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ON THE OSCILLATION OF A CLASS OF FOURTH ORDER DIFFERENTIAL EQUATIONS

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1. **Introduction.** This paper is concerned with fourth order differential equations of the form

$$(p(x)y'')'' - q(x)y'' - r(x)y = 0, \tag{L}$$

where p, q and r are assumed to be continuous, real-valued functions on the interval $[a, \infty)$. In addition, it will be assumed throughout that $p > 0$, $q \geqq 0$ and $r \geqq 0$ on $[a, \infty)$, with r not identically zero on any subinterval. If q is a (non-negative) constant, then (L) is self-adjoint; otherwise (L) is non-self-adjoint.

The objective of the paper is to study the oscillatory behavior of the solutions of (L). A non-trivial solution y is *oscillatory* if the set of zeros of y is not bounded above. If the set of zeros of y is bounded above, which implies y has only finitely many zeros, then y is *non-oscillatory*. Hereafter, the term "solution" will be interpreted to mean non-trivial solution.

Various special cases of (L) have been studied in detail. In particular, we refer to the fundamental work of W. Leighton and Z. Nehari [5, Part I] on the self-adjoint equation

$$(p(x)y'')'' - r(x)y = 0. \tag{1}$$

M. Keener [3, Part I] continued the investigation of (1), concentrating on the oscillatory behavior of solutions. S. Hastings and A. Lazer [2] considered the self-adjoint equation

$$y^{(4)} - r(x)y = 0, \tag{2}$$

showing that (2) has a linearly independent pair of bounded oscillatory solutions when it is assumed that $r \in C'[a, \infty)$, with $r > 0$ and $r' \geqq 0$ on $[a, \infty)$. S. Ahmad [1] has also studied (2), giving necessary and sufficient conditions for the existence of a linearly independent pair of oscillatory solutions. Finally, we refer to the work of V. Pudei [6], [7] in which the equation

$$y^{(4)} - q(x)y'' - r(x)y = 0 \tag{3}$$

is considered.

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The results in this paper may be viewed as a continuation of the work in [6] and [7] in coordination with the techniques and results in [1], [2], [3] and [5]. In contrast to [3], and [6] and [7], where the existence of oscillatory solutions is assumed, we give, in § 4, some oscillation criteria for (L).

2. **Preliminary Results.** As a notational convenience in discussing the solutions of (L), we introduce the differential operators

$$D_0y(x) = y(x), D_1y(x) = y'(x), D_2y(x) = p(x)y''(x),$$
$$D_3y(x) = [p(x)y''(x)]', D_4y(x) = [p(x)y''(x)]''.$$

Our first result is essential in the work which follows. Corresponding results for (1), (2), and (3) are given in [3], [1] and [7], respectively. The proof is straightforward and can be modeled on the proofs of Lemmas 2.1, 2.2 in [5].

THEOREM 2.1. *If y is a solution of (L) such that $D_iy(b) \geqq 0$ $(\leqq 0)$, $i = 0, 1, 2, 3$, at some point $b \geqq a$, with strict inequaltiy for at least one i, then $D_iy > 0$ (<0), $i = 0, 1, 2, 3$, $D_4y \geqq 0$ $(\leqq 0)$ on (b, ∞) and*

$$\lim_{x\to\infty} D_iy(x) = \infty\ (-\infty), i = 0, 1, 2.$$

If z is a solution of (L) such that $(-1)^iD_iz(c) \geqq 0 (\leqq 0)$, $i = 0, 1, 2, 3$, at some point $c > a$, with strict inequality for at least one i, then $(-1)^iD_iz > 0$ (<0), $i = 0, 1, 2, 3$, and $D_4z \geqq 0$ $(\leqq 0)$ on $[a, c)$.

The behavior of a solution of (L) having a zero of multiplicity greater than 1 at some point on $[a, \infty)$ is now an immediate consequence of Theorem 2.1. In particular, if y is a solution of (L) with a "triple zero" at $x = b$, i.e., $D_iy(b) = 0, i = 0, 1, 2$, and if $D_3y(b) > 0$, then $(-1)^{i+1}D_iy > 0$, $i = 0, 1, 2, 3$, on $[a, b)$ (if $b > a$), and $D_iy > 0$, $i = 0, 1, 2, 3$, on (b, ∞), with $\lim_{x\to\infty} D_iy(x) = \infty$, $i = 0, 1, 2$. If y has a "double zero" at $x = b$, i.e., $D_iy(b) = 0$, $i = 0, 1$, and if $D_2y(b) > 0$, then either $(-1)^iD_iy > 0$, $i = 0, 1, 2, 3$, on $[a, b)$ (if $b > a$), or $D_iy > 0$, $i = 0, 1, 2, 3$, on (b, ∞), with $\lim_{x\to\infty} D_iy(x) = \infty$, $i = 0, 1, 2$.

We point out here that the results in [5, §§ 2 and 3] concerning (1) do not depend on the form of the equation, but rather on the fact that the analogue of Theorem 2.1 holds. Thus the results contained in Lemma 2.1 through Theorem 3.8 of [5], giving separation properties of zeros of solutions and a characterization of the conjugate points of a, are also valid for (L).

