, \quad u \in \mathbb{R}^n. \tag{6}$$

We shall require the obvious facts that for compact, convex $K, L \subset \mathbb{R}^n$,

$$K \subset L \quad \text{if and only if} \quad h_K \le h_L, \tag{7}$$

and that for $c > 0$ and $x \in \mathbb{R}^n$,

$$h_{cK}(x) = c h_K(x) \quad \text{and} \quad h_K(cx) = c h_K(x). \tag{8}$$

If $K_i \in \mathcal{K}^n$, we say that $K_i \to K \in \mathcal{K}^n$ in the Hausdorff metric provided

$$\|h_{K_i} - h_K\|_\infty := \max_{u \in S^{n-1}} \left|h_{K_i}(u) - h_K(u)\right| \to 0. \tag{9}$$

For a Borel set $\omega \subset S^{n-1}$, the surface area measure $S_K(\omega) = S(K, \omega)$ of the convex body K is the $(n-1)$-dimensional Hausdorff measure of the set of all boundary points of K for which there exists a normal vector of K belonging to ω. Observe that for the surface area measure of cK we have

$$S_{cK} = c^{n-1} S_K, \quad c > 0. \tag{10}$$

We will use the fact that S_K is weakly continuous in K [49]; i.e., for $K_i \in \mathcal{K}^n$,

$$K_i \to K \in \mathcal{K}^n \quad \Longrightarrow \quad S_{K_i} \to S_K, \quad \text{weakly, as} \quad i \to +\infty. \tag{11}$$

For $K, L \in \mathcal{K}^n$, the mixed volume $V_1(K, L)$ may be defined by

$$V_1(K, L) = \frac{1}{n} \int_{S^{n-1}} h_L \, dS_K. \tag{12}$$

In particular, for $K \in \mathcal{K}^n$,

$$V_1(K, K) = V(K),$$

or equivalently,

$$V(K) = \frac{1}{n} \int_{S^{n-1}} h_K \, dS_K. \tag{13}$$

Minkowski's inequality states that for convex bodies K, L:

$$V_1(K, L)^n \ge V(K)^{n-1} V(L) \tag{14}$$

with equality if and only if K and L are homothetic.

We will use the following simple fact:

Let $f, f_1, \ldots \in C(S^{n-1})$, $\mu, \mu_1, \ldots$ be finite measures on S^{n-1}. If $f_i \to f$ uniformly on S^{n-1}, and $\mu_i \to \mu$ weakly on S^{n-1}, then

$$\lim_{i\to+\infty} \int_{S^{n-1}} f_i(u)\,d\mu_i(u) = \int_{S^{n-1}} f(u)\,d\mu(u). \tag{15}$$

3 Orlicz Norms

Definition 3.1 [18] Let $\phi : [0,\infty) \to [0,\infty)$ be continuous, strictly increasing, continuously differentiable on $(0,\infty)$ with positive derivative, and satisfy $\lim_{t\to\infty} \phi(t) = \infty$. Let μ be a finite Borel measure on the sphere S^{n-1}. For a continuous function $f : S^{n-1} \to [0,\infty)$, the Orlicz norm $\|f\|_{\phi,\mu}$ is defined by

$$\|f\|_{\phi,\mu} = \inf\left\{\lambda > 0 : \frac{1}{|\mu|}\int_{S^{n-1}} \phi\left(\frac{f}{\lambda}\right) d\mu \le \phi(1)\right\}. \tag{16}$$

As in [18], the Orlicz norm of a function f not only depends on μ but also depends on ϕ. The usual L_p norm is obtained by taking $\phi(t) = t^p$.

As was shown in [18], the following properties for continuous $f : S^{n-1} \to [0,\infty)$ hold:

$$\|cf\|_{\phi,\mu} = c\|f\|_{\phi,\mu}, \quad c > 0. \tag{17}$$

In particular

$$\|c\|_{\phi,\mu} = c, \quad c > 0. \tag{18}$$

Moreover, the monotonicity of ϕ guarantees that for continuous $f, g : S^{n-1} \to [0,\infty)$,

$$f \le g \quad\Longrightarrow\quad \|f\|_{\phi,\mu} \le \|g\|_{\phi,\mu}. \tag{19}$$

Lemma 3.2 [18, Lemma 3] *Suppose μ is a finite Borel measure on S^{n-1} and the function $f : S^{n-1} \to [0,\infty)$ is continuous and such that $\mu(\{f \ne 0\}) > 0$. Then the Orlicz norm $\|f\|_{\phi,\mu}$ is positive and*

$$\|f\|_{\phi,\mu} = \lambda_0 \quad\Longleftrightarrow\quad \frac{1}{|\mu|}\int_{S^{n-1}} \phi\left(\frac{f}{\lambda_0}\right) d\mu = \phi(1).$$

