Research Article

On the (p, q)th Relative Order Oriented Growth Properties of Entire Functions

Luis Manuel Sánchez Ruiz,[1] **Sanjib Kumar Datta,**[2]
Tanmay Biswas,[3] **and Golok Kumar Mondal**[4]

[1] *ETSID, Departamento de Matemática Aplicada & CITG, Universitat Politècnica de València, 46022 Valencia, Spain*
[2] *Department of Mathematics, University of Kalyani, Kalyani, District Nadia, West Bengal 741235, India*
[3] *Rajbari, Rabindrapalli, R. N. Tagore Road, P.O.-Krishnagar, District Nadia, West Bengal 741101, India*
[4] *Dhulauri Rabindra Vidyaniketan (H.S.), Vill +P.O. Dhulauri, P.S. Domkal, District Murshidabad, West Bengal 742308, India*

Correspondence should be addressed to Luis Manuel Sánchez Ruiz; lmsr@mat.upv.es

The relative order of growth gives a quantitative assessment of how different functions scale each other and to what extent they are self-similar in growth. In this paper for any two positive integers p and q, we wish to introduce an alternative definition of relative (p, q)th order which improves the earlier definition of relative (p, q)th order as introduced by Lahiri and Banerjee (2005). Also in this paper we discuss some growth rates of entire functions on the basis of the improved definition of relative (p, q)th order with respect to another entire function and extend some earlier concepts as given by Lahiri and Banerjee (2005), providing some examples of entire functions whose growth rate can accordingly be studied.

1. Introduction

A single valued function of one complex variable which is analytic in the finite complex plane is called an integral (entire) function. For example, exp, sin, cos, and so forth are all entire functions. In 1926 Rolf Nevanlinna initiated the value distribution theory of entire functions which is a prominent branch of Complex Analysis and is the prime concern of this paper. Perhaps the Fundamental Theorem of Classical Algebra which states that "If f is a polynomial of degree n with real or complex coefficients, then the equation $f(z) = 0$ has at least one root" is the most well known value distribution theorem, and consequently any such given polynomial can take any given, real or complex, value. In the value distribution theory one studies how an entire function assumes some values and, conversely, what is the influence in some specific manner of taking certain values on a function. It also deals with various aspects of the behavior of entire functions, one of which is the study of their comparative growth.

For any entire function f, the so-called maximum modulus function, denoted by M_f, is defined on each nonnegative real value r as

$$M_f(r) = \max_{|z|=r} |f(z)|. \tag{1}$$

And given two entire functions f and g the ratio $M_f(r)/M_g(r)$ as $r \rightarrow \infty$ is called the growth of f with respect to g in terms of their maximum moduli.

The *order* of an entire function f which is generally used in computational purpose is defined in terms of the growth of f with respect to the exponential function as

$$\rho_f = \lim \sup_{r \to \infty} \frac{\log \log M_f(r)}{\log \log M_{\exp z}(r)} = \lim \sup_{r \to \infty} \frac{\log \log M_f(r)}{\log(r)}. \tag{2}$$

Bernal [1, 2] introduced the relative order between two entire functions to avoid comparing growth just with $\exp z$. Extending the notion of relative order as cited in the

reference, in this paper we extend some results related to the growth rates of entire functions on the basis of avoiding some restriction, introducing a new type of relative order (p, q), and revisiting ideas developed by a number of authors including Lahiri and Banerjee [3].

2. Notation and Preliminary Remarks

Our notation is standard within the theory of Nevanlinna's value distribution of entire functions. For short, given a real function h and whenever the corresponding domain and range allow it, we will use the notation

$$
h^{[0]}(x) = x,
$$
$$
h^{[k]}(x) = h\left(h^{[k-1]}(x)\right) \quad \text{for } k = 1, 2, 3, \ldots
$$

(3)

omitting the parenthesis when h happens to be the log or exp function. Taking this into account the *order* (resp., *lower order*) of an entire function f is given by

$$
\rho_f = \lim_{r \to \infty} \sup \frac{\log^{[2]} M_f(r)}{\log r},
$$
$$
\left(\text{resp. } \lambda_f = \lim_{r \to \infty} \inf \frac{\log^{[2]} M_f(r)}{\log r}\right).
$$

(4)

Let us recall that Juneja et al. [4] defined the order (p, q) and lower order (p, q) of an entire function f, respectively, as follows:

$$
\rho_f(p, q) = \lim_{r \to \infty} \sup \frac{\log^{[p]} M_f(r)}{\log^{[q]} r},
$$
$$
\lambda_f(p, q) = \lim_{r \to \infty} \inf \frac{\log^{[p]} M_f(r)}{\log^{[q]} r},
$$

(5)

where p, q are any two positive integers with $p \geq q$. These definitions extended the generalized order $\rho_f^{[l]}$ and generalized lower order $\lambda_f^{[l]}$ of an entire function f considered in [5] for each integer $l \geq 2$ since these correspond to the particular case $\rho_f^{[l]} = \rho_f(l, 1)$ and $\lambda_f^{[l]} = \lambda_f(l, 1)$. Clearly $\rho_f(2, 1) = \rho_f$ and $\lambda_f(2, 1) = \lambda_f$.