It is clear from Theorem 2.1 that (L) has unbounded, non-oscillatory solutions. In fact for any point b on $[a, \infty)$, the four solutions $y_i(x, b)$, $i = 0, 1, 2, 3$, determined by the initial conditions

$$D_j y_i(b, b) = \delta_{ij} = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$$

$$i, j = 0, 1, 2, 3,$$

are monotone increasing (on $[b, \infty)$), unbounded solutions, forming a solution basis for (L); a so-called *canonical basis.*

The next theorem provides the existence of a bounded, non-oscillatory solution. The technique employed in establishing this result is well-known (see [4, Theorem 1.1] or [1, Theorem 2]) and, consequently, the proof is omitted.

THEOREM 2.2. *There exists a solution* $w(x)$ *of* (L) *such that*
(i) $\prod_{i=0}^{3} D_i w(x) \neq 0$ *on* $[a, \infty)$,
(ii) $\operatorname{sgn} D_0 w = \operatorname{sgn} D_2 w \neq \operatorname{sgn} D_1 w = \operatorname{sgn} D_3 w$ *on* $[a, \infty)$,
(iii) $\lim_{x \to \infty} D_i w(x) = 0$, $i = 1, 2, 3$, $\lim_{x \to \infty} |D_0 w(x)| = k \geqq 0$.

3. **Properties of Non-Oscillatory Solutions.** In this section we present three Lemmas which describe the behavior of non-oscillatory solutions of (L).

LEMMA 3.1. *If* y *is a non-oscillatory solution of* (L), *then* $\prod_{i=0}^{2} D_i y \neq 0$ *on* $[c, \infty)$ *for some* $c \geqq a$.

PROOF. Let y be a non-oscillatory solution of (L) and assume $y > 0$ on $[b, \infty)$, $b \geqq a$. It is sufficient to show that $D_2 y$ has at most a finite number of zeros on $[b, \infty)$. Put $J(x) = D_2 y(x) \cdot D_3 y(x)$. Differentiating J and integrating the result from s to t, $b \leqq s \leqq t$, we obtain

$$J(t) = J(s) + \int_s^t \{[D_3 y]^2 + D_2 y[qy'' + ry]\}$$

$$= J(s) + \int_s^t \{[D_3 y]^2 + pq[y'']^2 + ry \cdot D_2 y\}.$$

Now, if we regard s and t as being consecutive zeros of $D_2 y$, then it follows that $D_2 y < 0$ on (s, t). Theorem 2.1 rules out the possibility of $D_2 y$ having infinitely many "double zeros," and the lemma is proved.

LEMMA 3.2. *Assume that* (L) *has an oscillatory solution and let* y *be a non-oscillatory solution. If* u *is an oscillatory solution, and if there is a point* b, $b \geqq a$, *such that* $D_i y(b) = D_i u(b) = 0$, *for some* i, $0 \leqq i \leqq 3$, *then there exists a point* c, $c \geqq b$, *such that* $\operatorname{sgn} D_0 y = \operatorname{sgn} D_1 y = \operatorname{sgn} D_2 y = \operatorname{sgn} D_3 y$ *on* $[c, \infty)$, *and* $\lim_{x \to \infty} |D_j y(x)| = \infty$, $j = 0, 1, 2$.

PROOF. Assume that $y(x) \neq 0$ for all $x \in [d, \infty)$, $d \geqq b$. Since u oscillates and y does not, there exists a linear combination $z(x) = y(x) - ku(x)$ which has a double zero at some point $e \in (d, \infty)$ [**5**, Lemma 1.2]. Since $D_i z(b) = 0$, it follows from Theorem 2.1 that the functions $D_j z$, $j = 0, 1, 2, 3$, all have the same sign on (e, ∞), and the functions $D_j z$, $j = 0, 1, 2$, are unbounded. The lemma now follows using Lemma 3.1 and the fact that u is oscillatory.

LEMMA 3.3. *If (L) has an oscillatory solution, and if y is a non-oscillatory solution, then there exists a point b, $b \geqq a$, such that* $\operatorname{sgn} y = \operatorname{sgn} D_2 y$ *on* $[b, \infty)$.

PROOF. Assume that y is eventually positive. By Lemma 3.1, there exists b, $b \geqq a$, such that $\prod_{i=0}^{2} D_i y \neq 0$ on $[b, \infty)$. Suppose $D_2 y < 0$ on $[b, \infty)$. Then $D_1 y > 0$ on $[b, \infty)$, for otherwise $D_1 y < 0$ on $[b, \infty)$ implying $\lim_{x \to \infty} y(x) = -\infty$. Let w be the non-oscillatory solution established by Theorem 2.2. It is easily seen that the Wronskian of y and w, $W[y, w] = D_0 w D_1 y - D_0 y D_1 w > 0$ on $[b, \infty)$, and thus every non-trivial linear combination of y and w is non-oscillatory. To obtain the desired contradiction let u be an oscillatory solution of (L) with a zero at $x = c$, $c \geqq a$, choose a linear combination of y and w which vanishes at $x = c$ and apply Lemma 3.2.