Lemma 3.3 [18, Lemma 4] *Suppose $f : S^{n-1} \to [0,\infty)$ is a continuous function with $\mu(\{f \ne 0\}) > 0$ and $\{f_i\}$ is a sequence of nonnegative functions in $C(S^{n-1})$ with $\mu(\{f_i \ne 0\}) > 0$. If*

$$f_i \to f \quad \text{in } C\big(S^{n-1}\big),$$

then

$$\|f_i\|_{\phi,\mu} \to \|f\|_{\phi,\mu}.$$

Lemma 3.4 *Suppose $f : S^{n-1} \to [0,\infty)$ is a continuous function with $\mu(\{f \neq 0\}) > 0$ and $\{\mu_i\}$ is a sequence of finite Borel measures on S^{n-1}. If*

$$\mu_i \to \mu \quad \text{weakly on } S^{n-1}, \tag{20}$$

then

$$\|f\|_{\phi,\mu_i} \to \|f\|_{\phi,\mu}. \tag{21}$$

Proof First, we will show that the sequence $\{\|f\|_{\phi,\mu_i}\}$ is bounded. The fact $f \in C(S^{n-1})$ guarantees that there exists a real $C > 0$ such that $f(u) \leq C$ for all $u \in S^{n-1}$. Together with (18) and (19), it follows that

$$0 \leq \|f\|_{\phi,\mu_i} \leq \|C\|_{\phi,\mu_i} = C$$

for all Borel measures μ_i on S^{n-1}, $i \in \mathbb{N}$. Thus the boundedness of the sequence $\{\|f\|_{\phi,\mu_i}\}$ is established.

The Bolzano–Weierstrass theorem guarantees that the sequence $\{\|f\|_{\phi,\mu_i}\}$ has a convergent subsequence. In order to show that the sequence $\{\|f\|_{\phi,\mu_i}\}$ converges to $\|f\|_{\phi,\mu}$, it suffices to prove that every convergent subsequence converges to $\|f\|_{\phi,\mu}$. To simplify the notation, we denote an arbitrary convergent subsequence of $\{\|f\|_{\phi,\mu_i}\}$ by $\{\|f\|_{\phi,\mu_i}\}$ as well.

Next, we will show $\lim_{i\to\infty} \|f\|_{\phi,\mu_i} > 0$. Suppose it is not true; i.e., $\|f\|_{\phi,\mu_i} \to 0$. The condition $\mu(\{f > 0\}) > 0$ yields the result that there exists a sufficiently small $c > 0$ such that $\mu(\{f \geq c\}) > 0$. Given $M > 0$, there exists a sufficiently small $\delta > 0$ such that $\phi(\frac{c}{\delta}) > \frac{M|\mu|}{\mu(\{f\geq c\})}$, since ϕ is strictly increasing and $\lim_{t\to\infty} \phi(t) = \infty$. Then the assumption $\lim_{i\to\infty} \|f\|_{\phi,\mu_i} = 0$ implies there exists N_1 such that $\|f\|_{\phi,\mu_i} < \delta$ whenever $i > N_1$ for the above given δ. Thus $\phi(\frac{f}{\|f\|_{\phi,\mu_i}}) \geq \phi(\frac{f}{\delta})$ for $i > N_1$. Note that since $\mu(\{f \neq 0\}) > 0$, $f \in C(S^{n-1})$ and $\mu_i \to \mu$ weakly on S^{n-1}, there exists N_2 such that $\mu_i(\{f \neq 0\}) > 0$ whenever $i > N_2$. From Lemma 3.2, it follows that

$$\|f\|_{\phi,\mu_i} > 0, \tag{22}$$

and

$$\frac{1}{|\mu_i|}\int_{S^{n-1}} \phi\left(\frac{f}{\|f\|_{\phi,\mu_i}}\right) d\mu_i = \phi(1), \tag{23}$$

whenever $i > N_2$.

We therefore deduce

$$\begin{aligned}
\phi(1) &= \lim_{i\to\infty} \frac{1}{|\mu_i|}\int_{S^{n-1}} \phi\left(\frac{f}{\|f\|_{\phi,\mu_i}}\right) d\mu_i \\
&\geq \liminf_{i\to\infty} \frac{1}{|\mu_i|}\int_{S^{n-1}} \phi\left(\frac{f}{\delta}\right) d\mu_i \\
&= \frac{1}{|\mu|}\int_{S^{n-1}} \phi\left(\frac{f}{\delta}\right) d\mu \quad \text{(by (20))}
\end{aligned}$$

$$\geq \frac{1}{|\mu|}\int_{\{f\geq c\}}\phi\left(\frac{f}{\delta}\right)d\mu$$
$$\geq \phi\left(\frac{c}{\delta}\right)\frac{1}{|\mu|}\int_{\{f\geq c\}}d\mu$$
$$> \frac{M|\mu|}{\mu(\{f\geq c\})}\cdot\frac{\mu(\{f\geq c\})}{|\mu|}=M,$$

since $M>0$ is arbitrary, this is a contradiction.

Since $\lim_{i\to\infty}\|f\|_{\phi,\mu_i}>0$, by (22), there exists a real $c>0$ such that $\|f\|_{\phi,\mu_i}>c$ whenever $i>N_2$. Let $\lambda_i:=\|f\|_{\phi,\mu_i}$, then we may rewrite (23) as

$$\int_{S^{n-1}}\phi\left(\frac{f}{\lambda_i}\right)d\bar{\mu}_i=\phi(1), \tag{24}$$

where $d\bar{\mu}_i=d\mu_i/|\mu_i|$, for $i>N_2$.