In this connection let us recall that if $0 < \rho_f(p, q) < \infty$, then the following properties hold:

$$
\rho_f(p - n, q) = \infty, \quad \text{for } n < p,
$$
$$
\rho_f(p, q - n) = 0, \quad \text{for } n < q,
$$

(6)

$$
\rho_f(p + n, q + n) = 1, \quad \text{for } n = 1, 2, \ldots.
$$

Similarly for $0 < \lambda_f(p, q) < \infty$, one can easily verify that

$$
\lambda_f(p - n, q) = \infty, \quad \text{for } n < p,
$$
$$
\lambda_f(p, q - n) = 0, \quad \text{for } n < q,
$$

(7)

$$
\lambda_f(p + n, q + n) = 1, \quad \text{for } n = 1, 2, \ldots.
$$

Recalling that for any pair of integer numbers m, n the Kronecker function is defined by $\delta_{m,n} = 1$ for $m = n$ and $\delta_{m,n} = 0$ for $m \neq n$, the aforementioned properties provide the following definition.

Definition 1 (see [4]). An entire function f is said to have index-pair $(1, 1)$ if $0 < \rho_f(1, 1) < \infty$. Otherwise, f is said to have index-pair $(p, q) \neq (1, 1)$, $p \geq q \geq 1$, if $\delta_{p-q,0} < \rho_f(p, q) < \infty$ and $\rho_f(p - 1, q - 1) \notin \mathbb{R}^+$.

Definition 2 (see [4]). An entire function f is said to have lower index-pair $(1, 1)$ if $0 < \lambda_f(1, 1) < \infty$. Otherwise, f is said to have lower index-pair $(p, q) \neq (1, 1)$, $p \geq q \geq 1$, if $\delta_{p-q,0} < \lambda_f(p, q) < \infty$ and $\lambda_f(p - 1, q - 1) \notin \mathbb{R}^+$.

An entire function f of index-pair (p, q) is said to be of regular (p, q)-growth if its (p, q)th order coincides with its (p, q)th lower order; otherwise f is said to be of irregular (p, q)-growth.

Given a nonconstant entire function f defined in the open complex plane $\mathbb{C}$ its maximum modulus function M_f is strictly increasing and continuous. Hence there exists its inverse function $M_f^{-1} : (|f(0)|, \infty) \to (0, \infty)$ with $\lim_{s \to \infty} M_f^{-1}(s) = \infty$.

Then Bernal [1, 2] introduced the definition of relative order of f with respect to g, denoted by $\rho_g(f)$ as follows:

$$
\rho_g(f) = \inf\left\{\mu > 0 : M_f(r) < M_g(r^\mu), \ \forall r > r_0(\mu) > 0\right\}
$$
$$
= \lim_{r \to \infty} \sup \frac{\log M_g^{-1} M_f(r)}{\log r}.
$$

(8)

This definition coincides with the classical one [6] if $g = \exp z$. Similarly one can define the relative lower order of f with respect to g denoted by $\lambda_g(f)$ as

$$
\lambda_g(f) = \lim_{r \to \infty} \inf \frac{\log M_g^{-1} M_f(r)}{\log r}.
$$

(9)

Lahiri and Banerjee [7] gave a more generalized concept of relative order in the following way.

Definition 3 (see [7]). If $k \geq 1$ is a positive integer, then the kth generalized relative order of f with respect to g, denoted by $\rho_g^k(g)$, is defined by

$$
\rho_g^k(f) = \inf\left\{\mu > 0 : M_f(r) < M_g\left(\exp^{[k-1]} r^\mu\right), \right.
$$
$$
\left. \forall r > r_0(\mu) > 0\right\}
$$
$$
= \lim_{r \to \infty} \sup \frac{\log^{[k]} M_g^{-1} M_f(r)}{\log r}.
$$

(10)

Clearly, $\rho_g^1(f) = \rho_g(f)$ and $\rho_{\exp z}^1(f) = \rho_f$.

In the case of relative order, it was then natural for Lahiri and Banerjee [3] to define the relative (p, q)th order of entire functions as follows.

Definition 4 (see [3]). Let p and q be any two positive integers with $p > q$. The relative (p, q)th order of f with respect to g is defined by

$$\rho_g^{(p,q)}(f) = \inf \left\{ \mu > 0 : M_f(r) < M_g\left(\exp^{[p-1]}\left(\mu \log^{[q]} r\right)\right), \right.$$

$$\left. \forall r > r_0(\mu) > 0 \right\}$$

$$= \lim_{r \to \infty} \sup \frac{\log^{[p-1]} M_g^{-1} M_f(r)}{\log^{[q]} r}. \tag{11}$$

Then $\rho_{\exp z}^{(p,q)}(f) = \rho_f(p, q)$ and $\rho_g^{(k+1,1)}(f) = \rho_g^k(f)$ for any $k \geq 1$.

In this paper we give an alternative definition of (p, q)th relative order $\rho_g^{(p,q)}(f)$ of an entire function f with respect to another entire function g, in the light of index-pair. Our next definition avoids the restriction $p > q$ and gives the more natural particular case $\rho_g^{(k,1)}(f) = \rho_g^k(f)$.

