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On the p-Rank of the Adjacency Matrices of Strongly Regular Graphs

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Abstract. Let Γ be a strongly regular graph with adjacency matrix A. Let I be the identity matrix, and J the all-1 matrix. Let p be a prime. Our aim is to study the p-rank (that is, the rank over $\mathbb{F}_p$, the finite field with p elements) of the matrices $M = aA + bJ + cI$ for integral a, b, c. This note is based on van Eijl [8].

Keywords: p-rank, strongly regular graphs

1. The Smith normal form

Let us write $M \sim N$ for integral matrices M and N of order n, if there are integral matrices P and Q of determinant ± 1 such that $N = PMQ$. Clearly, $\sim$ is an equivalence relation. Given the matrix M, we can find a diagonal matrix $S(M)$ with $S(M) \sim M$ and $S(M) = \text{diag}\ (s_1, \cdots, s_n)$ with $s_1 | s_2 | \cdots | s_n$. The matrix $S(M)$ is uniquely determined up to the signs of the s_i (we might require $s_i \geq 0$, but that is not always convenient), and is called the (more precisely, a) *Smith normal form* of M. (Thus, "$S(M) = A$" is to be interpreted as stating that A is a Smith normal form of M.) Clearly, $\prod_i s_i = \det S(M) = \pm\det M$, and more generally $\prod_{i=1}^{k} s_i$ is the g.c.d. of all minors of M of order k. The p-rank $\text{rk}_p(M)$ of M equals the number of s_i not divisible by p, and the $\mathbb{Q}$-rank $\text{rk}(M)$ of M equals the number of nonzero s_i. In particular, $\text{rk}_p M \leq \text{rk} M$. If $p^e \| \det M$, then $\text{rk}_p M \geq n - e$.

SNF0. *Let M be an integral matrix of order 2, and g the g.c.d. of its four elements. Then $S(M) = diag\ (g, (det M)/g)$.*

SNF1. $S(J_n + cI_n) = diag\ (1, c^{n-2}, c(c+n))$ *for* $c \in \mathbb{Z}$. *(We use exponents to denote multiplicities.) More generally* $S(bJ_n + cI_n) = diag\ (g, c^{n-2}, (bn + c)c/g)$, *where* $g = gcd(b, c)$.

SNF2. *Let A be the adjacency matrix of the $n \times n$ grid graph. Then*

$$S(A) = \text{diag}\ (1^{2n-2}, 2^{(n-2)^2}, (2n-4)^{2n-3}, 2(n-1)(n-2)).$$

More generally, we have

$$A + (c+2)I \sim \mathrm{diag}\ (I_n, (J_n + cI_n)^{n-2}, (n+c)(2J_n + cI_n)),$$

so that for example $S(A + 2I) = \mathrm{diag}\ (1^{2n-2}, 2n, 0^{(n-1)^2})$ *and* $S(A - (n-2)I) = \mathrm{diag}\ (1^{2n-2}, n^{(n-2)^2}, 0^{2n-2})$. [For the somewhat boring proof, see van Eijl [8].]

The $n \times n$ grid graph (also known as the lattice graph $L_2(n)$ or the Hamming graph $H(2,n)$) is strongly regular with parameters $(\nu, k, \lambda, \mu) = (n^2, 2(n-1), n-2, 2)$ and has spectrum $(2n-2)^1 (n-2)^{2n-2} (-2)^{(n-1)^2}$.

For $n = 4$, the Shrikhande graph has the same parameters and the same Smith normal form $S(A) = \mathrm{diag}\ (1^6, 2^4, 4^5, 12)$, but here $S(A + 2I) = \mathrm{diag}\ (1^6, 2^1, 0^9)$ and $S(A-2I) = \mathrm{diag}\ (1^6, 2^1, 4^2, 8^1, 0^6)$, while the 4×4 grid graph has $S(A+2I) = \mathrm{diag}\ (1^6, 8^1, 0^9)$ and $S(A - 2I) = \mathrm{diag}\ (1^6, 4^4, 0^6)$.