Suppose that for the subsequence $\{\lambda_i\}$ we have $\lambda_i\to\lambda$.

Finally, we prove $\lambda=\|f\|_{\phi,\mu}$. Observe that

$$\left|\frac{f(u)}{\lambda_i}-\frac{f(u)}{\lambda}\right|=\frac{f(u)|\lambda_i-\lambda|}{\lambda_i\lambda}\leq\frac{C|\lambda_i-\lambda|}{c^2}\quad\text{whenever } i>N_2,$$

which means that $f(u)/\lambda_i$ uniformly converges to $f(u)/\lambda$ for all $u\in S^{n-1}$. Note that $|f/\lambda_i|\leq C/c$ for $i>N_2$, and ϕ is uniformly continuous on $[0,C/c]$, hence the function $\phi(f(u)/\lambda_i)$ uniformly converges to $\phi(f(u)/\lambda)$ on S^{n-1}. Moreover, $\mu_i\to\mu$ weakly on S^{n-1} implies $\bar{\mu}_i\to\bar{\mu}$ weakly on S^{n-1}, by (15), we obtain from (24) that

$$\int_{S^{n-1}}\phi\left(\frac{f}{\lambda}\right)d\bar{\mu}=\phi(1).$$

Then, Lemma 3.2 again yields the desired result. □

Corollary 3.5 *Suppose* $f:S^{n-1}\to[0,\infty)$ *is a continuous function with* $\mu(\{f\neq 0\})>0$, $\{f_i\}$ *is a sequence of nonnegative functions in* $C(S^{n-1})$ *and* $\{\mu_i\}$ *is a sequence of finite Borel measures on* S^{n-1}. *If*

$$f_i\to f\quad \text{in } C\left(S^{n-1}\right),$$

and

$$\mu_i\to\mu\quad \text{weakly on } S^{n-1},$$

then

$$\|f_i\|_{\phi,\mu_i}\to\|f\|_{\phi,\mu}.$$

Proof Let $\varepsilon>0$ be given, by Lemma 3.3, there exists a sufficiently small $\delta>0$ such that

$$\left|\|f+\delta\|_{\phi,\mu}-\|f\|_{\phi,\mu}\right|<\frac{\varepsilon}{2} \tag{25}$$

and

$$\big|\|f_0\|_{\phi,\mu} - \|f\|_{\phi,\mu}\big| < \frac{\varepsilon}{2}, \tag{26}$$

where $f_0 := \max\{f - \delta, 0\}$ satisfies $\mu(f_0 > 0) > 0$.

Evidently, $f_0 : S^{n-1} \to [0, \infty)$ is a continuous function. Since $f_i \to f \in C(S^{n-1})$, there exists N_1 such that $f_0 \le f_i \le f + \delta$ for $i \ge N_1$. Using (19), we obtain $\|f_0\|_{\phi,\mu_i} \le \|f_i\|_{\phi,\mu_i} \le \|f + \delta\|_{\phi,\mu_i}$ for $i \ge N_1$.

We now apply Lemma 3.4 to the function $f + \delta$: there exists N_2 such that

$$\big|\|f + \delta\|_{\phi,\mu_i} - \|f + \delta\|_{\phi,\mu}\big| < \frac{\varepsilon}{2}, \quad \text{whenever } i \ge N_2. \tag{27}$$

Similarly, there exists N_3 such that

$$\big|\|f_0\|_{\phi,\mu_i} - \|f_0\|_{\phi,\mu}\big| < \frac{\varepsilon}{2} \quad \text{whenever } i \ge N_3. \tag{28}$$

For $N = \max\{N_1, N_2, N_3\}$, the inequalities (25) and (27) yield

$$\big|\|f + \delta\|_{\phi,\mu_i} - \|f\|_{\phi,\mu}\big| < \varepsilon \quad \text{whenever } i \ge N.$$

Similarly, the inequalities (26) and (28) yield

$$\big|\|f_0\|_{\phi,\mu_i} - \|f\|_{\phi,\mu}\big| < \varepsilon \quad \text{whenever } i \ge N.$$

Since $\|f_0\|_{\phi,\mu_i} \le \|f_i\|_{\phi,\mu_i} \le \|f + \delta\|_{\phi,\mu_i}$, the two inequalities above give

$$\big|\|f_i\|_{\phi,\mu_i} - \|f\|_{\phi,\mu}\big| < \varepsilon$$

whenever $i > N$. □

Recall (6) that $h_{\hat{v}}(u) = (u \cdot v)_+ = \frac{|u\cdot v| + u\cdot v}{2}$.