Definition 5. Let f and g be any two entire functions with index-pair (m, q) and (m, p), respectively, where p, q, m are positive integers such that $m \geq \max(p, q)$. Then the (p, q)th relative order of f with respect to g is defined as

$$\rho_g^{(p,q)}(f) = \lim_{r \to \infty} \sup \frac{\log^{[p]} M_g^{-1} M_f(r)}{\log^{[q]} r}. \tag{12}$$

The (p, q)th relative lower order of f with respect to g is defined by

$$\lambda_g^{(p,q)}(f) = \lim_{r \to \infty} \inf \frac{\log^{[p]} M_g^{-1} M_f(r)}{\log^{[q]} r}. \tag{13}$$

The previous definitions are easily generated as particular cases; for example, if f and g have got index-pair $(m, 1)$ and (m, k), respectively, then Definition 5 reduces to Definition 3. If the entire functions f and g have the same index-pair $(p, 1)$, where p is any positive integer, we get the definition of relative order introduced by Bernal [1] and if $g = \exp^{[m-1]} z$, then $\rho_g(f) = \rho_f^{[m]}$ and $\rho_g^{(p,q)}(f) = \rho_f(m, q)$. And if f is an entire function with index-pair $(2, 1)$ and $g = \exp z$, then Definition 5 becomes the classical one given in [6].

3. Some Examples

In this section we present some examples of entire functions in connection with definitions given in the previous section.

Example 6 (order of exp). Given any natural number n, the exponential function $f(z) = \exp z^n$ has got $M_f(r) = \exp r^n$, and therefore $\log^{[2]} M_f(r) / \log r$ is constantly equal to n and, consequently,

$$\rho_f = \lambda_f = n. \tag{14}$$

Example 7 (generalized order). Given any natural numbers k, n, the function $f(z) = \exp^{[k]} z^n$ has got $M_f(r) = \exp^{[k]} r^n$. Therefore $\log^{[k]} M_f(r) / \log r$ is constantly equal to n for each natural $k \geq 2$, following that

$$\rho_f^{[k+1]} = \lambda_f^{[k+1]} = n. \tag{15}$$

Note that $\rho_f^{[l]} = \lambda_f^{[l]} = +\infty$ for $2 \leq l \leq k$ and $\rho_f^{[l]} = \lambda_f^{[l]} = 0$ for $l > k + 1$.

Example 8 (index-pair). Given any four positive integers k, n, p, q with $p \geq q$, then function $f(z) = \exp^{[k]} z^n$ generates a constant quotient $\log^{[p]} M_f(r) / \log^{[q]} r$, and clearly

$$\rho_f(p, q) = \lambda_f(p, q) = n, \quad \text{for } (p, q) = (k + 1, 1) \tag{16}$$

but

$$\rho_f(p, q) = \lambda_f(p, q)$$

$$= \begin{cases} 1, & \forall (p, q) \text{ such that } p = q + k, q > 1, \\ \infty, & \forall (p, q) \text{ such that } p < q + k, \\ 0, & \forall (p, q) \text{ such that } p > q + k. \end{cases} \tag{17}$$

Thus f is a regular function with growth $(k + 1, 1)$.

Example 9 (regular function of growth (1,1)). Given any positive integer n, and nonnull real number a, the power function $f(z) = az^n$ generates a constant quotient $\log^{[p]} M_f(r) / \log^{[q]} r$, and clearly

$$\rho_f(p, q) = \lambda_f(p, q) = n, \quad \text{for } (p, q) = (1, 1) \tag{18}$$

but

$$\rho_f(p, q) = \lambda_f(p, q)$$

$$= \begin{cases} 1, & \forall (p, q) \text{ such that } p = q, q > 1, \\ \infty, & \forall (p, q) \text{ such that } p < q, \\ 0, & \forall (p, q) \text{ such that } p > q. \end{cases} \tag{19}$$

Thus f is a regular function with growth $(1, 1)$.

Example 10 (relative order between functions). From the above examples it follows that given the natural numbers m, n the functions

$$f(z) = \exp z^m, \qquad g(z) = \exp z^n \tag{20}$$

are of regular growth $(2, 1)$. In order to find their relative order of growth we evaluate

$$\frac{\log M_g^{-1} M_f(r)}{\log r} = \frac{\log \left[\log \left[\exp r^m\right]\right]^{1/n}}{\log r}, \tag{21}$$

which happens to be constant. Its upper and lower limits provide

$$\rho_g(f) = \lambda_g(f) = \frac{m}{n}. \tag{22}$$

Example 11 (relative order (p, q) between functions). Let k, m, n be any three positive integers and let $f(z) = \exp^{[k]} z^m$ and $g = \exp^{[k]} z^n$. Then f and g are regular functions with $(k + 1, 1)$-growth for which

$$\rho_f (k + 1, 1) = m, \qquad \rho_g (k + 1, 1) = n. \tag{23}$$

In order to find out their $(1, 1)$ relative order we evaluate

$$\frac{\log M_g^{-1} M_f (r)}{\log r} = \frac{\log (1/n) \left\{ \log^{[k]} \left(\exp^{[k]} r^m \right) \right\}^{1/n}}{\log r}, \tag{24}$$

which happens to be constant. By taking limits, we easily get that

$$\rho_g^{(1,1)} (f) = \lambda_g^{(1,1)} (f) = \frac{m}{n}. \tag{25}$$

The orders obtained in the last two examples will be easy consequences of the results given in Section 4.

4. Results

In this section we state the main results of the paper. We include the proof of the first main theorem for the sake of completeness. The others are basically omitted since they are easily proven with the same techniques or with some easy reasoning.