SNF3. *Let A be the adjacency matrix of the triangular graph $T(n)$. Then*

$$S(A) = \begin{cases} \mathrm{diag}\ (1^{n-2}, 2^{(n-2)(n-3)/2}, (2n-8)^{n-2}, (n-2)(n-4)) \\ \qquad \text{if } n \text{ is even } (n \geq 2) \\ \mathrm{diag}\ (1^{n-1}, 2^{(n-1)(n-4)/2}, (2n-8)^{n-2}, 2(n-2)(n-4)) \\ \qquad \text{if } n \text{ is odd } (n \geq 5). \\ \mathrm{diag}\ (1^2, 2) \quad \text{for } n = 3 \end{cases}$$

[Again, a detailed proof is given in van Eijl [8].]

Using the fact that $A + 2I = NN^T$, where N is the pair-point incidence matrix, we easily find

$$S(A + 2I) = \begin{cases} \mathrm{diag}\ (1^{n-2}, 2^2, 0^{n(n-3)/2}) & \text{if } n \text{ is even } (n \geq 4) \\ \mathrm{diag}\ (1^{n-1}, 4, 0^{n(n-3)/2}) & \text{if } n \text{ is odd } (n \geq 3). \end{cases}$$

The graph $T(n)$ (also known as the Johnson graph $J(n,2)$) is strongly regular with parameters $(\nu, k, \lambda, \mu) = (\frac{1}{2}n(n-1), 2(n-2), n-2, 4)$ and has spectrum $(2n-4)^1 (n-4)^{n-1} (-2)^{n(n-3)/2}$. For $n = 8$, the three Chang graphs have the same parameters, but different Smith normal form: instead of $\mathrm{diag}\ (1^6, 2^{15}, 8^6, 24^1)$ each of the Chang graphs has $S(A) = \mathrm{diag}\ (1^8, 2^{12}, 8^7, 24^1)$. In particular, $T(8)$ has 2-rank 6, but the three Chang graphs have 2-rank 8.

In fact it is more natural to consider $A+2I$ and $A-4I$. For $T(8)$ we find $S(A+2I) = \mathrm{diag}\ (1^6, 2^2, 0^{20})$ and $S(A-4I) = \mathrm{diag}\ (1^6, 2^2, 6^{13}, 0^7)$, while the three Chang graphs have $S(A + 2I) = \mathrm{diag}\ (1^8, 0^{20})$ and $S(A - 4I) = \mathrm{diag}\ (1^8, 6^{12}, 24^1, 0^7)$.

Remark. If one wants to determine the Smith normal form by computer, very large entries occur in the intermediate results, so that one needs multiple length arithmetic, even in moderately sized cases like the Chang graphs. However, as Aart Blokhuis remarked, doing all computations modulo some integer m yields the sequence $gcd(s_i, m)(1 \leq i \leq n)$. When M is symmetric, with known integral eigenvalues θ_i, then we have an equation of the form $Mp(M) = hI$ for some polynomial $p()$, where $h = \prod_i \theta_i$, and by Blokhuis and Calderbank [2] (Lemma 2.2) it follows that each nonzero s_i divides h. In particular, if $h = \prod_i p_i^{e_i}$ then working modulo $m = \prod_i p_i^{e_i+1}$ we find the actual Smith normal form.

2. Reduction mod p

In this section we want to show that if an integral matrix M is diagonalizable over $\mathbb{C}$, and also over a field F of characteristic p, then the F-spectrum of M is obtained from the $\mathbb{C}$-spectrum of M by reduction mod p. In particular, in this case the p-rank will be equal to the sum of the multiplicities of the eigenvalues that do not vanish mod p. We have to be a little careful, because the eigenvalues of M need not be integers.

