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ON THE OSCILLATORY BEHAVIOR OF SOLUTIONS OF SECOND ORDER NONLINEAR DIFFERENTIAL EQUATIONS

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INTRODUCTION

In this paper we are interested in obtaining results on the oscillatory behavior of solutions of a broad class of second order nonlinear differential equations. During the last twenty-five years there has been a great deal of work on the asymptotic behavior of solutions of equations of the type

$$(E_1) \quad (a(t)x')' + q(t)f(x) = 0, \quad q'(t) \geq 0.$$

Considerably less is known about the behavior of solutions of (E_1) when $q(t)$ is allowed to change signs,

$$(E_2) \quad (a(t)x')' + q(t)f(x) = 0, \quad q(t) \text{ oscillatory,}$$

and as recent contributions to this study we cite the papers of Chen [1, 2], Grace and Lalli [4], Graef and Spikes [5], Grammatikopoulos, et al. [6, 9], Kamenev [8], Kura [10], Kusano, et al. [11], Kwong and Wong [12, 13], Mahfoud and Rankin [14], Philos [15, 16], Staikos and Sficas [17], and Yeh [19–21]. There are a number of papers which discuss the behavior of just the bounded solutions of (E_2) , but since we are interested in obtaining results for all solutions, we chose not to list those papers here. Recently there has been interest in obtaining results on the asymptotic behavior of solutions of nonlinear equations of the form

$$(E_3) \quad (a(t)\psi(x)x')' + q(t)f(x) = 0$$

where $\psi(x) > 0$ for $x \neq 0$. Such results can be found in the papers of Domshlak [3] and Jannelli [7] as well as the references [4], [9], [14], [17], and [21] mentioned above.

Here we consider the equation

$$(E) \quad (a(t)\psi(x)x')' + q(t)f(x) = r(t)$$

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where $q(t)$ is allowed to change signs and we give sufficient conditions for any solution $x(t)$ of (E) to be either oscillatory or satisfy $\liminf_{t \rightarrow \infty} |x(t)| = 0$. Two other results give sufficient conditions for all solutions of (E) to be oscillatory in the case when $r(t) \equiv 0$. The results presented here differ in several respects from those of other authors. Due first to the fact that q may change signs and secondly to the presence of the function ψ our results differ from those previously obtained for equations (E_1) and (E_2) respectively. Moreover, our results will cover all solutions not just the bounded ones (for example, see some of the results in [3] and [11]). In addition, we do not require $\int^\infty q(s) ds = \infty$ as many authors do (see some of the results in [3], [11], or [17]), and in that respect even when $\psi(x) \equiv 1$, our results differ from some of those previously known for equation (E_2) . Some comparisons between our theorems and those of other authors are indicated, and some examples illustrating our results are also included.

OSCILLATORY AND ASYMPTOTIC BEHAVIOR

Consider the equation

$$(1) \quad (a(t) \psi(x) x')' + q(t) f(x) = r(t)$$

where $a, q, r: [t_0, \infty) \rightarrow R$ and $f, \psi: R \rightarrow R$ are continuous, $a(t) > 0$, and $\psi(x) > 0$ for $x \neq 0$. The results in this paper pertain only to the continuable solutions of (1). Such a solution $x(t)$ is said to be *oscillatory* if it has arbitrarily large zeros and *nonoscillatory* otherwise. The following conditions will be utilized as they are needed:

$$(2) \quad \int_{t_0}^{\infty} [1/a(s)] ds = \infty ;$$

$$(3) \quad x f(x) > 0 \quad \text{for all } x \neq 0 ;$$

and

$$(4) \quad \int_{t_0}^{\infty} |r(s)| ds < \infty .$$

Also, to simplify notation we let $W(t) = a(t) \psi(x(t)) x'(t)/f(x(t))$ for any nonoscillatory solution $x(t)$ of (1).

We first extend a result of Kwong and Wong [12, Lemma 2]. For the proof we need the following lemma which can be found in [18, p. 14].

