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ON THE p-ADIC HEIGHTS OF SOME ABELIAN VARIETIES

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ABSTRACT. For an abelian variety defined over an algebraic number field, different definitions of p-adic heights have been given by several authors. In this note, we shall prove that the p-adic height defined by A. Néron and that by P. Schneider coincide.

1. In this section we shall recall briefly the construction of p-adic heights. Let K be a finite extension field of the p-adic rational number field $\mathbf{Q}_p$, where p is a prime number. Let $\mathcal{O}$ be the integer ring of K, $\mathfrak{p}$ the maximal ideal of $\mathcal{O}$, and let $k = \mathcal{O}/\mathfrak{p}$ be the residue field of $\mathcal{O}$. Let A be an abelian variety of dimension d defined over K, $\delta = \mathrm{id}_A$ the identity map of A. Let 0 be the identity point of A and let $t_1, \dots, t_d$ be a $\mathfrak{p}$-admissible system of local coordinates of A at 0 in the sense of A. Néron; namely, $t_1, \dots, t_d$ are K-rational functions on A which constitute a local coordinate system at 0 and their reductions modulo $\mathfrak{p}$ are k-rational functions on $A \times_{\mathcal{O}} k$ also constitute a local coordinate system at the identity point of $A \times_{\mathcal{O}} k$.

Let $A(K)$ be the group of K-rational points of A and let U be an open neighborhood of 0 in $A(K)$. A function $\Phi\colon U \to K$ is called $\mathcal{O}$-analytic on U if for $a \in U$ we have

$$\Phi(a) = \phi(t_1(a), \dots, t_d(a)),$$

where $\phi = \phi(\tau_1, \dots, \tau_d)$ is a d-variable power series with coefficients in $\mathcal{O}$ which converges at $(t_1(a), \dots, t_d(a))$ for all $a \in U$.

A K-rational divisor Δ on A is called disjoint from $0 \bmod \mathfrak{p}$ if any component of the set obtained from the reduction mod $\mathfrak{p}$ of the support of Δ does not contain the identity point of $A \times_{\mathcal{O}} k$.

For a subgroup G of $A(K)$, let $\Lambda = \mathbf{Z}[G]$ be the group ring of G with coefficients in $\mathbf{Z}$, which is also the group of 0-cycles with components in G. Let $I \subset \Lambda$ be the augmentation ideal of Λ; i.e., the ideal generated by the cycles of the form $(a) - (0)$ with $a \in G$. Let $I^2 \subset \Lambda$ be the ideal generated by the cycles of the form $(a+b) - (a) - (b) + (0)$ with $a, b \in G$. As $(a+b) - (a) - (b) + (0) = ((a) - (0)) * ((b) - (0))$, where $*$ is the multiplication in Λ, I^2 coincides with the square of I; $I^2 = \{\mathfrak{a} * \mathfrak{b} \mid \mathfrak{a}, \mathfrak{b} \in I\}$.

A divisor and a 0-cycle are called disjoint if their supports are disjoint. For a divisor $\Delta = \mathrm{div}(f)$ linearly equivalent to 0, and a 0-cycle $\mathfrak{a} \in I$ which are disjoint, we define a pairing $[\Delta, \mathfrak{a}]$ by $[\Delta, \mathfrak{a}] = \prod f(a_i)^{m_i}$ for $\mathfrak{a} = \sum m_i(a_i)$, as usual. For a divisor Δ algebraically equivalent to 0 and a 0-cycle $\mathfrak{a} = \sum m_i(a_i)$, let $\Delta * a = \sum m_i \Delta_{a_i}$, where Δ_{a_i} is the translate of Δ by a_i, and let $\mathfrak{a}^- = \sum m_i(-a_i)$. If $\mathfrak{a} \in I$, then $\Delta * \mathfrak{a}$ is linearly equivalent to 0. So for $\mathfrak{a} = \mathfrak{b} * \mathfrak{c} \in I^2$, Δ algebraically

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equivalent to 0, the value $[\Delta * \mathfrak{b}^-, \mathfrak{c}]$ has a meaning. We define the pairing $[\Delta, \mathfrak{a}]$ by $[\Delta, \mathfrak{a}] = [\Delta * \mathfrak{b}^-, \mathfrak{c}]$. By a certain reciprocity law, this pairing is known to be well defined.