4. **Oscillatory Solutions.** Our first result gives a necessary and sufficient condition for the existence of oscillatory solutions of (L). Ahmad [1, Theorem 3] established this result for (2). By using the Lemmas of the previous section, Ahmad's proof is valid for (L).

THEOREM 4.1. *The following two statements are equivalent:*

(a) *(L) has an oscillatory solution.*

(b) *If y is a non-oscillatory solution of (L), then either*

  (i) *there exists a point b, $b \geqq a$, such that* $\operatorname{sgn} D_0 y = \operatorname{sgn} D_1 y = \operatorname{sgn} D_2 y = \operatorname{sgn} D_3 y$ *and* $\lim_{x \to \infty} |D_i y(x)| = \infty$, $i = 0, 1, 2$,

or (ii) $\prod_{i=0}^{3} D_i y(x) \neq 0$ *for all x on* $[a, \infty)$ *and* $\operatorname{sgn} D_0 y = \operatorname{sgn} D_2 y \neq \operatorname{sgn} D_1 y = \operatorname{sgn} D_3 y$ *on* $[a, \infty)$.

EXAMPLE. The differential equation

$$y^{(4)} - \left[\frac{6 + x^4 \ln(x)}{x^2[1 + x^2 \ln(x)]}\right] y'' - \left[\frac{x^2 - 6}{x^2[1 + x^2 \ln(x)]}\right] y = 0$$

on $[\sqrt{6}, \infty)$ has $y(x) = \ln(x)$ and $z(x) = e^x$ as solutions. Since y is a non-oscillatory solution satisfying $D_0 y > 0$, $D_1 y > 0$, $D_2 y < 0$, and $D_3 y > 0$ on $[\sqrt{6}, \infty)$, we can conclude, from Theorem 4.1, that all solutions of the equation are non-oscillatory.

We now present two sets of hypotheses, each of which is sufficient for the existence of oscillatory solutions. The approach is through Theorem 4.1.

(H_1) $\int_a^\infty [1/p(x)]\, dx = \infty$, q bounded on $[a, \infty)$, $\int_a^\infty r(x)\, dx = \infty$.

(H_2) $\int_a^\infty [1/p(x)]\, dx = \infty$, $q > 0$ on $[a, \infty)$, $\liminf_{x\to\infty} p \cdot r/q = m > 0$, and either $\int_a^\infty [q(x)/p(x)]\, dx = \infty$ or $\int_a^\infty x r(x)\, dx = \infty$.

THEOREM 4.2. *Each of* (H_1) and (H_2) implies (b) *of Theorem* 4.1. *Consequently, if either* (H_1) *or* (H_2) *holds, then* (L) *has an oscillatory solution.*

PROOF. Suppose (H_1) holds. Let y be a non-oscillatory solution of (L) and assume $y > 0$ on $[b, \infty)$, $b \geqq a$. By Lemma 3.1 we may also assume that $D_1 y$ and $D_2 y$ are non-zero on $[b, \infty)$.

We show first that $D_2 y > 0$ on $[b, \infty)$. Assume, therefore, that $D_2 y < 0$ on this interval. Then $y'' < 0$ and $D_1 y > 0$ on $[b, \infty)$. Let M be an upper bound of q on $[a, \infty)$ and integrate qy'' from b to x to obtain the inequality

$$\int_b^x q(s)y''(s)\, ds \geqq M \int_b^x y''(s)\, ds = M[D_1 y(x) - D_1 y(b)].$$

Since $\lim_{x\to\infty} D_1 y$ exists, we conclude that $\int_b^\infty q(s)y''\, ds$ is finite. Integrating (L) from b to x and using the fact that y is increasing on $[b, \infty)$, we have

$$D_3 y(x) - D_3 y(b) = \int_b^x q(s)y''(s)\, ds + \int_b^x r(s)y(s)\, ds$$

$$\geqq \int_b^x q(s)y''(s)\, ds + y(b)\int_b^x r(s)\, ds.$$

Since $\int_b^\infty r(s)\, ds = \infty$, it follows that $\lim_{x\to\infty} D_3 y(x) = \infty$. But this implies $\lim_{x\to\infty} D_2 y(x) = \infty$, contradicting our assumption. Thus $D_2 y > 0$ on $[b, \infty)$.