Lemma 1. *Let $k(t, s, z)$ be a real-valued function of t and s in $[T, C)$ and z in $[T_1, C_1]$ such that for fixed t and s , k is a nondecreasing function of z . Let $g(t)$ be a given function on $[T, C)$, and let u and v be functions on $[T, C)$ satisfying $u(s)$ and $v(s)$ are in $[T_1, C_1]$ for all s in $[T, C)$, $k(t, s, v(s))$ and $k(t, s, u(s))$ are locally integrable in s for fixed t , and for all t in $[T, C)$*

$$v(t) = g(t) + \int_T^t k(t, s, v(s)) ds ,$$

and

$$u(t) \geq g(t) + \int_T^t k(t, s, u(s)) \, ds.$$

Then $v'(t) \leq u(t)$ for all t in $[T, C)$.

Lemma 2. Suppose that (3) holds and that

$$(5) \quad f'(x) \geq 0 \quad \text{for } x \neq 0.$$

Let $x(t)$ be a positive (negative) solution of (1) on $[T_1, C)$ for some positive T_1 such that $t_0 \leq T_1 < C \leq \infty$. If there exist T in $[T_1, C)$ and a positive constant A_1 such that

$$(6) \quad -W(T_1) + \int_{T_1}^t [q(s) - r(s)/f(x(s))] \, ds + \int_{T_1}^t [f'(x(s)) W^2(s)/a(s) \psi(x(s))] \, ds \geq A_1$$

for all t in $[T, C)$, then $a(t) \psi(x(t)) x'(t) \leq -A_1 f(x(T)) (a(t) \psi(x(t)) x'(t) \leq -A_1 f(x(T)))$ for all t in $[T, C)$.

Proof. Let $x(t)$ be a solution of (1) satisfying the hypotheses of the lemma. Since

$$W'(t) + f'(x(t)) W^2(t)/a(t) \psi(x(t)) = r(t)/f(x(t)) - q(t),$$

we have

$$-W(t) = -W(T_1) + \int_{T_1}^t [q(s) - r(s)/f(x(s))] \, ds + \int_{T_1}^t [f'(x(s)) W^2(s)/a(s) \psi(s)] \, ds$$

for $T_1 \leq t < C$, and thus from (6) we see that

$$(7) \quad -W(t) \geq A_1 + \int_T^t [f'(x(s)) W^2(s)/a(s) \psi(x(s))] \, ds$$

for $T \leq t < C$. Since the integral in (7) is nonnegative, we have $x(t) x'(t) < 0$ on $[T, C)$.

If $x(t)$ is positive, let $u(t) = -a(t) \psi(x(t)) x'(t)$. Then (7) becomes

$$u(t) \geq A_1 f(x(t)) + \int_T^t [f(x(t)) f'(x(s)) (-x'(s)) u(s)/f^2(x(s))] \, ds.$$

Define $k(t, s, z) = f(x(t)) f'(x(s)) (-x'(s)) z/f^2(x(s))$ for t and s in $[T, C)$ and z in $[0, \infty)$. Notice that, in its domain of definition, $k(t, s, z)$ is nondecreasing in z for fixed t and s . Hence, Lemma 1 applies, with $g(t) = A_1 f(x(t))$, and we have $u(t) \geq v(t)$, where $v(t)$ satisfies the equation

$$v(t) = A_1 f(x(t)) + \int_T^t [f(x(t)) f'(x(s)) (-x'(s)) v(s)/f^2(x(s))] \, ds$$

provided $v(s)$ is in $[0, \infty)$ for each s in $[T, C)$. Multiplying the last equation by $1/f(x(t))$ and then differentiating, we obtain $v'(t)/f(x(t)) \equiv 0$ so that $v(t) \equiv v(T) =$

$= A_1 f(x(T))$ for all t in $[T, C)$. Thus, by Lemma 1, $a(t) \psi(x(t)) x'(t) \leq -A_1 f(x(T))$ for $T \leq t < C$.

The proof for $x(t)$ negative follows by a similar argument by taking $u(t) = a(t) \psi(x(t)) x'(t)$ and $g(t) = -A_1 f(x(t))$.