LEMMA A. *Let N be a matrix of order n with distinct eigenvalues θ_i over a field F. Equivalent are: (i) N is diagonalizable, (ii) $\sum_i$ dim ker $(N - \theta_i I) = n$, and (iii) $\prod_i (N - \theta_i I) = 0$. When this is the case, the θ_j-eigenspace of N is the column space of $\prod_{i \neq j}(N - \theta_i I)$, and, more generally, the column space of $\prod_{i \in E}(N - \theta_i I)$ is the direct sum of the θ_j-eigenspaces of N for $j \notin E$.* □

[This is standard linear algebra; the referee told us to refer to Hoffman and Kunze [11], §6.]

More generally, let N have minimal polynomial g over F, with $g(x) = (x - \theta)^e g_1(x)$ and $g_1(\theta) \neq 0$. Then the kernel of $(N - \theta I)^e$ is the column space of $g_1(N)$. In particular, if $e = 1$ then the θ-eigenspace of N is the column space of $g_1(N)$.

If M is a matrix with entries in an integral domain Q with quotient field Q_0, we shall write $\text{rk}_Q(M)$ for the Q_0-rank of M, and $m_Q(\theta)$ for the Q_0-dimension of the θ-eigenspace of M.

Let R, S be integral domains, and let $h : R \to S$ be a ring homomorphism (with $h(1) = 1$). We shall write $\overline{r} := h(r)$ for $r \in R$. If M is a matrix with entries in R, we write $\overline{M}$ for its entrywise image under h. Clearly, $\text{rk}_S(\overline{M}) \leq \text{rk}_R(M)$.

PROPOSITION. *Let M be a square matrix over R, with pairwise distinct eigenvalues $\rho_i \in R$. Then $\text{rk}_S(\overline{M}) \geq \sum\{m_R(\rho_i) | \overline{\rho_i} \neq 0\}$.*

Proof. This says that the dimension (over S_0) of the null-space of $\overline{M}$ is at most the multiplicity of 0 as a root of the characteristic polynomial det $(xI - \overline{M})$. □

PROPOSITION. *Let M be a matrix of order n over R, and suppose that $\prod_i(M-\rho_i I) = 0$, where the ρ_i are pairwise distinct elements of R. Suppose moreover, that $\overline{M}$ has minimal polynomial g over S, where $g(x) = (x-\sigma)^e g_1(x)$ and $g_1(\sigma) \neq 0$. Then* dim ker $(\overline{M} - \sigma I)^e = \sum\{m_R(\rho_i) | \overline{\rho_i} = \sigma\}$. *In particular, if σ is a simple root of g, then $m_S(\sigma) = \sum\{m_R(\rho_i) | \overline{\rho_i} = \sigma\}$.*

Proof. First of all, since the minimal polynomial f of M over R factorizes completely: $f(x) = \prod(x-\rho_i)$ and $g(x)$ divides $\overline{f}(x) = \prod(x - \overline{\rho_i})$, also $g(x)$ factorizes completely: $g(x) = \prod(x-\sigma_j)^{e_j}$. By the (remark following the) above lemma, dim ker $(\overline{M} - \sigma I)^e = \mathrm{rk}_S(g_1(\overline{M}))$. Since $\overline{M} g_1(\overline{M}) = \sigma g_1(\overline{M})$, and hence $g_0(\overline{M}) g_1(\overline{M}) = g_0(\sigma) g_1(\overline{M})$ for any polynomial g_0, we have $\mathrm{rk}_S(g_1(\overline{M})^d) = \mathrm{rk}_S(g_1(\overline{M}))$ for any integer $d \geq 1$. Thus, if we put $f_1(x) = \prod_{\overline{\rho_i} \neq \sigma}(x - \rho_i)$, then

$$\dim \ker (\overline{M} - \sigma I)^e = \mathrm{rk}_S(g_1(\overline{M})) = \mathrm{rk}_s(\overline{f_1}(\overline{M})) \leq \mathrm{rk}_R(f_1(M)) = \textstyle\sum_{\overline{\rho_i} = \sigma} m_R(\rho_i),$$

and since both left-hand side and right-hand side sum to n, we must have equality everywhere. □