In [**4**], A. Néron has shown the following fact: Let G be an open subgroup of $A(K)$ such that $t_i(a) \in p\mathcal{O}$ for $a \in G$, $i = 1, \dots, d$, where $t_1, \dots, t_d$ are $\mathfrak{p}$-admissible system of local coordinates of A at 0. Let Δ be a K-rational divisor on A algebraically equivalent to 0 and disjoint from 0 mod $\mathfrak{p}$. Then for $\mathfrak{a} \in I = I_G$,

$$\theta'_{\Delta,0}(\mathfrak{a}) = \lim_{\nu \to \infty} [\Delta, p^\nu \mathfrak{a} - p^\nu \delta \mathfrak{a}]^{1/p^\nu}$$

converges and satisfies $\theta'_{\Delta,0}(0) = 1$, $\theta'_{\Delta,0}(\mathfrak{a}) = [\Delta, \mathfrak{a}]$ for $\mathfrak{a} \in I^2$. The function $\theta'_{\Delta,0}(a) = \theta'_{\Delta,0}((a) - (0))$ on $a \in G$ is an $\mathcal{O}$-analytic function with values in $1 + p\mathcal{O}$ (cf. [**4**, Théorème and its proof]).

In general, for a K-rational divisior Δ not necessarily disjoint from 0 mod $\mathfrak{p}$, take a K-rational function f on A such that $\operatorname{div}(f) + \Delta = \Delta'$ is disjoint from 0 mod $\mathfrak{p}$, and put $\theta'_{\Delta,0}(\mathfrak{a}) = \theta'_{\Delta',0}(\mathfrak{a}) f(\mathfrak{a})^{-1}$.

REMARK. Note that for a divisor $\Delta = \operatorname{div}(f)$ linearly equivalent to 0, Neron's $\mathfrak{p}$-adic theta function $\theta'_{\Delta,0}$ differs from the function f slightly (cf. [**4**, §4(b)]). So there are some ambiguities in the above definition of $\theta'_{\Delta,0}$ in the case of Δ not disjoint from 0 mod $\mathfrak{p}$. However from op. cit., for $\mathfrak{a} \in I$, $\theta'_{\Delta,0}(\mathfrak{a})$ is defined up to units so this ambiguity disappears when we take the image of $\theta'_{\Delta,0}(\mathfrak{a})$ by a homomorphism which is trivial on the unit group.

In §3 we shall prove the following Proposition.

PROPOSITION. *Let Ω be the completion of the algebraic closure of K, R the integer ring of Ω, and $\mathfrak{m}$ the maximal ideal of R. Let $\hat{A}(R) = \{a \in A(\Omega) | t_i(a) \in \mathfrak{m}$, $i = 1, \dots, d\}$. Assume that A has ordinary good reduction* mod $\mathfrak{p}$. *Let Δ be a K-rational divisor on A algebraically equivalent to* 0 *and disjoint from* 0 mod $\mathfrak{p}$. *Then the limit*

$$\theta'_{\Delta,0}(\mathfrak{a}) = \lim_{\nu \to \infty} [\Delta, p^\nu \mathfrak{a} - p^\nu \delta \mathfrak{a}]^{1/p^\nu}$$

converges for $\mathfrak{a} = (a) - (0)$ with $a \in \hat{A}(R)$, and $\theta'_{\Delta,0}(a) = \theta'_{\Delta,0}((a) - (0))$ defines an $\mathcal{O}$-analytic function on $\hat{A}(R)$ with values in $1 + \mathfrak{m}$, satisfying $\theta'_{\Delta,0}(\mathfrak{a}) = [\Delta, \mathfrak{a}]$ for $\mathfrak{a} \in I^2 = I^2_{\hat{A}(R)}$. Moreover, $\theta'_{\Delta,0}$ is the unique $\mathcal{O}$-analytic function on $\hat{A}(R)$ which satisfies $\theta'_{\Delta,0}(\mathfrak{a}) = [\Delta, \mathfrak{a}]$ for $\mathfrak{a} \in I^2$.