Lemma 3. *Suppose that (2)–(5) hold and that*

$$(8) \quad \int_{t_0}^{\infty} q(s) \, ds \text{ converges}$$

and

$$(9) \quad f(x) \rightarrow \pm \infty \text{ as } x \rightarrow \pm \infty.$$

If $x(t)$ is a solution of (1) such that $\liminf_{t \rightarrow \infty} |x(t)| > 0$, then

$$(10) \quad \int^{\infty} [f'(x(s)) W^2(s)/a(s) \psi(x(s))] \, ds < \infty,$$

$$(11) \quad W(t) \rightarrow 0 \text{ as } t \rightarrow \infty,$$

and

$$(12) \quad W(t) = \int_t^{\infty} [f'(x(s)) W^2(s)/a(s) \psi(x(s))] \, ds + \int_t^{\infty} [q(s) - r(s)/f(x(s))] \, ds$$

for all sufficiently large t .

Proof. Let $x(t)$ be a nonoscillatory solution of (1) satisfying $\liminf_{t \rightarrow \infty} |x(t)| > 0$. Then there exist $m > 0$, $M > 0$, and $t_1 > t_0$ such that $|x(t)| \geq m$ and $|f(x(t))| \geq M$ for $t \geq t_1$. This, together with (4), implies that

$$(13) \quad \left| \int_{t_1}^t [r(s)/f(x(s))] \, ds \right| \leq \int_{t_1}^t |r(s)/f(x(s))| \, ds \leq N_1$$

for some $N_1 > 0$ and all $t \geq t_1$.

Now suppose that (10) does not hold. Then, in view of (8), there exist $A_1 > 0$ and $t_2 > t_1$ such that (6) holds for all $t \geq t_2$. If $x(t) > 0$ on $[t_1, \infty)$, it follows from Lemma 2 and its proof that $x'(t) < 0$ and $a(t) \psi(x(t)) x'(t) \leq -A_1 f(x(t_2))$ for $t \geq t_2$. Since $x(t)$ is positive and decreasing on $[t_2, \infty)$, $0 < \psi(x(t)) \leq A_2$ on $[t_2, \infty)$ for some positive constant A_2 . Thus $x'(t) \leq -A_1 f(x(t_2))/A_2 a(t)$ and integrating we have $x(t) \leq x(t_2) - (A_1 f(x(t_2))/A_2) \int_{t_2}^t [1/a(s)] \, ds$ which, in view of (2), contradicts the assumption that $x(t)$ is positive on $[t_1, \infty)$. A similar argument handles the case when $x(t) < 0$ on $[t_1, \infty)$.

Since

$$W'(t) + f'(x(t)) W^2(t)/a(t) \psi(x(t)) = r(t)/f(x(t)) - q(t),$$

we have

$$(14) \quad W(z) + \int_t^z [f'(x(s)) W^2(s)/a(s) \psi(x(s))] \, ds = W(t) + \int_t^z [r(s)/f(x(s)) - q(s)] \, ds.$$

From (8), (10), (13), and (14), we see that $\lim_{z \rightarrow \infty} W(z)$ exists, say $W(z) \rightarrow A_3$ as $z \rightarrow \infty$, so that from (14) we have

$$(15) \quad W(t) = A_3 + \int_t^\infty [q(s) - r'(s)/f(x(s))] ds + \int_t^\infty [f'(x(s)) W^2(s)/a(s) \psi(x(s))] ds$$

for $t \geq t_1$.

To show that (11) and (12) hold we must show that $A_3 = 0$. Suppose first that $x(t) > 0$ on $[t_1, \infty)$. If $A_3 < 0$, then from (8), (10), and (13) there exists $T_1 > t_1$ such that $|\int_{T_1}^t q(s) ds| \leq -A_3/8$ and

$$\left| \int_{T_1}^t [r'(s)/f(x(s))] ds \right| \leq -A_3/8 \quad \text{for } t \geq T_1, \quad \text{and}$$

$$\int_{T_1}^\infty [f'(x(s)) W^2(s)/a(s) \psi(x(s))] ds < -A_3/8.$$

From (15) we see that (6) holds on $[T_1, \infty)$ with $T = T_1$. But then, as argued above, Lemma 2 and its proof contradict the assumption that $x(t)$ is positive on $[t_1, \infty)$. If $A_3 > 0$, it follows from (8), (10), (13), and (15) that $W(t) \rightarrow A_3$ as $t \rightarrow \infty$ so there exists $T_2 > t_1$ such that $a(t) \psi(x(t)) x'(t)/f(x(t)) \geq A_3/2$ for $t \geq T_2$. Thus