Theorem 12. *Let f and g be any two entire functions with index-pair (m, q) and (m, p), respectively, where p, q, m are all positive integers such that $m \geq p$ and $m \geq q$. Then*

$$\frac{\lambda_f (m, q)}{\rho_g (m, p)} \leq \lambda_g^{(p,q)} (f) \leq \min \left\{ \frac{\lambda_f (m, q)}{\lambda_g (m, p)}, \frac{\rho_f (m, q)}{\rho_g (m, p)} \right\}$$

$$\leq \max \left\{ \frac{\lambda_f (m, q)}{\lambda_g (m, p)}, \frac{\rho_f (m, q)}{\rho_g (m, p)} \right\} \leq \rho_g^{(p,q)} (f) \tag{26}$$

$$\leq \frac{\rho_f (m, q)}{\lambda_g (m, p)}.$$

Proof. From the definitions of $\rho_f(m, q)$ and $\lambda_f(m, q)$ we have for all sufficiently large values of r that

$$M_f (r) \leq \exp^{[m]} \left\{ \left(\rho_f (m, q) + \varepsilon \right) \log^{[q]} r \right\}, \tag{27}$$

$$M_f (r) \geq \exp^{[m]} \left\{ \left(\lambda_f (m, q) - \varepsilon \right) \log^{[q]} r \right\} \tag{28}$$

and also for a sequence of values of r tending to infinity we get that

$$M_f (r) \geq \exp^{[m]} \left\{ \left(\rho_f (m, q) - \varepsilon \right) \log^{[q]} r \right\}, \tag{29}$$

$$M_f (r) \leq \exp^{[m]} \left\{ \left(\lambda_f (m, q) + \varepsilon \right) \log^{[q]} r \right\}. \tag{30}$$

Similarly from the definitions of $\rho_g(m, p)$ and $\lambda_f(m, q)$ it follows for all sufficiently large values of r that

$$M_g (r) \leq \exp^{[m]} \left\{ \left(\rho_g (m, p) + \varepsilon \right) \log^{[p]} r \right\}$$

$$\text{i.e., } r \leq M_g^{-1} \left[\exp^{[m]} \left\{ \left(\rho_g (m, p) + \varepsilon \right) \log^{[p]} r \right\} \right] \tag{31}$$

$$\text{i.e., } M_g^{-1} (r) \geq \exp^{[p]} \left[\frac{\log^{[m]} r}{\left(\rho_g (m, p) + \varepsilon \right)} \right],$$

$$M_g (r) \geq \exp^{[m]} \left\{ \left(\lambda_g (m, p) - \varepsilon \right) \log^{[p]} r \right\}$$

$$\text{i.e., } M_g^{-1} (r) \leq \exp^{[p]} \left[\frac{\log^{[m]} r}{\left(\lambda_g (m, p) - \varepsilon \right)} \right] \tag{32}$$

and for a sequence of values of r tending to infinity we obtain that

$$M_g (r) \geq \exp^{[m]} \left\{ \left(\rho_g (m, p) - \varepsilon \right) \log^{[p]} r \right\}$$

$$\text{i.e., } M_g^{-1} (r) \leq \exp^{[p]} \left[\frac{\log^{[m]} r}{\left(\rho_g (m, p) - \varepsilon \right)} \right], \tag{33}$$

$$M_g (r) \leq \exp^{[m]} \left\{ \left(\lambda_g (m, p) + \varepsilon \right) \log^{[p]} r \right\}$$

$$\text{i.e., } M_g^{-1} (r) \geq \exp^{[p]} \left[\frac{\log^{[m]} r}{\left(\lambda_g (m, p) + \varepsilon \right)} \right]. \tag{34}$$

Now from (29) and in view of (31), for a sequence of values of r tending to infinity we get that

$$\log^{[p]} M_g^{-1} M_f (r)$$

$$\geq \log^{[p]} M_g^{-1} \left[\exp^{[m]} \left\{ \left(\rho_f (m, q) - \varepsilon \right) \log^{[q]} r \right\} \right]$$

$$\text{i.e., } \log^{[p]} M_g^{-1} M_f (r)$$

$$\geq \log^{[p]} \exp^{[p]} \left[\frac{\log^{[m]} \exp^{[m]} \left\{ \left(\rho_f (m, q) - \varepsilon \right) \log^{[q]} r \right\}}{\left(\rho_g (m, p) + \varepsilon \right)} \right]$$

$$= \frac{\left(\rho_f (m, q) - \varepsilon \right)}{\left(\rho_g (m, p) + \varepsilon \right)} \log^{[q]} r$$

$$\text{i.e., } \frac{\log^{[p]} M_g^{-1} M_f (r)}{\log^{[q]} r} \geq \frac{\left(\rho_f (m, q) - \varepsilon \right)}{\left(\rho_g (m, p) + \varepsilon \right)}. \tag{35}$$

As $\varepsilon (> 0)$ is arbitrary, it follows that

$$\rho_g^{(p,q)} (f) \geq \frac{\rho_f (m, q)}{\rho_g (m, p)}. \tag{36}$$

Analogously from (28) and in view of (34) it follows for a sequence of values of r tending to infinity that

$$\log^{[p]} M_g^{-1} M_f(r)$$
$$\geq \log^{[p]} M_g^{-1} \left[\exp^{[m]} \left\{ \left(\lambda_f(m,q) - \varepsilon \right) \log^{[q]} r \right\} \right]$$

i.e., $\log^{[p]} M_g^{-1} M_f(r)$

$$\geq \log^{[p]} \exp^{[p]} \left[\frac{\log^{[m]} \exp^{[m]} \left\{ \left(\lambda_f(m,q) - \varepsilon \right) \log^{[q]} r \right\}}{\left(\lambda_g(m,p) + \varepsilon \right)} \right]$$