A. Néron has extended his $\mathfrak{p}$-adic theta function $\theta'_{\Delta,0}$ to $A(K)$ in the following way. Let Δ be a K-rational divisor on A algebraically equivalent to 0 and let $\mathfrak{a} = (a) - (0)$ be a 0-cycle disjoint from Δ with $a \in A(K)$. Let m be the exponent of the finite group $A(K)/G$ (G is an open subgroup of $A(K)$ such that $t_i(a) \in p\mathcal{O}$ for $a \in G$, $i = 1, \dots, d$). Then define the extension $\tilde{\theta}'_{\Delta,0}$ of $\theta'_{\Delta,0}$ by

$$[\Delta, m\mathfrak{a} - m\delta\mathfrak{a}] = \tilde{\theta}'_{\Delta,0}(\mathfrak{a})^m \theta'_{\Delta,0}(m\delta\mathfrak{a})^{-1}.$$

So $\tilde{\theta}'_{\Delta,0}(a) = \tilde{\theta}'_{\Delta,0}((a) - (0))$ is defined up to mth root of unity (with value in the algebraic closure of K).

Now let K be an algebraic number field of finite degree. Let A be an abelian variety defined over K. For a nonarchimedean prime v of K, let K_v be the completion of K at v, $\mathcal{O}_v$ the integer ring of K_v, $\mathfrak{p}_v$ the maximal ideal of $\mathcal{O}_v$, and let

$k_v = \mathcal{O}_v/\mathfrak{p}_v$ be the residue field at v. Let $\mathbf{A}_K^\times$ be the group of finite ideles of K; i.e., $\mathbf{A}_K^\times$ is the restricted direct product $\prod' K_v^\times$ where the product ranges through all of the nonarchimedean primes of K and (x_v) is in $\mathbf{A}_K^\times$ if $x_v \in \mathcal{O}_v^\times$ for all but a finite number of primes v. We have a canonical homomorphism $K^\times \to \mathbf{A}_K^\times$ and we identify $K^\times$ with its image. Let p be a fixed prime number such that A has ordinary good reduction at all primes of K over p. Let $\rho\colon \mathbf{A}_K^\times \to \mathbf{Z}_p$ be a nontrivial continuous homomorphism which is trivial on $K^\times$ corresponding to a $\mathbf{Z}_p$-extension of K by the class field theory. For a nonarchimedean prime v, let ρ_v be the component of ρ at v.

Let Δ be a K-rational divisor on A algebraically equivalent to 0 and $a \in A(K)$, which are disjoint. For a nonarchimedean prime v, we can construct the v-adic theta function $\tilde{\theta}'_{\Delta,0,v}(a)$ substituting A, K_v instead of A, K in the above construction.

Then A. Néron's p-adic height corresponding to the homomorphism ρ is defined by

$$h_{\Delta,\rho}(a) = \sum \rho_v(\tilde{\theta}'_{\Delta,0,v}(a)),$$

where the summation ranges through all of the nonarchimedean primes of K, and as $\tilde{\theta}'_{\Delta,0,v}(a)^m \in K_v^\times$ for some m, we put

$$\rho_v(\tilde{\theta}'_{\Delta,0,v}(a)) = \frac{1}{m}\rho_v(\tilde{\theta}'_{\Delta,0,v}(a)^m).$$

It can be shown that this function is well defined and depends only on the linear equivalence class of Δ.

Let $A' = \operatorname{Pic}^0(A)$ be the dual abelian variety. Then we obtain a pairing from $A(K) \times A'(K)$ to $\mathbf{Q}_p$ by putting

$$\langle a, \operatorname{cl}(\Delta)\rangle_N = h_{\Delta,\rho}(a),$$

here $\operatorname{cl}(\Delta)$ is the image of Δ in A'.