$$\int_{T_2}^t [f'(x(s)) W^2(s)/a(s) \psi(x(s))] ds = \int_{T_2}^t [f'(x(s)) (x'(s))^2 a(s) \psi(x(s))/f^2(x(s))] ds \geq$$

$$\geq (A_3/2) \int_{T_2}^t [f'(x(s)) x'(s)/f(x(s))] ds = (A_3/2) \ln \frac{f(x(t))}{f(x(T_2))}.$$

But this, together with (9) and (10), implies that $x(t)$ is bounded above and, hence, $0 < \psi(x(t)) \leq A_4$ for some positive constant A_4 . Since $a(t) \psi(x(t)) x'(t) \geq (A_3/2) f(x(t))$, then $x'(t) > 0$ for $t \geq T_2$ which, together with (5), implies that $f(x(t)) \geq f(x(T_2))$ for $t \geq T_2$. Therefore $x'(t) \geq A_3 f(x(T_2))/2A_4 a(t)$ on $[T_2, \infty)$ and integrating we have

$$x(t) \geq x(T_2) + (A_3 f(x(T_2))/2A_4) \int_{T_2}^t [1/a(s)] ds$$

for $t \geq T_2$ which, in view of (2), contradicts the boundedness of $x(t)$. This completes the proof that $A_3 = 0$ for case the $x(t) > 0$ on $[t_1, \infty)$. The proof that $A_3 = 0$ when $x(t) < 0$ on $[t_1, \infty)$ is similar and will be omitted.

Remark. Lemma 3 generalizes Lemma 1 in [5].

Before stating our first theorem we observe that if (4) and (8) hold, then

$$h_0(t) = \int_t^\infty [q(s) - P|r(s)|] ds/a^{1/2}(t)$$

is a well-defined function on $[t_0, \infty)$ for every positive constant P in the sense that the improper integral converges. As long as the improper integrals involved converge,

we can define

$$h_1(t) = \int_t^\infty h_0^2(s) \, ds$$

and

$$h_{n+1}(t) = \int_t^\infty [h_0(s) + Lh_n(s)/a^{1/2}(s)]^2 \, ds$$

for $n = 1, 2, 3, \dots$, where L is any positive constant.

In the next two theorems we will need the condition that for every constant $L > 0$ there exists a positive integer N such that

$$(16) \quad h_n \text{ exists for } n = 0, 1, 2, \dots, N-1 \text{ and } h_N \text{ does not exist.}$$

Theorem 4. Suppose that (2)–(5), (8), and (16) hold, and for any $\lambda_1 > 0$ there exists $\lambda_2 > 0$ such that

$$(17) \quad f'(x)/\psi(x) \geq \lambda_2 \quad \text{for all } |x| \geq \lambda_1.$$

Suppose, furthermore, that for any $P > 0$

$$(18) \quad h_0(t) \geq 0$$

for all sufficiently large t . Then any solution $x(t)$ of (1) is either oscillatory or satisfies $\liminf_{t \rightarrow \infty} |x(t)| = 0$.

Proof. Assume that the conclusion of the theorem is false. Then there is a solution $x(t)$ of (1) such that $\liminf_{t \rightarrow \infty} |x(t)| > 0$. It then follows from (5) that $|f(x(t))| \geq M$ for some $M > 0$ and all $t \geq t_1$ for some $t_1 \geq t_0$. From (12) and (17) we then have

$$(19) \quad W(t) \geq a^{1/2}(t) h_0(t) + L \int_t^\infty [W^2(s)/a(s)] \, ds$$

for $t \geq t_1$ and some $L > 0$. Now (10) implies that

$$(20) \quad \int_{t_1}^\infty [W^2(s)/a(s)] \, ds < \infty$$

and (18) and (19) imply that $W(t) \geq a^{1/2}(t) h_0(t) \geq 0$ so

$$(21) \quad W^2(t)/a(t) \geq h_0^2(t).$$

If $N = 1$, then (20) and (21) imply that $h_1(t) = \int_t^\infty h_0^2(s) \, ds < \infty$ which contradicts the nonexistence of $h_N(t) = h_1(t)$. If $N = 2$, then (19) and (21) yield