$$= \frac{\left(\lambda_f(m,q) - \varepsilon \right)}{\left(\lambda_g(m,p) + \varepsilon \right)} \log^{[q]} r$$

i.e., $\dfrac{\log^{[p]} M_g^{-1} M_f(r)}{\log^{[q]} r} \geq \dfrac{\left(\lambda_f(m,q) - \varepsilon \right)}{\left(\lambda_g(m,p) + \varepsilon \right)}.$

$$(37)$$

Since $\varepsilon(> 0)$ is arbitrary, we get from above that

$$\rho_g^{(p,q)}(f) \geq \frac{\lambda_f(m,q)}{\lambda_g(m,p)}. \tag{38}$$

Again in view of (32) we have from (27) for all sufficiently large values of r that

$$\log^{[p]} M_g^{-1} M_f(r)$$
$$\leq \log^{[p]} M_g^{-1} \left[\exp^{[m]} \left\{ \left(\rho_f(m,q) + \varepsilon \right) \log^{[q]} r \right\} \right]$$

i.e., $\log^{[p]} M_g^{-1} M_f(r)$

$$\leq \log^{[p]} \exp^{[p]} \left[\frac{\log^{[m]} \exp^{[m]} \left\{ \left(\rho_f(m,q) + \varepsilon \right) \log^{[q]} r \right\}}{\left(\lambda_g(m,p) - \varepsilon \right)} \right]$$

$$= \frac{\left(\rho_f(m,q) + \varepsilon \right)}{\left(\lambda_g(m,p) - \varepsilon \right)} \log^{[q]} r$$

i.e., $\dfrac{\log^{[p]} M_g^{-1} M_f(r)}{\log^{[q]} r} \leq \dfrac{\left(\rho_f(m,q) + \varepsilon \right)}{\left(\lambda_g(m,p) - \varepsilon \right)}.$

$$(39)$$

Since $\varepsilon(> 0)$ is arbitrary, we obtain that

$$\rho_g^{(p,q)}(f) \leq \frac{\rho_f(m,q)}{\lambda_g(m,p)}. \tag{40}$$

Again from (28) and in view of (31) with the same reasoning we get that

$$\lambda_g^{(p,q)}(f) \geq \frac{\lambda_f(m,q)}{\rho_g(m,p)}. \tag{41}$$

Also in view of (33), we get from (27) for a sequence of values of r tending to infinity that

$$\log^{[p]} M_g^{-1} M_f(r)$$
$$\leq \log^{[p]} M_g^{-1} \left[\exp^{[m]} \left\{ \left(\rho_f(m,q) + \varepsilon \right) \log^{[q]} r \right\} \right]$$

i.e., $\log^{[p]} M_g^{-1} M_f(r)$

$$\leq \log^{[p]} \exp^{[p]} \left[\frac{\log^{[m]} \exp^{[m]} \left\{ \left(\rho_f(m,q) + \varepsilon \right) \log^{[q]} r \right\}}{\left(\rho_g(m,p) - \varepsilon \right)} \right]$$

$$= \frac{\left(\rho_f(m,q) + \varepsilon \right)}{\left(\rho_g(m,p) - \varepsilon \right)} \log^{[q]} r$$

i.e., $\dfrac{\log^{[p]} M_g^{-1} M_f(r)}{\log^{[q]} r} \leq \dfrac{\left(\rho_f(m,q) + \varepsilon \right)}{\left(\rho_g(m,p) - \varepsilon \right)}.$

$$(42)$$

Since $\varepsilon(> 0)$ is arbitrary, we get from above that

$$\lambda_g^{(p,q)}(f) \leq \frac{\rho_f(m,q)}{\rho_g(m,p)}. \tag{43}$$

Similarly from (30) and in view of (32) it follows for a sequence of values of r tending to infinity that

$$\log^{[p]} M_g^{-1} M_f(r)$$
$$\leq \log^{[p]} M_g^{-1} \left[\exp^{[m]} \left\{ \left(\lambda_f(m,q) + \varepsilon \right) \log^{[q]} r \right\} \right]$$

i.e., $\log^{[p]} M_g^{-1} M_f(r)$

$$\leq \log^{[p]} \exp^{[p]} \left[\frac{\log^{[m]} \exp^{[m]} \left\{ \left(\lambda_f(m,q) + \varepsilon \right) \log^{[q]} r \right\}}{\left(\lambda_g(m,p) - \varepsilon \right)} \right]$$

$$= \frac{\left(\lambda_f(m,q) + \varepsilon \right)}{\left(\lambda_g(m,p) - \varepsilon \right)} \log^{[q]} r$$

i.e., $\dfrac{\log^{[p]} M_g^{-1} M_f(r)}{\log^{[q]} r} \leq \dfrac{\left(\lambda_f(m,q) + \varepsilon \right)}{\left(\lambda_g(m,p) - \varepsilon \right)}.$

$$(44)$$

As $\varepsilon(> 0)$ is arbitrary, from above we obtain that

$$\lambda_g^{(p,q)}(f) \leq \frac{\lambda_f(m,q)}{\lambda_g(m,p)}. \tag{45}$$

The theorem follows from (36), (38), (40), (41), (43), and (45). □

Corollary 13. *Let f be an entire function with index-pair (m,q) and let g be an entire of regular (m,p)-growth, where p, q, m are all positive integers such that $m \geq p$ and $m \geq q$. Then*

$$\lambda_g^{(p,q)}(f) = \frac{\lambda_f(m,q)}{\rho_g(m,p)}, \qquad \rho_g^{(p,q)}(f) = \frac{\rho_f(m,q)}{\rho_g(m,p)}. \tag{46}$$

In addition, if $\rho_f(m, q) = \rho_g(m, p)$, then

$$\rho_g^{(p,q)}(f) = \lambda_f^{(q,p)}(g) = 1. \tag{47}$$

Remark 14. The first part of Corollary 13 improves [8, Theorem 2.1 and Theorem 2.2].