Next we recall the construction of P. Schneider's p-adic height pairing (as P. Schneider's p-adic height pairing coincides with Mazur and Tate's p-adic height pairing (cf. [**3**, §4.4]), we recall the latter).

Let K be an algebraic number field of finite degree and A an abelian variety defined over K. Let $\mathcal{O}$ be the integer ring of K and let $\mathcal{A}$ be the Néron model of A over $\mathcal{O}$. Let $\mathcal{A}^0$ be the connected component of the identity of $\mathcal{A}$ and let A' be the dual abelian variety of A. For a K-rational divisor Δ of A algebraically equivalent to 0 there corresponds an extension X of $\mathcal{A}^0$ by the multiplicative group $\mathbf{G}_m$ over $\mathcal{O}$ (cf. [**1**, §1]);

$$0 \to \mathbf{G}_m \to X \xrightarrow{\pi} \mathcal{A}^0 \to 0. \tag{1}$$

Here we note that for an extension field L of K, the group of L-rational points $X(L)$ of X may be described as follows (cf. [**3**, p. 210]): $X(L)$ is the set of pairs $(\mathfrak{a}, c)$ where $\mathfrak{a}$ is a 0-cycle of degree 0 with L-rational components and $c \in L^\times$; $c(\mathfrak{a}, 1) = (\mathfrak{a}, c)$, $(\mathfrak{a}_1, c_1)(\mathfrak{a}_2, c_2) = (\mathfrak{a}_1 + \mathfrak{a}_2, c_1c_2)$, when $(\mathfrak{a}, c)$ maps to $0 \in A(L)$ (i.e., when $\mathfrak{a} \in I^2$ is Néron's notation recalled as before), then $(\mathfrak{a}, c) = (0, [\Delta, \mathfrak{a}]c)$. For $a \in A(L)$, the map $\sigma_\Delta : a \mapsto ((a) - (0), 1)$ defines a K-rational section of π, and σ_Δ is regular on $A(L)\backslash|\Delta|$.

Let p be a fixed prime number such that A has ordinary good reduction at the primes of K over p. Let $\rho\colon \mathbf{A}_K^\times \to \mathbf{Z}_p$ be a nontrivial continuous homomorphism which is trivial on $K^\times$ corresponding to a $\mathbf{Z}_p$-extension of K, where $\mathbf{A}_K^\times$ is the group of finite ideles of K as before. For a nonarchimedean prime v of K, let ρ_v be the component of ρ at v.

By considering the K_v-rational points of (1) we obtain the following exact sequence:

$$0 \to K_v^\times \to X(K_v) \to A(K_v) \to 0$$

(note that $H^1(R, \mathbf{G}_m) = 0$ for a local ring R). In [**3**, Theorem (1.5)], Mazur and Tate have constructed a canonical ρ_v-splitting $\psi_{\rho,v}\colon X(K_v) \to \mathbf{Q}_p$ for the above exact sequence (in our case, canonical ρ_v-splitting is the restriction to the fiber at $\mathrm{cl}(\Delta) \in A'(K)$ of their canonical ρ_v-splitting of the biextension of $A(K_v) \times A'(K_v)$ by $K_v^\times$). We recall that a homomorphism $\psi_{\rho,v}\colon X(K_v) \to \mathbf{Q}_p$ is called a ρ_v-splitting if it satisfies

$$\psi_{\rho,v}(cx) = \rho_v(c) + \psi_{\rho,v}(x) \quad \text{for } c \in K_v^\times,\ x \in X(K_v).$$

(The characterizing property of canonicalness will be explained later.)

Now let $a \in A(K)$ and take a point $x \in X(K)$ mapping to a. Mazur and Tate have defined their p-adic height pairing by

$$\langle a, \mathrm{cl}(\Delta)\rangle_S = \sum \psi_{\rho,v}(x),$$

where the summation ranges through the nonarchimedean primes of K. Here we note that the sum on the right is well defined and independent of the choice of x and depends only on the linear equivalence class of Δ.