3. The p-rank of strongly regular graphs

Let M be an integral matrix of order ν with integral eigenvalues θ_i with (geometric) multiplicities $m_i (0 \leq i \leq t)$. As we saw in the previous section, $\mathrm{rk}_p M \geq \sum\{m_i | \theta_i \not\equiv 0 \pmod p\}$ and $\mathrm{rk}_p M \leq \mathrm{rk} M$. Thus, if all eigenvalues θ_i are nonzero (mod p) then M has full rank ν. If the eigenvalue θ_j is divisible by p, then $\mathrm{rk}_p M \leq \nu - m_j$ since $\mathrm{rk}_p M = \mathrm{rk}_p(M - \theta_j I) \leq \mathrm{rk}(M - \theta_j I) = \nu - m_j$. Thus, if precisely one eigenvalue θ_j is divisible by p, then $\mathrm{rk}_p M = \nu - m_j$.

Not let Γ be a strongly regular graph with adjacency matrix A, and assume that A has integral eigenvalues k, r, s with multiplicities $1, f, g$, respectively. Let $M = A + bJ + cI$. Then M has eigenvalues $\theta_0 = k + b\nu + c, \theta_1 = r + c, \theta_2 = s + c$, with multiplicities $1, f, g$, respectively. We have to study the case where at least two of the θ_i vanish (mod p).

Suppose $\theta_0 \equiv \theta_1 \equiv 0 \pmod p$, $\theta_2 \not\equiv 0 \pmod p$. Then we can apply Lemma A with the two eigenvalues 0 and θ_2. Since $\mathrm{rk}_p(M - \theta_2 I) = \nu - g$, and $g \leq \mathrm{rk}_p M \leq g + 1$, it follows that $\mathrm{rk}_p M = g$ if and only if $M(M - \theta_2 I) \equiv 0 \pmod p$. But using $(A - rI)(A - sI) = \mu J$ and $r + s = \lambda - \mu$, we find $M(M - \theta_2 I) \equiv (A + bJ - rI)(A + bJ - sI) = eJ$, where

$$e := \mu + b^2\nu + 2bk + b(\mu - \lambda).$$

Thus: if $p|e$, then $\mathrm{rk}_p M = g$, and $\mathrm{rk}_p M = g + 1$ otherwise.

Similarly, if $\theta_0 \equiv \theta_2 \equiv 0 \pmod p), \theta_1 \not\equiv 0 \pmod p$, then $\mathrm{rk}_p M = f$ if $p|e$, and $\mathrm{rk}_p M = f + 1$ otherwise.

In the particular case $b = c = 0$ we find (using $rs = \mu - k$): If $k \equiv r \equiv 0 \pmod p$ and $s \not\equiv 0 \pmod p$, then $\mathrm{rk}_p A = g$. If $k \equiv s \equiv 0 \pmod p$ and $r \not\equiv 0 \pmod p$, then $\mathrm{rk}_p A = f$.

Thus, up to now the p-rank of M was easily computable from the parameters of Γ. It follows that the only interesting case (where the structure of Γ plays a rôle) is that where p divides both θ_1 and θ_2, so that $p|(r-s)$. In particular, only finitely many primes are of interest. In this case we only have the upper bound $\mathrm{rk}_p M \le \min(f+1, g+1)$.

Looking at the idempotents sometimes improves this bound by 1: We have $E_1 = (r-s)^{-1}(A-sI-(k-s)\nu^{-1}J)$ and $E_2 = (s-r)^{-1}(A-rI-(k-r)\nu^{-1}J)$. Thus, if $k-s$ and ν are divisible by the same power of p (so that $(k-s)/\nu$ can be interpreted in $\mathbb{F}_p$), then $\mathrm{rk}_p(A-sI-(k-s)\nu^{-1}J) \le \mathrm{rk}E_1 = f$, and, similarly, if $k-r$ and ν are divisible by the same power of p then $\mathrm{rk}_p(A-rI-(k-r)\nu^{-1}J) \le \mathrm{rk}E_2 = g$. For $M = A + bJ + cI$ and $p|(r+c), p|(s+c)$ we have $ME_1 = JE_1 = 0$, (over $\mathbb{F}_p$) so that $\mathrm{rk}_p\langle M, \mathbf{1}\rangle \le g+1$, and hence $\mathrm{rk}_p M \le g$ (and similarly $\mathrm{rk}_p M \le f$) in case $\mathbf{1} \notin \langle M\rangle$.