We now have $y > 0$ and $y'' > 0$ on $[b, \infty)$. From (L), $D_4 y \geqq 0$ on $[b, \infty)$. Thus $D_3 y$ is eventually of one sign. If $D_3 y > 0$ on $[c, \infty)$, $c \geqq b$, then $\lim_{x\to\infty} D_2 y = \infty$, and, in particular, $D_2 y = p(x)y''(x) \geqq 1$ on $[d, \infty)$, $d \geqq c$. Therefore

$$D_1 y(x) - D_1 y(d) = \int_d^x y''(s)\, ds \geqq \int_d^x 1/p(s)\, ds,$$

and we conclude $\lim_{x\to\infty} D_1 y(x) = \lim_{x\to\infty} D_0 y(x) = \infty$, implying that (i) of Theorem 4.1 (b) holds. If $D_3 y < 0$ on $[c, \infty)$, $c \geqq b$, then we claim that $D_1 y < 0$ on $[b, \infty)$. For if $D_1 y > 0$ on $[b, \infty)$, then y is increasing on $[b, \infty)$, and, from (L),

$$D_4y(x) = q(x)y''(x) + r(x)y(x) \geqq y(b)r(x).$$

Integrating this inequality, we obtain

$$D_3y(x) - D_3y(b) \geqq y(b) \int_b^x r(s)\, ds,$$

so that $\lim_{x\to\infty} D_3y(x) = \infty$ which contradicts $D_3y < 0$ on $[c, \infty)$. Thus $D_3y < 0$ on $[c, \infty)$, implies that the inequalities $D_0y > 0$, $D_1y < 0$, $D_2y > 0$, $D_3y < 0$ hold on $[c, \infty)$. From Theorem 2.1, these same inequalities must also hold on $[a, c)$, so that we have (ii) of Theorem 4.1 (b).

Suppose (H_2) holds. Let y be a non-oscillatory solution of (L) such that $\prod_{i=0}^{2} D_iy(x) \neq 0$ for all x on $[b, \infty)$, $b \geqq a$, with $y > 0$ on this interval. As in the first part of the proof, we want to show that $D_2y > 0$ on $[b, \infty)$. Assume, therefore, that $D_2y < 0$ and $D_1y > 0$ on $[b, \infty)$. We claim that D_3y is eventually of one sign. Suppose, to the contrary, that D_3y changes sign infinitely many times on $[b, \infty)$. Then D_2y has infinitely many maxima on $[b, \infty)$, and, since $D_2y < 0$ $\limsup_{x\to\infty} D_2y \leqq 0$. In particular, if $\limsup_{x\to\infty} D_2y < 0$, then there exist numbers $e < 0$ and $c \geqq b$ such that $p(x)y''(x) \leqq e$ on $[c, \infty)$, and

$$D_1y(x) - D_1y(c) = \int_c^x y''(s)\, ds \leqq e \int_c^x 1/p(s)\, ds.$$

But this implies $\lim_{x\to\infty} D_1y(x) = -\infty$, a contradiction. Thus $\limsup_{x\to\infty} D_2y = 0$, and there exists a sequence $\{x_n\}$ of maxima of D_2y such that $x_n \to \infty$ and $D_2y(x_n) \to 0$ as $n \to \infty$. Now write (L) in the form

$$D_4y(x) = \frac{q(x)}{p(x)} \left[D_2y(x) + \frac{p(x)\cdot r(x)}{q(x)} y(x) \right]. \tag{4}$$

Since y is increasing on $[b, \infty)$ and $\liminf_{x\to\infty} [p\cdot r/q] = m > 0$, we can choose n large enough such that $D_4y(x_n) = [D_2y(x_n)]'' < 0$, contradicting the fact that x_n is a relative maximum of D_2y. Thus D_3y cannot change sign infinitely many times on $[b, \infty)$, and because $D_2y < 0$ it is easy to see that D_3y must, in fact, be non-negative on $[c, \infty)$ for some $c \geqq b$. We can now conclude that $\lim_{x\to\infty} D_2y(x) = 0$. But, from (4), it follows that $D_4y(x) > 0$ on some interval $[d, \infty)$, $d \geqq c$. Thus $D_4y(x) > 0$ and $D_3y(x) > 0$ on $[d, \infty)$, which implies $\lim_{x\to\infty} D_2y(x) = \infty$ and contradicts $D_2y < 0$ on $[b, \infty)$. Consequently $D_2y > 0$ on $[b, \infty)$.

Now $y > 0$, $D_2y > 0$ on $[b, \infty)$ implies $D_4y \geqq 0$ on $[b, \infty)$. If y is bounded, then, as in the first part of the proof, we must have $D_3y < 0$ and $D_1y < 0$ on some interval $[c, \infty)$, $c \geqq b$, which implies that the inequalities $y > 0$, $D_1y < 0$, $D_2y > 0$, $D_3 < 0$ must hold on $[a, \infty)$ and (ii) of Theorem 4.1 (b) holds. If, on the other hand, y is not bounded, then $D_1y > 0$ on $[b, \infty)$, and, using (L), it is readily verified that either of the conditions $\int_a^\infty [q(x)/p(x)]\,dx = \infty$ or $\int_a^\infty xr(x)\,dx = \infty$ yields $\lim_{x\to\infty} D_3y(x) = \infty$. Thus y unbounded implies (i) of Theorem 4.1 (b) holds. This completes the proof of the theorem.

The case $p(x) \equiv 1$, $q(x) \equiv 0$ yields the equation (2) studied by Hastings and Lazer [2]. Hypothesis (H_1) is weaker than the hypotheses which they use to establish the existence of oscillatory solutions [2, Theorem 1].

The next theorem provides additional information concerning the behavior of bounded non-oscillatory solutions.