2. The result of this note is the following Theorem.

THEOREM. *Let K be an algebraic number field of finite degree and A an abelian variety defined over K. Let p be a prime number such that A has ordinary good reduction at all primes of K over p. Let $\rho\colon \mathbf{A}_K^\times \to \mathbf{Z}_p$ be a nontrivial continuous homomorphism trivial on $K^\times$, corresponding to a $\mathbf{Z}_p$-extension of K. Then for $a \in A(K)$ and Δ a K-rational divisior on A algebraically equivalent to* 0, *we have*

$$\langle a, \mathrm{cl}(\Delta)\rangle_N = \langle a, \mathrm{cl}(\Delta)\rangle_S.$$

PROOF. Let the notations be the same as in §1. Let v be a nonarchimedean prime of K. We take a divisor from the linear equivalence class of Δ, which is disjoint from 0 mod v for all primes v over p (for simplicity, we write this divisor by Δ). We compare the local components of Néron's and Schneider's p-adic height pairings.

First we treat the case where v is a prime not above p. In this case ρ_v is trivial on $\mathcal{O}_v^\times$ as ρ is continuous. Let q be the residue characteristic of v, and let G be an open subgroup of $\mathcal{A}^0(\mathcal{O}_v)$ such that $t_i(a) \in q\mathcal{O}_v$ for $a \in G$, $i = 1, \dots, d$; here $t_1, \dots, t_d$ are v-admissible systems of local coordinates of A at 0.

If $\Delta = \mathrm{div}(f)$ is linearly equivalent to 0, then from [**3**, (2.2.2)], we have $\psi_{\rho,v}((\mathfrak{a}, 1)) = \rho_v(f(\mathfrak{a}))$ for $\mathfrak{a} \in I = I_G$, whenever the right-hand side is defined. On the other hand from the Remark before the Proposition in §1 (cf. also [**4**, §4(b)]), $\rho_v(\theta'_{\Delta,0}(\mathfrak{a})) = \rho_v(f(\mathfrak{a}))$.

In general let Δ be a divisor algebraically equivalent to 0. Take a K_v-rational function on A such that $\mathrm{div}(f) + \Delta = \Delta'$ is disjoint from 0 mod v. We consider the objects of §1 defined with respect to Δ'. Now in the case we are considering, the characterizing property of canonicalness of ρ_v-splitting is that $\psi_{\rho,v} = 0$ on $X(\mathcal{O}_v)$, where X is the extension of $\mathcal{A}^0$ by $\mathbf{G}_m$ corresponding to Δ'. Let $m = [A(K_v) \colon G]$, where G is an open subgroup of $\mathcal{A}^0(\mathcal{O}_v)$ as above. Then we have for $\mathfrak{a} = (a) - (0)$, $a \in A(K_v)$.

$$\begin{aligned}\psi_{\rho,v}((\mathfrak{a}, 1)) &= \frac{1}{m}\psi_{\rho,v}((m\mathfrak{a}, 1)) \\ &= \frac{1}{m}\psi_{\rho,v}((0, [\Delta', m\mathfrak{a} - m\delta\mathfrak{a}])) + \frac{1}{m}\psi_{\rho,v}((m\delta\mathfrak{a}, 1)) \\ &= \frac{1}{m}\rho_v([\Delta', m\mathfrak{a} - m\delta\mathfrak{a}]),\end{aligned}$$

as $m\delta\mathfrak{a}$ has support in $\mathcal{A}^0(\mathcal{O}_v)$.

On the other hand from the equation defining the v-adic theta function $\tilde{\theta}'_{\Delta',0,v}$, we have

$$\frac{1}{m}\rho_v([\Delta', m\mathfrak{a} - m\delta\mathfrak{a}]) = \rho_v(\tilde{\theta}'_{\Delta',0,v}(\mathfrak{a})),$$

as $\theta'_{\Delta',0,v}(m\delta\mathfrak{a}) \in 1 + q\mathcal{O}_v$.

From the multiplicativity of the local pairings with respect to Δ, we see that in this case the local components of Néron's and Schneider's p-adic height pairings coincide.