3.1. Duality; submodules of dimension or codimension 1

Let M be a matrix over some field F. Then we have a nondegenerate pairing $(x^T M, My) \mapsto x^T M y$ between the row space and the column space of M. In particular, when $M = M^T$, then the lattice of submodules of the column space of M has a natural duality.

Let us denote the column space of M by $\langle M\rangle$ (or by $\langle M\rangle_F$ if it is desirable to mention the field F explicitly, or by $\langle M\rangle_p$ in case $F = \mathbb{F}_p$). [We shall also use notations like $\langle M, u\rangle (= \langle M\rangle + \langle u\rangle)$ for the span of the vector u and the columns of M.]

Of particular interest are the submodules of dimension or codimension 1. Let $\langle M\rangle^+$ be the subspace of $\langle M\rangle$ generated by the differences of columns of M, so that it consists of the linear combinations of the columns of M with coefficients summing to zero. Clearly, either $\dim \langle M\rangle^+ = \dim \langle M\rangle$ or $\dim \langle M\rangle^+ = \dim \langle M\rangle - 1$, where the latter holds if and only if $\mathbf{1} \in \langle M^T\rangle$. Moreover, $\langle M + bJ\rangle^+ = \langle M\rangle^+$.

Often one has to decide whether $\mathbf{1} \in \langle M\rangle_p$. Two easy criteria are as follows: suppose that M has order ν and constant row sums m. If $p \nmid m$, then $\mathbf{1} \in \langle M\rangle$. If $p|m$ and $p \nmid \nu$, then $\mathbf{1} \notin \langle M\rangle$, since $\langle M\rangle \subseteq \mathbf{1}^\perp$, and $\mathbf{1} \notin \mathbf{1}^\perp$.

If M has nonzero order, and $\langle M\rangle^+ = \langle M\rangle$, then since any column of M is in $\langle M\rangle^+$, the zero column can be written as a linear combination of the columns of M with coefficients summing to 1, and $\mathbf{1} \in \langle M + bJ\rangle$ for all $b \ne 0$. In particular this holds when $\langle M\rangle$ is a nontrivial irreducible G-module for some group G.

If $\mathbf{1} \notin \langle M\rangle$ and $\mathbf{1} \in \langle M + bJ\rangle$ for some $b \ne 0$, then $\mathbf{1} \in \langle M + bJ\rangle$ for all $b \ne 0$ (mod p). Thus, either $\mathrm{rk}_p(M + bJ)$ is independent of b, or there is precisely one value of b for which this rank is one lower that for all other values. If $p \nmid \nu$, we are in the latter case.

In the special case where $M = A - rI + bJ$, where A is the adjacency matrix of a strongly regular graph, we have $MJ = (k - r + b\nu)J$ and $M(A - sI) = (\mu + bk - bs)J$,

so certainly $\mathbf{1} \in \langle M \rangle$ when $p \nmid (k - r + b\nu)$ or when $p \nmid (\mu + b(k - s))$.
To conclude this section, a few useful lemmas.

LEMMA. *The 2-rank of a symmetric integral matrix with zero diagonal is even.*

[This is standard linear algebra; a reference is Kaplansky [12], p. 22.]