$$(22) \quad W(t) \geq a^{1/2}(t) h_0(t) + L \int_t^\infty h_0^2(s) \, ds = a^{1/2}(t) h_0(t) + Lh_1(t)$$

so

$$W^2(t)/a(t) \geq [h_0(t) + Lh_1(t)/a^{1/2}(t)]^2.$$

In view of (20), an integration of the above inequality would give a contradiction to the nonexistence of $h_N(t) = h_2(t)$. A similar argument leads to a contradiction for any integer $N > 2$.

As an example of an equation satisfying the hypotheses of Theorem 4 consider

$$\begin{aligned} (x^2 x')' + [(2 + \sin t - 2t \cos t)/2t^{3/2}] x^3 &= \\ = 4/t^5 + (2 + \sin t - 2t \cos t)/2t^{9/2}, \quad t \geq 1 \end{aligned}$$

which has the nonoscillatory solution $x(t) = 1/t$. Here $f'(x)/\psi(x) \equiv 3$ and

$$\int_t^\infty q(s) ds = (2 + \sin t)/t^{1/2} \geq 1/t^{1/2}.$$

Now $|r(t)| \leq 7/t^3$ so

$$\int_t^\infty |r(s)| ds \leq 7/2t^2$$

and, hence, $h_0(t) \geq 0$ for all sufficiently large t . Since

$$\int_t^\infty h_0^2(s) ds \geq \int_t^\infty [1/s^{1/2} - 7P/2s^2]^2 ds = \infty$$

we have that $N = 1$.

Although Staikos and Sficas [17] obtained results similar to Theorem 4 for equation (1), none of the results in [17] apply to the preceding example. Also, Theorem 4 generalizes Theorem 2 in [5].

Our next two theorems are oscillation results for the case when $r(t) \equiv 0$. Observe that in this case equation (1) becomes

$$(23) \quad (a(t) \psi(x) x')' + q(t) f(x) = 0$$

and

$$h_0(t) = \int_t^\infty q(s) ds / a^{1/2}(t).$$

Theorem 5. *Suppose that conditions (2)–(3), (8), (9), (16), and (18) hold and there exists $\lambda > 0$ such that*

$$(24) \quad f'(x)/\psi(x) \geq \lambda \quad \text{for all } x \neq 0.$$

Then all solutions of (23) are oscillatory.

Proof. Let $x(t)$ be a nonoscillatory solution of (23). Then there exists $t_1 \geq t_0$ such that $|x(t)| > 0$ for $t \geq t_1$, and since (24) implies that $f'(x) \geq 0$ for $x \neq 0$, we have $|f(x(t))| > 0$ for $t \geq t_1$. It is easy to see that Lemma 2 is valid for equation (23) with condition (6) replaced by

$$(6') \quad -W(T_1) + \int_{T_1}^t q(s) ds + \int_{T_1}^T [f'(x(s)) W^2(s)/a(s) \psi(x(s))] ds \geq A_1.$$

Proceeding as in the proof of Lemma 3, we again obtain (10), i.e.,

$$(10) \quad \int_0^\infty [f'(x(s)) W^2(s)/a(s) \psi(x(s))] ds < \infty$$

since (13) obviously holds. Using (14) with $r(t) \equiv 0$ and continuing as in the proof of Lemma 3, we again obtain

$$(11) \quad W(t) \rightarrow 0 \quad \text{as } t \rightarrow \infty$$

and (12) with $r(t) \equiv 0$, i.e.,

$$(12') \quad W(t) = \int_t^\infty [f'(x(s)) W^2(s)/a(s) \psi(x(s))] ds + \int_t^\infty q(s) ds$$

for all sufficiently large t . The remainder of the proof of this theorem is similar to the proof of Theorem 4 and will be omitted.

Notice that obtaining (10), (11), and (12') in the proof of Theorem 5 extends the lemmas in [2] and [8] and Corollary 4 and Theorem 1 in [12]. Also, Theorem 5 reduces to Theorem 1 in [2] when $\psi(x) \equiv 1$ and to Theorem 1 in [8] when $\psi(x) \equiv 1$ and $a(t) \equiv 1$.