In the case when v is a prime over p, we consider the fiber at v of the exact sequence (1). Taking the formal completion along the maximal torus of the special fiber (we denote this completion by superscript t) we obtain the following exact sequence (cf. [**3**, §5])

$$0 \to \mathbf{G}_m^t \to X^t \to A^t \to 0.$$

Mazur and Tate have proved that the above exact sequence has a unique splitting $\psi \colon X^t \to \mathbf{G}_m^t$. The characterizing property of canonicalness of ρ_v-splitting is that $\psi_{\rho,v} = \rho_v \circ \psi$ on $X^t(\mathcal{O}_v)$. Now take the formal completion along the 0-section of the special fiber of the exact sequence (1) (we denote this completion by $\hat{X}$ etc.); we obtain the following exact sequence:

$$0 \to \hat{\mathbf{G}}_m \to \hat{X} \to \hat{A} \to 0.$$

Let Ω_v be the completion of the algebraic closure of K_v, R_v the integer ring of Ω_v, and let $\mathfrak{m}_v$ be the maximal ideal of R_v. Considering the R_v-valued points we obtain the following exact sequence:

$$0 \to 1 + \mathfrak{m}_v \to \hat{X}(R_v) \to \hat{A}(R_v) \to 0. \tag{2}$$

From Mazur and Tate's splitting $\psi \colon X^t \to \mathbf{G}_m^t$, we obtain a splitting $\psi' \colon \hat{X}(R_v) \to 1 + \mathfrak{m}_v$ of the above exact sequence. Now Δ is disjoint from $0 \bmod v$. From the Proposition in §1, for $\mathfrak{a} = (a) - (0)$ with $a \in \hat{A}(R_v)$,

$$\theta'_{\Delta,0,v}(\mathfrak{a}) = \lim_{\nu\to\infty}[\Delta, p^\nu\mathfrak{a} - p^\nu\delta\mathfrak{a}]^{1/p^\nu}$$

converges and defines an $\mathcal{O}_v$-analytic function on $\hat{A}(R_v)$. From $\theta'_{\Delta,0,v}(\mathfrak{a}) = [\Delta, \mathfrak{a}]$ for $\mathfrak{a} \in I^2$, by putting $\mathfrak{a} = (a+b) - (a) - (b) + (0)$ with $a, b \in \hat{A}(R_v)$ we have

$$\theta'_{\Delta,0,v}(a+b)\theta'_{\Delta,0,v}(a)^{-1}\theta'_{\Delta,0,v}(b)^{-1} = [\Delta, \mathfrak{a}] = \sigma_\Delta(a+b)\sigma_\Delta(a)^{-1}\sigma_\Delta(b)^{-1};$$

here $\sigma_\Delta\colon a \mapsto ((a)-(0),1)$ is the K-rational section of $\pi\colon X \to A$ such that $\sigma_\Delta(0)=1$. Hence $\sigma_\Delta \cdot \theta'^{-1}_{\Delta,0,v}\colon \hat{A}(R_v) \to \hat{X}(R_v)$ is a section of (2) as abstract groups. As $\theta'_{\Delta,0,v}$ is $\mathcal{O}_v$-analytic and σ_Δ is K-rational, from [**8**, (4.2), Corollary 1], $\sigma_\Delta \cdot \theta'^{-1}_{\Delta,0,v}$ is in fact a formal group homomorphism defined over $\mathcal{O}_v$. We define $\psi''\colon \hat{X}(R_v) \to 1+\mathfrak{m}_v$ by

$$\psi''(x) = x/(\sigma_\Delta \cdot \theta'^{-1}_{\Delta,0,v}(\pi(x))),$$

then ψ'' is a splitting of (2) as abstract groups. As $\psi' \cdot \psi''^{-1}$ is trivial on $1+\mathfrak{m}_v$, it induces a formal group homomorphism $\psi' \cdot \psi''^{-1}\colon \hat{A} \to \hat{\mathbf{G}}_m$ defined over $\mathcal{O}_v$ (cf. ibid.). We need the following Lemma.