Corollary 15. *Let f and g be any two entire functions with regular (m, q)-growth and regular (m, p)-growth, respectively, where p, q, m are all positive integers with $m \geq \max\{p, q\}$. Then*

$$\lambda_g^{(p,q)}(f) = \rho_g^{(p,q)}(f) = \frac{\rho_f(m, q)}{\rho_g(m, p)}. \tag{48}$$

Corollary 16. *Let f and g be any two entire functions with regular (m, q)th growth and regular (m, p)th growth, respectively, where p, q, m are all positive integers with $m \geq p$ and $m \geq q$. Also suppose that $\rho_f(m, q) = \rho_g(m, p)$. Then*

$$\lambda_g^{(p,q)}(f) = \rho_g^{(p,q)}(f) = \lambda_f^{(q,p)}(g) = \rho_f^{(q,p)}(g) = 1. \tag{49}$$

Corollary 17. *Let f and g be any two entire functions with regular growth (m, q) and (m, p), respectively, where p, q, m are all positive integers such that $m \geq \max\{p, q\}$. Then*

$$\rho_g^{(p,q)}(f) \cdot \rho_f^{(q,p)}(g) = \lambda_g^{(p,q)}(f) \cdot \lambda_f^{(q,p)}(g) = 1. \tag{50}$$

Corollary 18. *Let f and g be any two entire functions with index-pair (m, q) and (m, p), respectively, where p, q, m are all positive integers such that $m \geq p$ and $m \geq q$. If either f is not of regular (m, q)th growth or g is not of regular (m, p)th growth, then*

$$\lambda_g^{(p,q)}(f) \cdot \lambda_f^{(q,p)}(g) < 1 < \rho_g^{(p,q)}(f) \cdot \rho_f^{(q,p)}(g). \tag{51}$$

Remark 19. Corollaries 17 and 18 can be regarded as an extension of the Corollaries of [8, Theorems 2.1 and 2.2].

Corollary 20. *Let f be an entire function with index-pair (m, q), where m, q are positive integers with $m \geq q$. Then for any entire function g,*

(i) $\lambda_g^{(p,q)}(f) = \infty$ *when* $\rho_g(m, p) = 0$,

(ii) $\rho_g^{(p,q)}(f) = \infty$ *when* $\lambda_g(m, p) = 0$,

(iii) $\lambda_g^{(p,q)}(f) = 0$ *when* $\rho_g(m, p) = \infty$,

(iv) $\rho_g^{(p,q)}(f) = 0$ *when* $\lambda_g(m, p) = \infty$,

where p is any positive integer with $m \geq p$.

Remark 21. The first part of Corollary 20 improves [8, Theorem 2.3].

Corollary 22. *Let g be an entire function with index-pair (m, p), where m, p are positive integers with $m \geq p$. Then for any entire function f,*

(i) $\rho_g^{(p,q)}(f) = 0$ *when* $\rho_f(m, q) = 0$,

(ii) $\lambda_g^{(p,q)}(f) = 0$ *when* $\lambda_f(m, q) = 0$,

(iii) $\rho_g^{(p,q)}(f) = \infty$ *when* $\rho_f(m, q) = \infty$,

(iv) $\lambda_g^{(p,q)}(f) = \infty$ *when* $\lambda_f(m, q) = \infty$,

where q is any positive integer such that $m \geq q$.

Example 23 (relative order between polynomials). To simplify let us consider any two given natural numbers m and n and $a \in \mathbb{R}$, $a \neq 0$, so that

$$f(z) = z^m, \qquad g(z) = az^n. \tag{52}$$

Then

$$\rho_f(1, 1) = \lambda_f(1, 1) = m, \qquad \rho_g(1, 1) = \lambda_g(1, 1) = n. \tag{53}$$

Now

$$\rho_g^{(1,1)}(f) = \lambda_g^{(1,1)}(f) = \frac{\rho_f(1, 1)}{\rho_g(1, 1)} = \frac{m}{n}. \tag{54}$$

Example 24 (relative order between exponentials of the same order). Let n be any natural number and a any positive real number and consider

$$f(z) = \exp z^n, \qquad g(z) = \exp(az)^n. \tag{55}$$

In this case f and g are two entire functions with regular $(2, 1)$ growth; thus

$$\lambda_g^{(1,1)}(f) = \rho_g^{(1,1)}(f) = \frac{\rho_f(2, 1)}{\rho_g(2, 1)} = \frac{n}{n} = 1. \tag{56}$$

Clearly

$$\rho_f^{(1,1)}(g) = \lambda_f^{(1,1)}(g) = 1. \tag{57}$$

Example 25 (relative order between exponential and power function). Let m, n be any two natural numbers and consider

$$f = \exp z^m, \qquad g = z^n. \tag{58}$$

Then

$$\rho_f = \lambda_f = m, \qquad \rho_g = \lambda_g = 0. \tag{59}$$

Now

$$\rho_g^{(1,1)}(f) = \lambda_g^{(1,1)}(f) = \infty,$$
$$\rho_f^{(1,1)}(g) = \lambda_f^{(1,1)}(g) = 0. \tag{60}$$

When f and g are any two entire functions with index-pair (m, q) and (n, p), respectively, where p, q, m, n are all positive integers such that $m \geq q$ and $n \geq p$, but $m \neq n$, the next definition enables studying their relative order.