Theorem 4.3. *Let either* (H_1) *or* (H_2) *hold. If* y *is a bounded non-oscillatory solution of* (L) *(e.g., the solution* w *specified in Theorem 2.2), then* $\lim_{x\to\infty} y(x) = 0$.

Proof. Let (H_1) hold and suppose y is a bounded non-oscillatory of (L). Then from Theorem 4.1, we may assume $y > 0$, $D_1y < 0$, $D_2y > 0$, $D_3y < 0$ on $[a, \infty)$. If $\lim y(x) = k > 0$, then $y(x) > k$ for all x on $[a, \infty)$. Thus upon solving (L) for $D_4y(x)$ and integrating, we get

$$D_3y(x) - D_3y(a) = \int_a^x q(s)y''(s)\,ds + \int_a^x r(s)y(s)\,ds$$

$$\geqq k\int_a^x r(s)\,ds,$$

which implies $\lim_{x\to\infty} D_3y(x) = \infty$, a contradiction.

Now suppose that (H_2) holds, and that y is a bounded non-oscillatory solution of (L) with $y > 0$, $D_1y < 0$, $D_2y > 0$, $D_3y < 0$ on $[a, \infty)$ and $\lim_{x\to\infty} y(x) = k > 0$. Since $\liminf_{x\to\infty} pr/q = m > 0$, there exists a point b, $b \geqq a$, such that $p(x)\cdot r(x)/q(x) \geqq m/2$ on $[b, \infty)$.

If we assume that $\int_a^\infty [q(x)/p(x)]\,dx = \infty$, then integrating (L) from b to x yields

$$D_3y(x) - D_3y(b) = \int_b^x [q(s)y''(s) + r(s)y(s)]\,ds$$

$$= \int_b^x \frac{q(s)}{p(s)}\left[D_2y(s) + \frac{p(s)r(s)}{q(s)}\,y(s)\right]\,ds$$

$$\geqq \frac{km}{2} \int_b^x [q(s)/p(s)]\, ds.$$

Therefore $\lim_{x\to\infty} D_3 y(x) = \infty$, a contradiction.

Finally, assuming that $\int_a^\infty x r(x)\, dx = \infty$, multiply (L) by x and integrate to obtain

$$\int_a^x t \cdot D_4 y(t)\, dt = \int_a^x t \cdot q(t) y''(t)\, dt + \int_a^x t \cdot r(t) y(t)\, dt$$

$$\geqq k \int_a^x t \cdot r(t)\, dt.$$

Integrating the left-hand side of this inequality by parts, we get

$$x D_3 y(x) - a D_3 y(a) - D_2 y(x) + D_2 y(a) \geqq k \int_a^x t r(t)\, dt;$$

which implies $\lim_{x\to\infty} x \cdot D_3 y(x) = \infty$ and contradicts the fact that $D_3 y < 0$ on $[a, \infty)$.

The next result is an immediate extension of [1, Theorem 4] and [3, Theorem 3.3].

THEOREM 4.4. *Assume that* (L) *has oscillatory solutions and let* $b \geqq a$. *Then*

(i) *for each integer* $j, 0 \leqq j \leqq 3$, *there exists a pair of linearly independent oscillatory solutions* u_j, v_j *of* (L) *such that* $D_j u_j(b) = D_j v_j(b) = 0$, *the zeros of* u_j *and* v_j *separate on* (b, ∞), *and every linear combination of* u_j *and* v_j *is oscillatory*; *and*

(ii) *there exist three linearly independent oscillatory solutions of* (L).

The final theorem of this section provides a characterization of the oscillatory solutions of (L). Keener [3, Theorems 4.4, 4.5 and 4.7] established similar results for (1) under the assumption that all oscillatory solutions are bounded.

THEOREM 4.5. *Assume that* (L) *has oscillatory solutions. Let* w *be the bounded non-oscillatory solution of* (L) *whose existence is guaranteed by Theorem* 2.2, *let* $b \geqq a$, *and let* u, v *be a linearly independent pair of oscillatory solutions of* (L) *such that* (i) $D_j u(b) = D_j v(b) = 0$ *for some* $j, 0 \leqq j \leqq 3$, (ii) *their zeros separate on* (b, ∞), *and* (iii) *every linear combination of* u *and* v *is oscillatory. Then the following are equivalent*:

(a) *Every oscillatory solution* z *of* (L) *such that* $D_j z(b) = 0$ *is a linear combination of* u *and* v.

(b) *Every oscillatory solution of* (L) *is a linear combination of* u, v *and* w.

(c) *If* z *is an oscillatory solution of* (L) *and* $y_i(x, c)$, $0 \leqq i \leqq 3, c \geqq a$, *is a member of the canonical basis for* (L) *at* $x = c$, *then* $z + ky_i$ *is non-oscillatory for every non-zero real number* k.