LEMMA. *Let Γ be a graph, x a nonisolated vertex of Γ, and Δ the subgraph of Γ induced by the nonneighbours of x. Let Γ and Δ have adjacency matrices A and C, respectively. Then* $\mathrm{rk}_p(A) \geq \mathrm{rk}_p(C) + 2$ *for any p, and* $\mathrm{rk}_p(A + cI) \geq \mathrm{rk}_p(C + cI) + 1$ *for all p and c.*

Proof. We have

$$A = \begin{bmatrix} 0 & \mathbf{1}^T & 0 \\ \mathbf{1} & B & N \\ 0 & N^T & C \end{bmatrix} \sim \begin{bmatrix} \mathbf{1} & N & B \\ 0 & C & N^T \\ 0 & 0 & \mathbf{1}^T \end{bmatrix}. \qquad \square$$

LEMMA. *Let $M = \begin{bmatrix} B & J \\ J^T & C \end{bmatrix}$ be a symmetric matrix, with square submatrices B and C (and a rectangular block J of all 1's). If $\mathbf{1} \in \langle B \rangle^+$ (or $\mathbf{1} \in \langle C \rangle^+$), then* $\mathrm{rk}M = \mathrm{rk}B + \mathrm{rk}C$.

Proof. If $\mathbf{1} \in \langle B \rangle^+$, then $\dim \langle B \rangle^+ = \dim \langle B \rangle - 1$ by duality, and

$$\begin{bmatrix} B & J \\ J^T & C \end{bmatrix} \sim \begin{bmatrix} b_0 & B^+ & J \\ \mathbf{1} & 0 & C \end{bmatrix} \sim \begin{bmatrix} b_0 & B^+ & 0 \\ \mathbf{1} & 0 & C \end{bmatrix}. \qquad \square$$

(Without condition we can also have $\mathrm{rk}M = \mathrm{rk}B + \mathrm{rk}C - 1$ or $\mathrm{rk}M = \mathrm{rk}B + \mathrm{rk}C + 1$, as can be seen by taking $B = C = J$ or $B = C = 0$.) This lemma will be applied for $p = 2$ to the submatrix of $J - A$ indexed by two adjacent vertices x, y (with adjacency matrix $B = J - I$) and all vertices nonadjacent to both (with adjacency matrix C) to conclude that $\mathrm{rk}_2(J - A) \geq 2 + \mathrm{rk}_2(J - C)$.

3.2. Switching

If Γ and Δ are switching equivalent graphs with adjacency matrices A and B, respectively, then by the definition of switching, the matrices $J - 2A + zI$ and $J - 2B + zI$ have the same Smith normal form for any integer z. (For a definition of switching, cf. Brouwer, Cohen and Neumaier [3], p. 15.) For odd p we may conclude that $\mathrm{rk}_p(A - \frac{1}{2}J + cI) = \mathrm{rk}_p(B - \frac{1}{2}J + cI)$ for all integers c. If $p = 2$, and Δ is obtained from Γ by switching with respect to a set with characteristic vector χ, then $\langle A + b_1 J + cI \rangle + \langle \mathbf{1}, \chi \rangle = \langle B + b_2 J + cI \rangle + \langle \mathbf{1}, \chi \rangle$.

(Hence, $|\mathrm{rk}_p(A + b_1 J + cI) - \mathrm{rk}_p(B + b_2 J + cI)| \leq 2$ for all integers b_1, b_2, c and all primes p.)

(i) In the special case where $p = 2$ and Δ has an isolated vertex δ, we have $\chi \in \langle A \rangle$, so that $\langle A \rangle + \langle \mathbf{1} \rangle = \langle B \rangle + \langle \mathbf{1}, \chi \rangle$. Now B contains a zero row, so $\mathbf{1} \notin \langle B \rangle$. Thus, if $\mathbf{1} \in \langle A \rangle$, then $\mathrm{rk}_2 A = \mathrm{rk}_2 B + 2$, and $\mathrm{rk}_2 A = \mathrm{rk}_2 B$ otherwise.
(ii) Let $\Delta' := \Delta \backslash \{\delta\}$ with adjacency matrix B' of order $\nu - 1$. Then $\mathrm{rk}_p B = \mathrm{rk}_p B'$ for all p, and $\mathrm{rk}_p(B + cI_\nu) = \mathrm{rk}_p(B' + cI_{\nu-1}) + 1$ when $c \not\equiv 0 \pmod p$. It may happen that $\mathrm{rk}_2(B' + I) = \mathrm{rk}_2(A + I) - 2$ (take for Γ a path of length 3 and for δ an internal vertex). However, if Δ' is strongly regular, then $|\mathrm{rk}_2(A + I) - \mathrm{rk}_2(B' + I)| \leq 1$ since in that case Δ' has even valency (cf. Brouwer, Cohen and Neumaier [3], Theorem 1.5.6), so that $\mathbf{1} \in \langle B + I \rangle_2$.