LEMMA. *Let $\hat{A}$ be the formal group of an abelian variety defined over K_v, which has good reduction* mod v. *Then any formal group homomorphism $\phi\colon \hat{A} \to \hat{\mathbf{G}}_m$ defined over $\mathcal{O}_v$ is trivial.*

PROOF OF THE LEMMA. If ϕ were nontrivial, there should exist a surjective homomorphism ψ such that $\phi = \psi^m$ for some m (cf. [**8**, (4.2), Corollary 1]). The reduction mod v of the homomorphism ψ is a surjective homomorphism of formal groups over k_v, $\psi/k_v\colon \hat{A}/k_v \to \hat{\mathbf{G}}_m/k_v$. Consider the characteristic polynomials of the Frobenius endomorphism of $\bar{k}_v$ over k_v acting on these p-divisible groups over k_v (cf. [**2**, §4.e]). The multiplicativity of the characteristic polynomials and the Riemann-Weil hypothesis lead to a contradiction.

By the Lemma, $\psi' = \psi''$. We apply the Mazur-Tate's construction of p-adic height pairing recalled as before. Let $m = [A(K_v)\colon \hat{A}(\mathcal{O}_v)]$, which is the cardinality of the group of rational points of the special fiber of A at v. As before, let $\mathfrak{a} = (a)-(0)$. Then we have

$$\begin{aligned}
\psi_{\rho,v}((\mathfrak{a},1)) &= \frac{1}{m}\psi_{\rho,v}((m\mathfrak{a},1)) \\
&= \frac{1}{m}\rho_v([\Delta, m\mathfrak{a} - m\delta\mathfrak{a}]) + \frac{1}{m}\rho_v \circ \psi'((m\delta\mathfrak{a},1)) \\
&= \frac{1}{m}\rho_v([\Delta, m\mathfrak{a} - m\delta\mathfrak{a}]) + \frac{1}{m}\rho_v \circ \psi''(\sigma_\Delta(ma)) \\
&= \frac{1}{m}\rho_v([\Delta, m\mathfrak{a} - m\delta\mathfrak{a}]) + \frac{1}{m}\rho_v \circ \theta'_{\Delta,0,v}(ma) \\
&= \rho_v(\tilde{\theta}'_{\Delta,0,v}(\mathfrak{a})).
\end{aligned}$$

So $\psi_{\rho,v}((\mathfrak{a},1)) = \rho_v(\tilde{\theta}'_{\Delta,0,v}(\mathfrak{a}))$ for all nonarchimedean primes v. The Theorem is proved.

3. In this section we prove the Proposition of §1. Let the notations be the same as in the Proposition. Let f be the K-rational function on A such that $\operatorname{div}(f) = (p\delta)^{-1}\Delta - p\Delta$, $f(0)=1$. We expand f as a power series in $t_1,\dots,t_d$ near 0, then from [**4**, p. 158], we see that $f \in \mathcal{O}[[t_1,\dots,t_d]]$ and $f-1 \equiv 0 \pmod{\deg 2}$. Also from the arguments used in ibid. p. 158, to prove the convergence of $\theta'_{\Delta,0}$, it suffices to show that for $a \in \hat{A}(R)$, $\operatorname{ord}_p((f(p^\nu a)-1)/p^\nu)$ diverges to ∞ as $\nu \to \infty$, where ord_p is the additive valuation such $\operatorname{ord}_p(p)=1$. Let $\hat{A}$ be the formal group of A, and let I be the integer ring of the completion of the maximal unramified extension of K. Then $\hat{A}$ is isomorphic to $\hat{\mathbf{G}}_m^d$ over I. Hence $\hat{A}(R) \simeq (1+\mathfrak{m})^d$ over I. As this isomorphism may be expressed by some invertible power series, we see that $\mathrm{ord}_p(t_i(p^\nu a)) = \mathrm{ord}_p((1+z)^{p^\nu} - 1)$ for some $z \in \mathfrak{m}$. We expand $(1+z)^{p^\nu}$ with binomial coefficients $\binom{p^\nu}{r}$. For $r < p^{\nu-1}$, using the fact that $\mathrm{ord}_p(r!) = \sum_{k=0}^{\infty}[r/p^k]$, where $[x]$ is the largest integer such that $[x] \le x$, after some computations we have $\mathrm{ord}_p \binom{p^\nu}{r} = \nu - \mathrm{ord}_p(r)$. Hence