Also notice that for the case $\psi(x) \equiv a(t) \equiv 1$ that (24) becomes

$$(24') \quad f'(x) \geq \lambda > 0, \quad x \neq 0.$$

For this case Kwong and Wong [12, Th. 3] obtained the conclusion of Theorem 5 under the hypotheses (3), (8), (24'),

$$(25) \quad \int_{\pm 1}^{\pm \infty} [1/f(x)] dx < \infty,$$

and $\limsup_{T \rightarrow \infty} \int_{t_0}^T (h_0(t) + \int_t^\infty H_+^2(s) ds) dt = \infty$ where $H_+(s) = \max \{h_0(s), 0\}$.

Observe that Theorem 5 implies that the equation

$$x'' + [(2 + \sin t - 2t \cos t)/2t^{3/2}] (x + x^{1/3}) = 0, \quad t > 0$$

is oscillatory whereas this result cannot be deduced from [12, Th. 3] since (25) fails.

Theorem 5 is also related to numerous other results contained in and referred to in our list of references. However, none of these appear to contain Theorem 5. We illustrate the independence of Theorem 5 and a number of these results by considering the examples

$$(26) \quad x'' + [(2 + \sin t - 2t \cos t)/2t^{3/2}] x = 0, \quad t > 0$$

$$(27) \quad x'' + [(2 + \sin t - 2t \cos t)/2t^{3/2}] (x + x^3) = 0, \quad t > 0$$

$$(28) \quad (x^2 x')' + [(2 + \sin t - 2t \cos t)/2t^{3/2}] x^3 = 0, \quad t > 0$$

and

$$(29) \quad x'' + (x + x^3)/t \ln^2 t = 0, \quad t > 1.$$

Theorem 5 implies that each of (26)–(29) is oscillatory. However, Theorem 2.1 in [4], Theorem 6 in [9], Corollary 1 in [13], the Corollary and Theorem 2 in [19], and Corollary 1 in [20] do not apply to either (26) or (27); none of the results in [16] apply to (27); Theorem 2.3 in [4], Theorems 4 and 5 in [14], and none of the results in [1] or [17] apply to (28); Corollary 3 in [12] and Corollary 2 in [20] do not apply to (29).

Theorem 6. *Suppose that conditions (2)–(3), (8), (18), and (24) hold,*

$$(30) \quad \int_1^{\infty} [\psi(s)/f(s)] ds < \infty \quad \text{and} \quad \int_{-1}^{-\infty} [\psi(s)/f(s)] ds < \infty$$

and for some positive integer N the functions h_n exist for $n = 0, 1, 2, \dots, N$. If for every $B > 0$

$$\int_1^{\infty} \{[h_0(s) + Bh_N(s)]/a^{1/2}(s)\} ds = \infty,$$

then all solutions of (23) are oscillatory.

Proof. Letting $x(t)$ be a nonoscillatory solution of (23) and proceeding as in the proof of Theorem 5 (also see (22) in the proof of Theorem 4), we eventually obtain

$$W(t) \geq a^{1/2}(t) h_0(t) + Bh_N(t)$$

for $t \geq t_1$ for some $t_1 \geq t_0$ and $B > 0$. Hence

$$\psi(x(t)) x'(t)/f(x(t)) \geq [h_0(t) + Bh_N(t)]/a^{1/2}(t)$$

and integrating we obtain

$$\int_{x(t_1)}^{x(t)} [\psi(s)/f(s)] ds \geq \int_{t_1}^t \{[h_0(s) + Bh_N(s)]/a^{1/2}(s)\} ds \rightarrow \infty$$

as $t \rightarrow \infty$ which contradicts (30).

Theorem 6 extends Theorem 2 in [2] and Theorem 2 in [8] and applies to equations to which Theorem 2 in [17] is not applicable.

It is interesting to observe that, in view of (18), Theorem 6 reduces to one case of Theorem 3 in [12] when $\psi(x) \equiv a(t) \equiv 1$ and $N = 1$. Notice also that because of (30), Theorem 6 is not applicable when $f(x) = x$ whereas Theorem 5 is.

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