Lemma 3.6 *If μ is a finite Borel measure on the sphere S^{n-1} whose support is not contained in a closed hemisphere, then there exists a real $c > 0$ such that $\|h_{\hat{v}}\|_{\phi,\mu} \ge c$ for every $v \in S^{n-1}$.*

Proof Since the support of μ is not contained in a closed hemisphere of S^{n-1}

$$\mu\big(\{h_{\hat{v}} > 0\}\big) = \mu\big(S^{n-1} \setminus H^-_{v,0}\big) > 0 \quad \text{for every } v \in S^{n-1}.$$

According to Lemma 3.2, we have $\|h_{\hat{v}}\|_{\phi,\mu} > 0$. Since S^{n-1} is compact, it suffices to show that the function $v \mapsto \|h_{\hat{v}}\|_{\phi,\mu}$ is continuous.

Suppose $v_i \in S^{n-1}$ and $v_i \to v$ as $i \to +\infty$. Note that $h_{\hat{v}}(u) = \frac{|u\cdot v| + u\cdot v}{2}$ implies $h_{\hat{v_i}} \to h_{\hat{v}}$ uniformly on S^{n-1}, and thus $\|h_{\hat{v_i}}\|_{\phi,\mu}$ converges to $\|h_{\hat{v}}\|_{\phi,\mu}$ by Lemma 3.3. So the desired continuity of $v \mapsto \|h_{\hat{v}}\|_{\phi,\mu}$ is established. □

Corollary 3.7 *If a sequence of finite discrete measures $\{\mu_i\}$ weakly converges to a finite Borel measure μ on S^{n-1}, and the supports of μ_i and μ are all not contained in a closed hemisphere, $i \in \mathbb{N}$. Then, there exists a real $c > 0$ such that $\|h_{\hat{v}}\|_{\phi,\mu_i} \geq c$ for every $v \in S^{n-1}$ and $i \in \mathbb{N}$.*

Proof Recall that a finite discrete measure μ_i on S^{n-1} is a Borel measure. Observe that $h_{\hat{v}} \leq 1$, combining (19) and (18), we have $\|h_{\hat{v}}\|_{\phi,\mu} \leq 1$ for any Borel measure μ. From Lemma 3.6, for each i, let

$$c_i = \min_{v \in S^{n-1}} \|h_{\hat{v}}\|_{\phi,\mu_i} = \|h_{\widehat{v_i}}\|_{\phi,\mu_i} > 0, \tag{29}$$

where $v_i \in S^{n-1}$ is any point where this minimum is attained. Similarly, there exists $c_0 > 0$ such that $\|h_{\hat{v}}\|_{\phi,\mu} \geq c_0$ for all $v \in S^{n-1}$. The compactness of $[0, 1]$ guarantees that the sequence $\{c_i\}$ has a convergent subsequence.

In order to prove this corollary, it suffices to show that no convergent subsequence of the sequence $\{c_i\}$ converges to 0. We argue by contradiction, assume there exists a subsequence $\{c_{i'}\}$ of the sequence $\{c_i\}$ such that $c_{i'} \to 0$. From (29), for the subsequence $\{c_{i'}\}$ we can write $c_{i'} = \|h_{\widehat{v_{i'}}}\|_{\phi,\mu_{i'}}$ for some $v_{i'} \in S^{n-1}$ as well. Since S^{n-1} is compact, there exists a subsequence $\{v_{i''}\}$ of the sequence $\{v_{i'}\}$ such that $v_{i''}$ converges to $v_0 \in S^{n-1}$. Thus we obtain a subsequence $\{c_{i''}\}$ of the sequence $\{c_{i'}\}$ such that $c_{i''}$ converges to 0 by the assumption, where $c_{i''} = \|h_{\widehat{v_{i''}}}\|_{\phi,\mu_{i''}}$. Meanwhile, $h_{\widehat{v_{i''}}}$ converges to $h_{\hat{v}}$ uniformly on S^{n-1} as shown in Lemma 3.6. By Corollary 3.5, we see that $c_{i''} = \|h_{\widehat{v_{i''}}}\|_{\phi,\mu_{i''}} \to \|h_{\widehat{v_0}}\|_{\phi,\mu}$. Note that $\|h_{\hat{v}}\|_{\phi,\mu} \geq c_0$ for all $v \in S^{n-1}$, while $c_{i''} = \|h_{\widehat{v_{i''}}}\|_{\phi,\mu_{i''}} \to 0$, this is the desired contradiction. □

4 The Orlicz Minkowski Problem for Discrete Measures

This section is devoted to the proof of Theorem 1.1.

Lemma 4.1 *Suppose $\varphi : (0, \infty) \to (0, \infty)$ is a continuous function such that $\phi(t) = \int_o^t \frac{1}{\varphi(s)}\, ds$ exists for every positive t and is unbounded as $t \to \infty$, and $\varphi(s)$ tends to infinity as $s \to 0^+$. Then,*

(i) $\lim_{t\to 0^+} \phi(t) = 0$, $\lim_{t\to 0^+} \phi'(t) = 0$.
(ii) *Let $\alpha(t) := \frac{1}{\varphi(t)}$, $\beta(t) := \frac{t}{\varphi(t)}$, both of them are continuous on $(0, \infty)$. Moreover, we can extend the domain of $\alpha(t)$, $\beta(t)$ right continuously to the origin; i.e., $\alpha(0) = \lim_{t\to 0^+} \alpha(t) = 0$, $\beta(0) = \lim_{t\to 0^+} \beta(t) = 0$.*
(iii) $\lim_{t\to 0^+} (\phi^{-1})'(\phi(h) - b\phi(t))$ *is bounded from above and below by positive reals, for any given $b, h > 0$.*

Proof First, we extend the domain of ϕ to $[0, \infty)$ by

$$\phi(t) = \int_o^t \frac{1}{\varphi(s)}\, ds, \quad \text{for } t > 0, \quad \text{and} \quad \phi(0) = \lim_{t\to 0^+} \phi(t).$$

Indeed, (i) and (ii) are easily verified.