$$\mathrm{ord}\, p((1+z)^{p^\nu} - 1) \ge \mathrm{Min}(p^{\nu-1}\,\mathrm{ord}_p\, z, \nu - \mathrm{ord}_p\, r + r\,\mathrm{ord}_p\, z),$$

where r ranges from 1 to $p^{\nu-1} - 1$.

As $\mathrm{ord}_p(t_i(p^\nu a))$ satisfies similar inequality, we see that

$$\mathrm{ord}_p((f(p^\nu a) - 1)/p^\nu) \ge \mathrm{Min}(2p^{\nu-1}\,\mathrm{ord}_p(z_i) - \nu, \nu - 2\,\mathrm{ord}_p\, r + 2r\,\mathrm{ord}_p(z_i)),$$

where r ranges from 1 to $p^{\nu-1} - 1$, i ranges from 1 to d, and z_i are elements in $\mathfrak{m}$.

From $\mathrm{ord}_p(z_i) > 0$, we see easily that the right-hand side of the above inequality diverges to ∞ when $\nu \to \infty$.

When $\mathrm{ord}_p(z_i)$ ranges through a compact subset of $\{x \in \mathbf{R} | x > 0\}$, this convergence is uniform. Hence $\theta'_{\Delta,0}$ defines an $\mathcal{O}$-analytic function on $\hat{A}(R)$.

Now let $\mathfrak{a} = (a+b) - (a) - (b) + (0) \in I^2$. By Néron's result we have $\theta'_{\Delta,0}(\mathfrak{a}) = [\Delta, \mathfrak{a}]$ for $a, b \in \hat{A}(R)$ sufficiently close to 0 (cf. [**4**, Théorème]). For a fixed b, both sides of the above equation are given by power series in coordinates of a. Hence we have $\theta'_{\Delta,0}(\mathfrak{a}) = [\Delta, \mathfrak{a}]$ for all $\mathfrak{a} \in I^2$. Uniqueness is clear from the Lemma in §2.

REFERENCES

1. S. Bloch, *A note on height pairings, Tamagawa numbers and the Birch and Swinnerton-Dyer conjecture*, Invent. Math. **58** (1980), 65–76.
2. B. Mazur, *Rational points of abelian varieties with values in towers of number fields*, Invent. Math. **18** (1972), 183–266.
3. B. Mazur and J. Tate, *Canonical height pairing via biextensions*, Arithmetic and Geometry (Volume dedicated to Shafarevich, vol. 1), Progress in Math., vol. 35, Birkhäuser, Boston, Basel, and Stuttgart, 1983, pp. 195–273.
4. A. Néron, *Fonctions thêta p-adiques et hauteurs p-adiques* (Séminaire de théorie de nombres, Paris 1980–1981), Progress in Math., vol. 22, Birkhäuser, Boston, Basel, and Stuttgart, 1982, pp. 149–174.
5. J. Oesterlé, *Construction de hauteurs archimediennes et p-adiques suivant la methode de Bloch* (Séminaire de théorie de nombres, Paris 1980–1981), Progress in Math., vol. 22, Birkhäuser, Boston, Basel, and Stuttgart, 1982, pp. 175–192.
6. B. Perrin-Riou, *Hauteurs p-adiques* (Séminaire de théorie de nombres, Paris 1982–1983), Progress in Math., vol. 51, Birkhäuser, Boston, Basel, and Stuttgart, 1984, pp. 233–257.
7. P. Schneider, *p-adic height pairings*. I, Invent. Math. **69** (1982), 401–409.
8. J. Tate, *p-divisible groups*, Proceedings of a Conference on Local Fields, NUFFIC Summer School held at Driebergen, 1966, Springer, Berlin, Heidelberg, and New York, 1967, pp. 158–183.

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