Definition 26. Let f and g be any two entire functions with index-pair (m, q) and (n, p), respectively, where p, q, m, n are all positive integers such that $m \geq q$ and $n \geq p$. If $m > n$, then the relative $(p+m-n, q)$th order (resp., relative $(p+m-n, q)$th lower) of f with respect to g is defined as

(i)

$$\rho_g^{(p+m-n,q)}(f) = \lim_{r \to \infty} \sup \frac{\log^{[p+m-n]} M_g^{-1} M_f(r)}{\log^{[q]} r},$$

$$\left(\text{resp. } \lambda_g^{(p+m-n,q)}(f) = \lim_{r \to \infty} \inf \frac{\log^{[p+m-n]} M_g^{-1} M_f(r)}{\log^{[q]} r} \right).$$
(61)

If $m < n$, then the relative $(p, q+n-m)$th order (resp., relative $(p, q + n - m)$th lower) of f with respect to g is defined as

(ii)

$$\rho_g^{(p,q+n-m)}(f) = \lim_{r \to \infty} \sup \frac{\log^{[p]} M_g^{-1} M_f(r)}{\log^{[q+n-m]} r},$$

$$\left(\text{resp. } \lambda_g^{(p,q+n-m)}(f) = \lim_{r \to \infty} \inf \frac{\log^{[p]} M_g^{-1} M_f(r)}{\log^{[q+n-m]} r} \right).$$
(62)

The following result is easy to check.

Theorem 27. *Under the hypothesis of Definition 26, for $m > n$:*

(i)

$$\rho_g^{(p+m-n, q)}(f) = \lim_{r \to \infty} \sup \frac{\log^{[m]} M_f(r)}{\log^{[q]} r},$$

$$\lambda_g^{(p+m-n, q)}(f) = \lim_{r \to \infty} \inf \frac{\log^{[m]} M_f(r)}{\log^{[q]} r},$$
(63)

and for $m < n$:

(ii)

$$\rho_g^{(p, q+n-m)}(f) = \lim_{r \to \infty} \sup \frac{\log^{[p]} r}{\log^{[n]} M_g(r)},$$

$$\lambda_g^{(p, q+n-m)}(f) = \lim_{r \to \infty} \inf \frac{\log^{[p]} r}{\log^{[n]} M_g(r)}.$$
(64)

The next example will make an alternative use of Theorem 27.

Example 28 (relative order between exponentials of different order). Let

$$f(z) = \exp^{[27]} z^5, \qquad g(z) = \exp^{[50]} z^{17}.$$
(65)

In this case f and g are entire functions of regular growth $(m, p) = (28, 1)$ and $(n, q) = (51, 1)$, respectively, with

$$\rho_f(28, 1) = \lambda_f(28, 1) = 5, \qquad \rho_g(51, 1) = \lambda_g(51, 1) = 17.$$
(66)

Now

$$\frac{\log^{[p]} M_g^{-1} M_f(r)}{\log^{[q+n-m]} r} = \frac{\log \left[\log^{[50]} \left(\exp^{[27]} r^5 \right) \right]^{1/17}}{\log^{[24]} r}$$
(67)

and by taking lim sup and lim inf, we get

$$\rho_g^{(p,q+n-m)}(f) = \frac{1}{17} = \lambda_g^{(p,q+n-m)}(f).$$
(68)

Obviously, the same limit is achieved if, by using Theorem 27, we consider the quotient

$$\frac{\log^{[p]} r}{\log^{[n]} M_g(r)} = \frac{\log r}{\log^{[51]} M_g(r)}.$$
(69)

Reciprocally, in order to evaluate $\rho_f^{(p+m-n,q)}(g)$ and $\lambda_f^{(p+m-n,q)}(g)$, we would take limits in either

$$\frac{\log^{[24]} M_f^{-1} M_g(r)}{\log r}$$

$$= \frac{\log^{[24]} \left[\log^{[27]} \left(\exp^{[50]} r^{17} \right) \right]^{1/5}}{\log r} \quad \text{or} \quad \frac{\log^{[51]} M_g(r)}{\log r},$$
(70)

obtaining that

$$\rho_f^{(p+m-n,q)}(g) = 17 = \lambda_f^{(p+m-n,q)}(g).$$
(71)

5. Conclusion

The main aim of the paper is to extend and modify the notion of order to relative order of higher dimensions in case of entire functions as the relative order of growth gives a quantitative assessment of how different functions scale each other and to what extent they are self-similar in growth, and in this connection we have established some theorems. In fact, some works on relative order of entire functions and the growth estimates of composite entire functions on the basis of it have been explored in [8–15]. Actually we are trying to generalize the growth properties of composite entire functions on the basis of relative (p, q)th order and relative (p, q)th lower order and, analogously, we may also define relative (p, q)th order of meromorphic functions in order to establish related growth properties, improving the results of [16–18]. For any two positive integers p and q, we are trying to establish the concepts of relative (p, q)th type and relative (p, q)th weak type of entire and meromorphic functions, too, in order to determine the relative growth of two entire or meromorphic functions having the same nonzero finite relative (p, q)th order or relative (p, q)th lower order with respect to another entire function, respectively. Moreover, the notion of relative order, relative type, and relative weak type of higher dimensions may also be applied in the field of slowly changing functions and also in case of entire or meromorphic functions of several complex variables.