Next, we will show (iii).

Note that the function ϕ is strictly increasing and continuously differentiable on $(0,\infty)$, and $\phi' > 0$. Since $\lim_{t\to 0^+}\phi(t) = 0$ and $\lim_{t\to\infty}\phi(t) = \infty$, ϕ has an inverse $\phi^{-1} : [0,\infty) \to [0,\infty)$ which is continuously differentiable on $(0,\infty)$.

Hence for any given $b, h > 0$, there exists $\delta > 0$ such that

$$\phi(h) - b\phi(t) \in \phi\left(\left(\frac{h}{2}, h\right)\right)$$

for all $t \in (0,\delta)$. Obviously, $\phi((\frac{h}{2}, h)) \subset (0,\infty)$, thus $(\phi^{-1})'(\phi(h) - b\phi(t))$ exists for a sufficiently small t. Observe $(\phi^{-1})'(\phi(s)) = \frac{1}{\phi'(s)} = \varphi(s)$ when $s \in (\frac{h}{2}, h)$, and the function φ is continuous on $(0,\infty)$, thus $\lim_{t\to 0^+}(\phi^{-1})'(\phi(h) - b\phi(t))$ is bounded from above and below by positive reals. □

By this lemma we can assume in the proofs of Theorems 1.1 and 1.2 that the domains of $\phi(t)$, $\alpha(t)$, and $\beta(t)$ contain zero; i.e., $\phi(0) = \lim_{t\to 0^+}\phi(t) = 0$, $\alpha(0) = \lim_{t\to 0^+}\alpha(t) = 0$, and $\beta(0) = \lim_{t\to 0^+}\beta(t) = 0$.

In the following, we denote by $\mathbb{R}^m_+$ the set of all $x = (x_1,\dots,x_m) \in \mathbb{R}^m$ with positive components. Recall that $H^-_{u,t} := \{y \in \mathbb{R}^n : u \cdot y \le t\}$ is the halfspace with exterior normal vector u and the distance $t \ge 0$ from the origin.

Lemma 4.2 [20, Lemma 3.2] *Let $u_1,\dots,u_m \in S^{n-1}$ be pairwise distinct vectors which are not contained in a closed hemisphere. For $x \in \mathbb{R}^m_+$, let $P(x) := \bigcap_{i=1}^m H^-_{u_i,x_i}$. Then $V(P(x))$ is of class C^1 and $\partial_i V(P(x)) = S(P(x),\{u_i\})$ for $i = 1,\dots,m$.*

With these lemmas in hand, we can complete the proof of Theorem 1.1.

Proof of Theorem 1.1 Let $\mathbb{R}^m_*$ be the set of all $x = (x_1,\dots,x_m) \in \mathbb{R}^m$ with nonnegative components, define the set

$$M := \left\{x \in \mathbb{R}^m_* : \sum_{i=1}^m \alpha_i\phi(x_i) = \sum_{i=1}^m \alpha_i\phi(1)\right\}.$$

Since ϕ is strictly increasing, the surface M is compact. For $x \in M$, we define $P(x)$ as the convex polytope

$$P(x) := \bigcap_{i=1}^m H^-_{u_i,x_i} = \bigcap_{i=1}^m H^-_{u_i,h_i^x}, \tag{30}$$

where $h_i^x := h(P(x),u_i)$ for $i = 1,\dots,m$.

Since M is compact and the function $x \mapsto V(P(x))$ is continuous, there is a point $z \in M$ such that

$$V\big(P(x)\big) \le V\big(P(z)\big) \quad \text{for all } x \in M.$$

Note that if we set $x^* = (1, \dots, 1) \in \mathbb{R}^m$, then clearly $x^* \in M$. Since $0 < V(B_2^n) \le V(P(x^*)) \le V(P(z))$, this gives $P(z) \in \mathcal{P}^n$.

We will prove that $P(z)$ is the desired polytope.