LEMMA. *If A is the adjacency matrix of the collinearity graph of a generalized quadrangle $GQ(s,t)$, then $\mathbf{1} \in \langle A - (s-1)I \rangle_{\mathbf{Z}}$. If $p|(s+1)$ then also $\mathbf{1} \in \langle A + 2I + bJ \rangle_p$ for all b.*

Proof. Add the columns corresponding to a line. □

LEMMA. *Let A be the adjacency matrix of the collinearity graph Γ of a generalized quadrangle $GQ(s,t)$, and let $p|s$. Let B be the submatrix (of order $s^2 t$) induced by the rows and columns corresponding to points nonadjacent to a fixed vertex x of Γ. Then $\mathrm{rk}_p(J - I - A) = \mathrm{rk}_p(J - I - B)$ and $\mathbf{1} \in \langle bJ - I - A \rangle_p$ for all b.*

Proof. Adding the columns of $J - I - A$ corresponding to the points of a line yields 0. This allows us to express the row (column) of $J - I - A$ corresponding to a point y on a line L in terms of the rows (columns) corresponding to the remaining points of L. In this way, column x of $J - I - A$ is expressed in terms of s^2 columns corresponding to points nonadjacent to x, and, in particular, $\mathbf{1}$ is the sum of s^2 columns of $J - I - B$. □

4. The half case

In the case where Γ has parameters $(\nu, k, \lambda, \mu) = (4t + 1, 2t, t - 1, t)$ for some integer t, the eigenvalues need not be integral. The eigenvalues are $2t$ with multiplicity 1, and $(-1 \pm \sqrt{\nu})/2$, both with multiplicity $(\nu - 1)/2$. Entirely analogously to the above, we find:

(i) If $p \nmid \theta_0\theta_1\theta_2$, then $\mathrm{rk}_p(A + cI) = \nu$. (Note that $\theta_0 = k + c$ and $\theta_1\theta_2 = c^2 - c - \mu$.)
(ii) If $p \nmid \theta_1\theta_2$ but $p|k + c$, then $\mathrm{rk}_p(A + cI) = \nu - 1$.

Now suppose that $p|c^2 - c - \mu$. Solving this for $c \pmod p$, we find several cases:

If $p = 2$, then there are no solutions if μ is odd, and $c = 0, 1$ if μ is even.

If $p > 2$, then $p|c^2 - c - \mu$ is equivalent to $(2c-1)^2 \equiv \nu \pmod p$, and there are 0, 1 or 2 solutions, depending on whether ν is a nonsquare, zero or a square (mod p).

(iii) If $\mu \equiv 0 \pmod p$, then $\mathrm{rk}_p A = (\nu - 1)/2$ and $\mathrm{rk}_p(A + I) = (\nu + 1)/2$.

[Indeed, in this case $A(A + I) \equiv 0 \pmod p$, so that A is diagonalizable over $\mathbb{F}_p$. The eigenvalues k and r vanish (mod p), and s becomes -1.]

(iv) If ν is a (nonzero) square (mod p) and $\nu \not\equiv 1 \pmod p$, then $\mathrm{rk}_p(A + cI) = (\nu + 1)/2$ for the two values of c satisfying $(2c-1)^2 \equiv \nu \pmod p$.

[Indeed, if c_1 and c_2 are these two values of c, then $(A+c_1I)(A+c_2I) \equiv \mu J \pmod p$, and $k + c_i \not\equiv 0 \pmod p$ since $\nu \not\equiv 0 \pmod p$, so that A is diagonalizable over $\mathbb{F}_p$.]