Proof. $(a) \Longrightarrow (b)$. Let z be an oscillatory solution of (L). If $D_j z(b) = 0$, then z is a linear combination of u and v. If $D_j z(b) \neq 0$, then there exists a number e such that $D_j z(b) - e \cdot D_j w(b) = 0$. Put $y(x) = z(x) - ew(x)$. If y is non-oscillatory, then, since $D_j y(b) = 0$, $\lim_{x \to \infty} |y(x)| = \infty$, by Theorem 4.1. But the fact that z is oscillatory and w is bounded produces a contradiction. Thus y is a linear combination of u and v, and (b) holds.

(b) $\Longrightarrow$ (c). Let z be an oscillatory solution of (L), let $c \geqq a$, and let $y_i(x, c)$ be one of the members of the canonical basis for (L) at $x = c$. Let $k \neq 0$ and put $y(x) = z(x) + ky_i(x, c)$. If y is oscillatory, then $y = c_1 u + c_2 v + c_3 w$. Also, since z is oscillatory, $z = d_1 u + d_2 v + d_3 w$. Thus, we have

$$ky_i(x, c) = (c_1 - d_1)u(x) + (c_2 - d_2)v(x) + (c_3 - d_3)w(x).$$

But, since $(c_1 - d_1)u + (c_2 - d_2)v$ is oscillatory (or identically zero) and w is bounded, we have a contradiction. Thus $y = z + ky_i$ is non-oscillatory and (c) holds.

(c) $\Longrightarrow$ (a). Let $y_i(x, b)$ be a member of the canonical basis at $x = b$ such that $D_j y_i(b, b) = 0$, and u, v and y_i are linearly independent. If z is an oscillatory solution of (L) such that $D_j z(b) = 0$, then

$$z(x) = c_1 u(x) + c_2 v(x) + c_3 y_i(x, b).$$

Since $c_1 u + c_2 v$ is oscillatory, we conclude, by (c), that $c_3 = 0$ and (a) holds.

Remark. If it is assumed that all oscillatory solutions of (L) are bounded (c.f., Keener [3, § 4]), then it is easy to show that either (a), or (b), or (c) of Theorem 4.5 holds (and, consequently, all three statements hold). This characterization of the oscillatory solutions of (L) extends Keener's results.

5. **Boundedness of Oscillatory Solutions.** The question of the boundedness of oscillatory solutions of (L) is somewhat complicated. Hastings and Lazer [2, Theorem 2] show that if $r(x) > 0$, $r'(x) \geqq 0$ and $\lim_{x \to \infty} r(x) = \infty$, then all oscillatory solutions of

$$y^{(4)} - r(x)y = 0 \tag{2}$$

have limit zero. All oscillatory solutions of the equation

$$y^{(4)} - y = 0 \tag{5}$$

are bounded, and no oscillatory solution has a limit at ∞. Finally, the set of solutions of

$$y^{(4)} - \frac{10}{x^2}y'' - \frac{60}{x^4}y = 0 \tag{6}$$

is given by

$$y(x) = c_1x^6 + c_2x^{-2} + c_3x\sin[\ln(x^2)] + c_4x\cos[\ln(x^2)], \tag{7}$$

where c_1, c_2, c_3 and c_4 are arbitrary constants, so that all oscillatory solutions of (6) are unbounded.

The following sets of hypotheses provide sufficient conditions for the oscillatory solutions of (L) to be bounded.

(H_3) $p, q, r \in C'[a, \infty)$, with $p'(x) \geqq 0$, $q'(x) \leqq 0$, $r'(x) \leqq 0$, and $\lim_{x\to\infty} r(x) \neq 0$.

(H_4) $p, q, r \in C'[a, \infty)$, with $r(a) > 0$, $p'(x) \geqq 0$, $q'(x) \leqq 0$ and $r'(x) \geqq 0$.

THEOREM 5.1. *Assume that (L) has oscillatory solutions and let either (H_3) or (H_4) hold. Then all oscillatory solutions of (L) are bounded.*

PROOF. For any solution y of (L), define $G[y]$ by

$$\begin{aligned} G[y(x)] &= p(x)[y''(x)]^2 + q(x)[y'(x)]^2 \\ &\quad + r(x)y^2(x) - 2y'(x)D_3y(x). \end{aligned} \tag{8}$$

Differentiating $G[y(x)]$ and integrating the result between b and x, for any $b \geqq a$, we obtain

$$\begin{aligned} G[y(x)] &= G[y(b)] \\ &\quad + \int_b^x \{r'(s)y^2(s) + q'(s)[y'(s)]^2 - p'(s)[y''(s)]^2\}\,ds. \end{aligned} \tag{9}$$

Let z be an oscillatory solution of (L) and let $\{b_n\}$ be the increasing sequence of zeros of z' on $[a, \infty)$.

Assume (H_3) holds, and let $b = a$ in (9). Then $G[z(x)]$ is a nonincreasing function on $[a, \infty)$, and

$$\begin{aligned} r(b_n)z^2(b_n) &\leqq p(b_n)[z''(b_n)]^2 + q(b_n)[z'(b_n)]^2 + r(b_n)z^2(b_n) \\ &= G[z(b_n)] \leqq G[z(a)]. \end{aligned}$$

Therefore

$$z^2(b_n) \leqq G[z(a)]/r(b_n).$$

and, since $r(x) \not\to 0$ as $x \to \infty$, we can conclude that z is bounded.