The results of this paper in connection with Nevanlinna's value distribution theory of entire functions on the basis of relative (p, q)th order and relative (p, q)th lower order may have a wide range of applications in complex dynamics, factorization theory of entire functions of single complex variable, the solution of complex differential equations, and so forth. In fact complex dynamics is a thrust area in modern function theory and it is solely based on the study of fixed points of entire functions as well as the normality of them. For further details in the progress of research in complex dynamics via Nevanlinna's value distribution theory one may see [19–24]. Factorization theory of entire functions is another branch of applications of Nevanlinna's theory which actually deals with how a given entire function can be factorized into other simpler entire functions in the sense of composition. Also Nevanlinna's value distribution theory has immense applications into the study of the properties of the solutions of complex differential equations and is still an active area of research.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

References

[1] L. Bernal, "Orden relativo de crecimiento de funciones enteras," *Collectanea Mathematica*, vol. 39, no. 3, pp. 209–229, 1988.

[2] L. Bernal, *Crecimiento relativo de funciones enteras. Contribución al estudio de lasfunciones enteras con índice exponencial finito [Doctoral Dissertation]*, University of Seville, Seville, Spain, 1984.

[3] B. K. Lahiri and D. Banerjee, "Entire functions of relative order (p, q)," *Soochow Journal of Mathematics*, vol. 31, no. 4, pp. 497–513, 2005.

[4] O. P. Juneja, G. P. Kapoor, and S. K. Bajpai, "On the (p, q)-order and lower (p, q)-order of an entire function," *Journal für die Reine und Angewandte Mathematik*, vol. 282, pp. 53–67, 1976.

[5] D. Sato, "On the rate of growth of entire functions of fast growth," *Bulletin of the American Mathematical Society*, vol. 69, pp. 411–414, 1963.

[6] E. C. Titchmarsh, *The Theory of Functions*, Oxford University Press, Oxford, UK, 2nd edition, 1968.

[7] B. K. Lahiri and D. Banerjee, "Generalised relative order of entire functions," *Proceedings of the National Academy of Sciences A*, vol. 72, no. 4, pp. 351–371, 2002.

[8] C. Roy, "On the relative order and lower relative order of an entire function," *Bulletin of the Calcutta Mathematical Society*, vol. 102, no. 1, pp. 17–26, 2010.

[9] B. C. Chakraborty and C. Roy, "Relative order of an entire function," *Journal of Pure Mathematics*, vol. 23, pp. 151–158, 2006.

[10] S. K. Datta and T. Biswas, "Growth of entire functions based on relative order," *International Journal of Pure and Applied Mathematics*, vol. 51, no. 1, pp. 49–58, 2009.

[11] S. K. Datta and T. Biswas, "Relative order of composite entire functions and some related growth properties," *Bulletin of the Calcutta Mathematical Society*, vol. 102, no. 3, pp. 259–266, 2010.

[12] S. K. Datta, T. Biswas, and R. Biswas, "On relative order based growth estimates of entire functions," *International Journal of Mathematical Sciences and Applications*, vol. 7, no. 2, pp. 59–67, 2013.

[13] S. K. Datta, T. Biswas, and R. Biswas, "Comparative growth properties of composite entire functions in the light of their relative order," *The Mathematics Student*, vol. 82, no. 1–4, pp. 1–8, 2013.

[14] S. K. Datta, T. Biswas, and D. C. Pramanick, "On relative order and maximum term-related comarative growth rates of entire functions," *Journal of Tripura Mathematical Society*, vol. 14, pp. 60–68, 2012.

[15] B. K. Lahiri and D. Banerjee, "A note on relative order of entire functions," *Bulletin of the Calcutta Mathematical Society*, vol. 97, no. 3, pp. 201–206, 2005.

[16] D. Banerjee, "A note on relative order of meromorphic functions," *Bulletin of the Calcutta Mathematical Society*, vol. 98, no. 1, pp. 25–30, 2006.

[17] D. Banerjee and S. Jana, "Meromorphic functions of relative order (p, q)," *Soochow Journal of Mathematics*, vol. 33, no. 3, pp. 343–357, 2007.

[18] B. K. Lahiri and D. Banerjee, "Relative order of meromorphic functions," *Proceedings of the National Academy of Sciences A*, vol. 75, no. 2, pp. 129–135, 2005.

[19] A. F. Beardon, *Iteration of Rational Functions*, vol. 132, Springer, Berlin, Germany, 1991.

[20] W. Bergweiler, "Periodic points of entire functions: proof of a conjecture of Baker," *Complex Variables*, vol. 17, no. 1-2, pp. 57–72, 1991.

[21] L. Carleson and T. W. Gamelin, *Complex Dynamics*, Springer, New York, NY, USA, 1993.

[22] P. C. Hua and C. C. Yang, *Dynamics of Transcendental Fucntions*, Gordon and Breach Science Publisher, 1998.

[23] J. Milnor, *Dynamics in One Complex Variable: Introductory Lectures*, Vieweg, Braunschweig, Germany, 2000.

[24] S. Morosawa, Y. Nishimura, M. Taniguchi, and T. Ueda, *Holomorphic Dynamics*, vol. 66, Cambridge University Press, Cambridge, UK, 2000.