First, we want to show

$$0 \in \operatorname{int}\big(P(z)\big). \tag{31}$$

Observe for any $x \in M$, $0 \in P(x)$, that is, either $0 \in \operatorname{bd}(P(z))$ or $0 \in \operatorname{int}(P(z))$. Recall that $h_i^z := h(P(z), u_i)$ for $i = 1, \dots, m$. Suppose $0 \in \operatorname{bd}(P(z))$, so there exist $h_1^z = \dots = h_k^z = 0$ and $h_{k+1}^z, \dots, h_m^z > 0$ for some $1 \le k < m$. We shall get a contradiction by showing that under this assumption there is some $z_t \in M$ such that $V(P(z_t)) > V(P(z))$, which contradicts the definition of z. For a small $t > 0$, we define

$$\begin{aligned} z_t := \big(&\phi^{-1}\big(\phi(z_1) + \phi(t)\big), \dots, \phi^{-1}\big(\phi(z_k) + \phi(t)\big), \\ &\phi^{-1}\big(\phi(z_{k+1}) - \alpha\phi(t)\big), \dots, \phi^{-1}\big(\phi(z_m) - \alpha\phi(t)\big)\big), \end{aligned}$$

where

$$\alpha := \frac{\sum_{i=1}^k \alpha_i}{\sum_{i=k+1}^m \alpha_i}.$$

Obviously, $z_t \in M$ if $t > 0$ is sufficiently small.

Analogously, for a small $t > 0$, we define

$$\begin{aligned} h_t^z := \big(&\phi^{-1}\big(\phi\big(h_1^z\big) + \phi(t)\big), \dots, \phi^{-1}\big(\phi\big(h_k^z\big) + \phi(t)\big), \\ &\phi^{-1}\big(\phi\big(h_{k+1}^z\big) - \alpha\phi(t)\big), \dots, \phi^{-1}\big(\phi\big(h_m^z\big) - \alpha\phi(t)\big)\big). \end{aligned}$$

Since $\phi(0) = 0$, by (30), we have

$$P\big(h_t^z\big) := \bigcap_{i=1}^k H^-_{u_i,t} \cap \bigcap_{i=k+1}^m H^-_{u_i,\phi^{-1}(\phi(h_i^z)-\alpha\phi(t))}.$$

A glance at (30) again shows $P(h_0^z) = P(z)$. Since $h_i^z \le z_i$ for all $i \le m$ and the function ϕ is strictly increasing, $P(h_t^z) \subset P(z_t)$ and $0 \in \operatorname{int}(P(h_t^z))$, if $t > 0$ is sufficiently small. For simplicity, we write

$$S_i := S\big(P(z), u_i\big) \quad \text{and} \quad S_i^t := S\big(P\big(h_t^z\big), u_i\big),$$

and thus for the n-dimensional polytope $P(h_t^z)$, $P(z)$, according to the discrete form of (13) and (12), it follows that

$$nV\big(P\big(h_t^z\big)\big) = t\sum_{i=1}^k S_i^t + \sum_{i=k+1}^m \phi^{-1}\big(\phi\big(h_i^z\big) - \alpha\phi(t)\big)S_i^t,$$

and

$$nV_1\big(P\big(h_t^z\big), P(z)\big) = 0\sum_{i=1}^k S_i^t + \sum_{i=k+1}^m h_i^z S_i^t.$$

Note that each open set intersecting $P(z)$ intersects $P(h_t^z)$ if $t>0$ is sufficiently small and each closed set having empty intersection with $P(z)$ has empty intersection with $P(h_t^z)$ if $t>0$ is sufficiently small. Hence $P(h_t^z)\to P(z)$ as $t\to 0^+$ [49, p. 57], and (11) gives $S_i^t\to S_i$ as $t\to 0^+$. Now, we conclude that

$$\begin{aligned}&\lim_{t\to 0^+}\frac{V(P(h_t^z))-V_1(P(h_t^z),P(z))}{t}\\&\quad=\frac{1}{n}\lim_{t\to 0^+}\left(\sum_{i=1}^{k}\frac{t-0}{t}S_i^t+\sum_{i=k+1}^{m}\frac{\phi^{-1}(\phi(h_i^z)-\alpha\phi(t))-h_i^z}{t}S_i^t\right)\\&\quad=\frac{1}{n}\sum_{i=1}^{k}S_i>0.\end{aligned}$$

Since $\lim_{t\to 0^+}\phi'(t)=0$ and $\lim_{t\to 0^+}(\phi^{-1})'(\phi(h_i^z)-\alpha\phi(t))$ is bounded by Lemma 4.1

$$\lim_{t\to 0^+}\frac{\phi^{-1}(\phi(h_i^z)-\alpha\phi(t))-h_i^z}{t}=\lim_{t\to 0^+}\bigl(\phi^{-1}\bigr)'\bigl(\phi\bigl(h_i^z\bigr)-\alpha\phi(t)\bigr)\bigl(-\alpha\phi'(t)\bigr)=0,$$

thus the last equality is obtained. By Minkowski's inequality (14) and $P(h_t^z)\to P(z)$ as $t\to 0^+$, we obtain

$$\begin{aligned}0&<\lim_{t\to 0^+}\frac{V(P(h_t^z))-V_1(P(h_t^z),P(z))}{t}\\&\le\liminf_{t\to 0^+}\frac{V(P(h_t^z))-V(P(h_t^z))^{1-\frac{1}{n}}V(P(z))^{\frac{1}{n}}}{t}\\&=V\bigl(P(z)\bigr)^{1-\frac{1}{n}}\liminf_{t\to 0^+}\frac{V(P(h_t^z))^{\frac{1}{n}}-V(P(z))^{\frac{1}{n}}}{t}.\end{aligned}$$

Consequently, $V(P(h_t^z))>V(P(z))$ if $t>0$ is sufficiently small. However, since $P(h_t^z)\subset P(z_t)$, the required contradiction $V(P(z_t))>V(P(z))$ follows.