Now assume that (H_4) holds and let $b = b_1$ in (9). Then, from (9),

$$G[z(x)] \leqq G[z(b_1)] + \int_{b_1}^{x} r'(s)z^2(s)\,ds. \tag{10}$$

Choose any positive integer $n > 1$, and let $\bar{x}$ be the point on $[b_1, b_n]$ at which $z^2(x)$ achieves its maximum value. We claim

$$z^2(\bar{x}) \leqq z^2(b_1) + \frac{p(b_1)[z''(b_1)]^2}{r(b_1)}. \tag{11}$$

Recall $r(a) > 0$ and $r'(x) \geqq 0$ implies $r(b_1) > 0$. Since $z'(b_1) = z'(b_n) = 0$, it follows that $z'(\bar{x}) = 0$. If $\bar{x} = b_1$, then, clearly, (11) holds. If $\bar{x} \geqq b_1$, then, from (10),

$$G[z(\bar{x})] \leqq G[z(b_1)] + z^2(\bar{x})[r(\bar{x}) - r(b_1)].$$

Using (8), we have

$$p(\bar{x})[z''(\bar{x})]^2 + r(\bar{x})z^2(\bar{x}) \leqq p(b_1)[z''(b_1)^2 + r(b_1)z^2(b_1) + \\ + z^2(\bar{x})[r(\bar{x}) - r(b_1)]$$

and (11) follows. Finally, since b_n may be chosen arbitrarily large on $[a, \infty)$, we conclude that z is bounded.

6. **The Adjoint of (L).** The adjoint of (L) is the equation

$$[p(x)y'' - q(x)y]'' - r(x)y = 0, \qquad \text{on } [a, \infty). \tag{L*}$$

It is convenient to introduce the differential operators

$$D_0^*y(x) = y(x),\ D_1^*y(x) = y'(x),\ D_2^*y(x) = p(x)y''(x) - q(x)y(x),$$

$$D_3^*y(x) = [p(x)y''(x) - q(x)y(x)]',\ D_4^*y(x) = [p(x)y''(x) - q(x)y(x)]''.$$

As in the case of (L), it can be verified that the analogues of Theorems 2.1 and 2.2 hold for (L^*). Thus the results in [5, Lemma 2.1 — Theorem 3.9] can also be established for (L^*).

Pudei [6, Theorem 55] has shown that (3) is oscillatory if and only if its adjoint

$$[y'' - q(x)y]'' - r(x)y = 0 \tag{12}$$

is oscillatory. With obvious modifications, his proof can be extended to the case of (L) and (L^*). An alternative proof of this fact can be accomplished by showing that the nth conjugate point of a with respect to (L) coincides with the nth conjugate point of a with respect to (L^*), and using [5, Theorem 3.8]. Therefore, if it is assumed that either (H_1) or (H_2) holds, then we can conclude that each of (L) and (L^*) is oscillatory.

In coordination with Theorem 4.2 and the remarks above, our final theorem shows that the solutions of (L) and (L^*) have essentially the same behavior.

THEOREM 6.1. *Let either* (H_1) *or* (H_2) *hold. If* y *is a non-oscillatory solution of* (L^*), *then* y *satisfies exactly one of the following two conditions:*

(i) *There is a point* b, $b \geqq a$, *such that*

$$\operatorname{sgn} D_0{}^*y(x) = \operatorname{sgn} D_1{}^*y(x) = \operatorname{sgn} D_2{}^*y(x) = \operatorname{sgn} D_3{}^*y(x)$$

on $[b, \infty)$, *and* $\lim_{x\to\infty}|D_i{}^*y(x)| = \infty$, $i = 0, 1, 2$.

(ii) $\prod_{i=0}^{3} D_i{}^*y(x) \neq 0$ *on* $[a, \infty)$, *and*

$$\operatorname{sgn} D_0{}^*y(x) = \operatorname{sgn} D_2{}^*y(x) \neq \operatorname{sgn} D_1{}^*y(x) = \operatorname{sgn} D_3{}^*y(x)$$

on $[a, \infty)$.

Moreover, if y *satisfies* (ii), *then* $\lim_{x\to\infty} y(x) = 0$.

PROOF. Suppose (H_1) holds. Let y be a non-oscillatory solution of (L^*) and assume $y > 0$ on $[b, \infty)$, $b \geqq a$. Then $D_4{}^*y \geqq 0$ on $[b, \infty)$ and is not identically zero on any subinterval. Therefore $D_3{}^*y$ has one sign on $[c, \infty)$ for some $c \geqq b$. *If* $D_3{}^*y > 0$ on $[c, \infty)$, then, as in the proof of Theorem 4.2, it is easily verified that $D_2{}^*y > 0$, $D_1{}^*y > 0$ on $[d, \infty)$, $d \geqq c$, and (i) of the theorem holds. If, on the other hand, $D_3{}^*y < 0$ on $[c, \infty)$, then $D_2{}^*y$ has one sign on $[d, \infty)$, $d \geqq c$. We want to show that $D_2{}^*y > 0$ on $[d, \infty)$, and so we shall assume $D_2{}^*y < 0$ on this interval. It is easy to see that $\liminf_{x\to\infty} y(x) = 0$, for otherwise the fact that $\int_a^\infty r(x)\,dx = \infty$ would imply $\lim_{x\to\infty} D_3{}^*y(x) = \infty$. Suppose $y' = D_1{}^*y$ has infinitely many changes in sign on $[d, \infty)$. Then there is a sequence $\{x_n\}$ such that $\lim_{n\to\infty} x_n = \infty$, y has a relative minimum at x_n, and $\lim_{n\to\infty} y(x_n) = 0$. Now, since $D_3{}^*y < 0$ and $D_2{}^*y < 0$ on $[d, \infty)$, it follows that