Next, we will prove the main conclusions (1) and (2). From (31), we have

$$z\in M_+:=\left\{x\in\mathbb{R}_+^m:\sum_{i=1}^{m}\alpha_i\phi(x_i)=\sum_{i=1}^{m}\alpha_i\phi(1)\right\}.$$

By the Lagrange multiplier rule there is some $\lambda\in\mathbb{R}$ such that

$$\nabla V\bigl(P(z)\bigr)=\lambda\nabla\left(\sum_{i=1}^{m}\alpha_i\phi(z_i)-\sum_{i=1}^{m}\alpha_i\phi(1)\right),$$

where $V(P(z))$ is differentiable by Lemma 4.2, and $\phi'(z_i)$ exists since $z_i>0$ for all $i=1,\dots,m$. Thus

$$S_i=\lambda\frac{\alpha_i}{\varphi(z_i)},\quad i=1,\dots,m.\tag{32}$$

The fact that $S_i > 0$ for some $i \in \{1, \dots, m\}$ and $\alpha_i, \varphi(z_i) > 0$ for all $i = 1, \dots, m$, shows $\lambda > 0$. Together with the above expression (32), it follows that $S_i > 0$ for all $i = 1, \dots, m$. Hence, $h(P(z), u_i) = z_i$ for all $i = 1, \dots, m$. By the discrete form of (13), and (32), it follows that

$$nV\big(P(z)\big) = \sum_{i=1}^{m} S_i z_i = \lambda \sum_{i=1}^{m} \frac{\alpha_i z_i}{\varphi(z_i)}.$$

Therefore, for $i = 1, \dots, m$,

$$S\big(P(z), u_i\big) = S_i = \frac{\alpha_i}{c\varphi(z_i)},$$

where $c = \frac{1}{nV(P(z))} \sum_{i=1}^{m} \frac{\alpha_i z_i}{\varphi(z_i)}$.

Indeed, together with $h(P(z), u_i) = z_i$, it follows that

$$\mu = \sum_{i=1}^{m} \alpha_i \delta_{u_i} = c\varphi\big(h\big(P(z), \cdot\big)\big) S\big(P(z), \cdot\big).$$

Observe the definition of M_+ and $h(P(z), u_i) = z_i$, hence

$$\sum_{i=1}^{m} \alpha_i \phi\big(h\big(P(z), u_i\big)\big) = \sum_{i=1}^{m} \alpha_i \phi(1).$$

Moreover, take $\mu = \sum_{i=1}^{m} \alpha_i \delta_{u_i}$ in Lemma 3.2 to conclude

$$\big\|h_{P(z)}\big\|_{\phi,\mu} = 1. \qquad \square$$

5 The Orlicz Minkowski Problem for General Measures

Proof of Theorem 1.2 As was shown in [49, Theorem 7.1.2], for a given Borel measure μ on S^{n-1} which is not concentrated in a closed hemisphere, one can construct a sequence of discrete measures $\{\mu_i\}$ on S^{n-1}, $i \in \mathbb{N}$, such that the support of μ_i is not contained in a closed hemisphere and $\mu_i \to \mu$ weakly as $i \to \infty$. By Theorem 1.1, for each $i \in \mathbb{N}$ there exists a polytope $P_i \in \mathcal{P}_0^n$ with

$$\mu_i = c_i \varphi\big(h(P_i, \cdot)\big) S(P_i, \cdot).$$

First, we claim that the sequence $\{P_i\}$ is bounded. For each i, let $v_i \in S^{n-1}$ be chosen such that $r_i v_i \in P_i$ with $|r_i v_i|$ maximal for suitable $r_i > 0$. Thus we derive $r_i h_{\hat{v}_i}(u) \le h_{P_i}(u)$ for all $u \in S^{n-1}$ from (8) and (7). Together with (17), (19), and (2), we see that

$$r_i \|h_{\hat{v}_i}\|_{\phi,\mu_i} = \|r_i h_{\hat{v}_i}\|_{\phi,\mu_i} \le \|h_{P_i}\|_{\phi,\mu_i} = 1. \tag{33}$$

By Corollary 3.7, there exists a real $c > 0$ such that $\|h_{\hat{v}}\|_{\phi,\mu_i} \geq c$ for every $v \in S^{n-1}$, $i \in \mathbb{N}$. It follows that the r_i's are bounded from above, and hence the sequence $\{P_i\}$ is bounded. We set $h_{P_i} \leq R$ for $i \in \mathbb{N}$.

Now Blaschke's selection theorem guarantees the existence of a convergent subsequence of $\{P_i\}$, which will also be denoted by $\{P_i\}$, with $\lim_{i\to\infty} P_i = K$. Now, $0 < V(B_2^n) \leq V(P_i)$ for all $i \in \mathbb{N}$ implies that $0 < V(B_2^n) \leq V(K)$, that is, $K \in \mathcal{K}^n$. Thus, from (9) it is clear that $h(P_i,\cdot) \to h(K,\cdot)$ uniformly on S^{n-1}. Since $0 \in \operatorname{int}(P_i)$ for all $i \in \mathbb{N}$, this gives $0 \in K$.

For a discrete measure μ_i, we rewrite $c_i = \frac{1}{nV(P_i)} \int_{S^{n-1}} \frac{h_{P_i}}{\varphi(h_{P_i})}\, d\mu_i$. We also set $c = \frac{1}{nV(K)} \int_{S^{n-1}} \frac{h_K}{\varphi(h_K)}\, d\mu$. From Lemma 4.1(ii) and $0 < h_{P_i} \leq R$, we find that $\beta(t) := \frac{t}{\varphi(t)}$ is uniformly continuous on $[0, R]$. Meanwhile, $h(P_i,\cdot) \to h(K,\cdot)$ uniformly on S^{n-1}, and it follows that $\frac{h(P_i,\cdot)}{\varphi(h(P_i,\cdot))}$ uniformly converges to $\frac{h(K,\cdot)}{\varphi(h(K,\cdot))}$ on S^{n-1}. Combining $V(P_i) \to V(K)$ and $\mu_i \to \mu$ weakly, as $i \to \infty$, by (15), we obtain $c_i \to c$ as $i \to \infty$.

Next, for a continuous function $f \in C(S^{n-1})$ and $i \in \mathbb{N}$, we have

$$\int_{S^{n-1}} \frac{f(u)}{\varphi(h(P_i,u))}\, d\mu_i(u) = c_i \int_{S^{n-1}} f(u)\, dS(P_i,u). \tag{34}$$

From Lemma 4.1(ii) and $0 < h_{P_i} \leq R$, we see that $\alpha(t) := \frac{1}{\varphi(t)}$ is uniformly continuous on $[0, R]$. Meanwhile, $h(P_i,\cdot) \to h(K,\cdot)$ uniformly on S^{n-1}, and it follows that $\frac{1}{\varphi(h(P_i,\cdot))}$ uniformly converges to $\frac{1}{\varphi(h(K,\cdot))}$ on S^{n-1}. Since $\mu_i \to \mu$ and $S(P_i,\cdot) \to S(K,\cdot)$ weakly by (11), as $i \to \infty$, using (15) again, it follows from (34) that

$$\int_{S^{n-1}} \frac{f(u)}{\varphi(h(K,u))}\, d\mu(u) = c \int_{S^{n-1}} f(u)\, dS(K,u). \tag{35}$$

The existence assertion (3) now follows, since (35) holds for any $f \in C(S^{n-1})$.

Moreover, since $\|h_{P_i}\|_{\phi,\mu_i} = 1$, and $h(P_i,\cdot) \to h(K,\cdot)$ uniformly on S^{n-1}, we obtain from Corollary 3.5 that

$$\|h_K\|_{\phi,\mu} = 1. \qquad \square$$

The following proof of Corollary 5.1 is similar to the one of [18, Corollary 2], which is known as the L_p-Minkowski problem. We include it for the sake of completeness.

Corollary 5.1 *If μ is a finite Borel measure on S^{n-1} which is not concentrated on a closed hemisphere, then*

(i) *for $p > 1$, there exists a convex body $K \in \mathcal{K}^n$ with $0 \in K$ such that*

$$h_K^{p-1}\, d\mu = c\, dS_K, \tag{36}$$

where $c = 1/V(K)$.

(ii) *for* $1 < p \neq n$, *there exists a convex body* $K \in \mathcal{K}^n$ *with* $0 \in K$ *such that*

$$h_K^{p-1}\, d\mu = dS_K. \tag{37}$$

Proof (i) Note that $\varphi(t) = t^{1-p}$ for $p > 1$ satisfies the requirements of Theorem 1.2. From (3) we have (36) and from (4) it follows that

$$\frac{1}{|\mu|}\int_{S^{n-1}} h_K^p\, d\mu = 1. \tag{38}$$

Using (13) and (3), we conclude

$$\int_{S^{n-1}} h_K^p\, d\mu = cnV(K). \tag{39}$$

From (38) and (39) it follows that $c = |\mu|/nV(K)$. Therefore

$$h_K^{p-1}\, d\mu = \frac{|\mu|}{nV(K)}\, dS_K.$$

In order to get the desired $c = 1/V(K)$, we will show that a dilation of K yields this c. Let $K = \lambda K'$, then the homogeneity properties of (8) and (10) imply

$$\lambda^p h_{K'}^{p-1}\, d\mu = \frac{|\mu|}{nV(K')}\, dS_{K'}.$$

Then K' is the desired convex body, by choosing $\lambda^p = |\mu|/n$.

(ii) If $K'' = \lambda K$ satisfies (37), then by (8) and (10) we have

$$h_K^{p-1}\, d\mu = \lambda^{n-p}\, dS_K.$$

Since there exists K satisfying (36), we can choose $\lambda^{n-p} = c$, where $n \neq p$ makes the reverse process feasible. □

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