$$D_2{}^*y(x) = p(x)y''(x) - q(x)y(x) \leqq \beta < 0 \qquad \text{on } [d, \infty). \tag{13}$$

But, since q is bounded and $\lim_{n\to\infty} y(x_n) = 0$, we can choose n large enough such that $q(x_n)y(x_n) < -\beta$, which implies $y''(x_n) < 0$ at a relative minimum of y, a contradiction. Thus y' cannot change sign infinitely many times on $[d, \infty)$. If y' is non-negative on $[e, \infty)$, $e \geqq d$, then $y \geqq \gamma > 0$ on $[e, \infty)$ and $\int_a^\infty r(x)\,dx = \infty$ yields $\lim_{x\to\infty} D_3{}^*y(x)$

$= \infty$, a contradiction. Therefore $y' \leqq 0$ on $[e, \infty)$, and $\lim_{x\to\infty} y(x) = 0$. But, from (13), we can conclude that $y'' < 0$ on $[f, \infty)$ for some $f \geqq e$, and $y'' < 0$, $y' \leqq 0$ implies $\lim_{x\to\infty} y(x) = -\infty$, a contradiction.

We now have $y > 0$, $D_4{}^*y \geqq 0$, $D_3{}^*y < 0$, $D_2{}^*y > 0$ on some interval $[b, \infty)$, $b \geqq a$. Since $D_2{}^*y > 0$, $y'' > 0$ on $[b, \infty)$, and so $y' = D_1{}^*y$ has one sign on $[c, \infty)$, $c \geqq b$. Since $y' > 0$ would lead to $\lim_{x\to\infty} D_3{}^*y(x) = \infty$, we must have $y' < 0$ on $[c, \infty)$, from which it follows that (ii) of the theorem holds.

Suppose (H_2) holds. Let y be a non-oscillatory solution of (L^*) and assume $y > 0$ on $[b, \infty)$, $b \geqq a$. Then $D_4{}^*y \geqq 0$ on $[b, \infty)$ from which we conclude that $D_3{}^*y$ has one sign on $[c, \infty)$ for some $c \geqq b$. The assumption $D_3{}^*y > 0$ yields (i) of the theorem. Suppose, therefore, that $D_3{}^*y < 0$ on $[c, \infty)$. Since

$$D_3{}^*y(x) - D_3{}^*y(a) = \int_a^x r(t)y(t)\,dt$$

and $\lim_{x\to\infty} D_3{}^*y$ exists, we have $\int_a^\infty r(t)y(t)\,dt < \infty$. Moreover, from the hypothesis (H_2),

$$\int_a^\infty \frac{q(x)y(x)}{p(x)}dx = \int_a^\infty \frac{q(x)}{p(x)r(x)} \cdot r(x)y(x)\,dx < \frac{1}{m}\int_a^\infty r(x)y(x)\,dx < \infty.$$

Now consider $D_2{}^*y$. If $D_2{}^*y < 0$ on $[d, \infty)$, $d \geqq c$, then

$$p(x)y''(x) - q(x)y(x) \leqq \beta < 0,$$

or

$$y''(x) - \frac{q(x)}{p(x)}y(x) \leqq \frac{\beta}{p(x)} \qquad \text{on } [d, \infty).$$

Therefore

$$y'(x) - y'(d) \leqq \int_d^x \frac{q(t)}{p(t)}y(t)\,dt + \int_d^x \frac{\beta}{p(t)}\,dt,$$

and we conclude that $\lim_{x\to\infty} D_1{}^*y(x) = \lim_{x\to\infty} y'(x) = -\infty$, contradicting $y > 0$ on $[b, \infty)$. It now follows that $D_2{}^*y > 0$ and $y'' > 0$ on $[d, \infty)$. If $y' > 0$ on $[e, \infty)$ for some $e \geqq d$, then the hypothesis $\int_a^\infty xr(x)\,dx = \infty$ yields $\lim_{x\to\infty} D_3{}^*y(x) = \infty$. Since this contradicts our assumption, we must have $y' < 0$ on $[e, \infty)$, and (ii) of the theorem holds.

Finally, if y is a bounded non-oscillatory solution of (L^*), then the fact that $\lim_{x\to\infty} y(x) = 0$ follows exactly as in Theorem 4.3.

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