Remains the single case $p|\nu$ for odd p, where we have no definite value, but only the upper bound $\mathrm{rk}_p(A + \frac{1}{2}I) \le (\nu + 1)/2$ (since $A + \frac{1}{2}I)^2 \equiv \mu J \pmod p$). It is likely that this last rank depends on the structure of Γ. In the special case of Paley graphs, we can compute it.

4.1. Paley graphs

Let q be a prime power, $q \equiv 1 \pmod 4$, and let Γ be the graph with vertex set $\mathbb{F}_q$ where two vertices are adjacent whenever their difference is a nonzero square. (Then Γ is called the Paley graph of order q.) In order to compute the p-rank of the Paley graphs, we first need a lemma.

LEMMA. *Let $p(x,y) = \sum_{i=0}^{d-1}\sum_{j=0}^{e-1} c_{ij}x^iy^j$ be a polynomial with coefficients in a field F. Let $A, B \subseteq F$, with $m := |A| \ge d$ and $n := |B| \ge e$. Consider the $m \times n$ matrix $P = (p(a,b))_{a\in A, b\in B}$ and the $d \times e$ matrix $C = (c_{ij})$. Then $\mathrm{rk}_F(P) = \mathrm{rk}_F(C)$.*

Proof. For any integer s and subset X of F, let $Z(s,X)$ be the $|X| \times s$ matrix $(x^i)_{x\in X, 0\le i\le s-1}$. Note that if $|X| = s$ then this is a Vandermonde matrix and hence invertible. We have $P = Z(d,A)CZ(e,B)^T$, so $\mathrm{rk}_F(P) \le \mathrm{rk}_F(C)$, but P contains a submatrix $Z(d,A')CZ(e,B')$ with $A' \subseteq A, B' \subseteq B, |A'| = d, |B'| = e$, and this submatrix has the same rank as C. □

For odd prime powers $q = p^e$, p prime, let Q be the $\{0, \pm 1\}$-matrix Q of order q with entries $Q_{xy} = \chi(y - x)(x, y \in \mathbb{F}_q, \chi$ the quadratic residue character, $\chi(0) = 0)$.

PROPOSITION. $\mathrm{rk}_p Q = ((p + 1)/2)^e$.

Proof. Applying the above lemma with $p(x,y) = \chi(y-x) = (y-x)^{(q-1)/2} = \sum_i (-1)^i \binom{(q-1)/2}{i} x^i y^{(q-1)/2-i}$, we see that $\mathrm{rk}_p Q$ equals the number of binomial coefficients $\binom{(q-1)/2}{i}$ with $0 \leq i \leq (q-1)/2$ not divisible by p. Now Lucas' Theorem says that if $l = \sum_i l_i p^i$ and $k = \sum_i k_i p^i$ are the p-ary expansions of l and k, then $\binom{l}{k} \equiv \prod_i \binom{l_i}{k_i} \pmod{p}$. Since $\frac{1}{2}(q-1) = \sum_i \frac{1}{2}(p-1)p^i$, this means that for each p-ary digit of i there are $(p+1)/2$ possibilities and the result follows. □

[For Lucas' Theorem, cf. MacWilliams and Sloane [15], §13.5, p. 404 (and references given there). Dickson [7], vol. I, p. 271, gives the reference Lucas [13] (for binomial coefficients, and refers to himself [p. 273] for the generalization to multinomial coefficients) but Lucas himself (cf. Lucas [14], p. 417, 503) seems to refer to Cauchy.]

Note that this proof shows that each submatrix of Q of order at least $(q+1)/2$ has the same rank as Q.

The relation between Q here and the adjacency matrix A of the Paley graph is $Q = 2A + I - J$. From $Q^2 = qI - J \equiv -J \pmod{p}$ and $(2A + I)^2 = qI + (q-1)J \equiv -J \pmod{p}$ it follows that both $\langle Q \rangle$ and $\langle 2A + I \rangle$ contain $\mathbf{1}$, so $\mathrm{rk}_p(A + \frac{1}{2}I) = \mathrm{rk}_p(2A + I) = \mathrm{rk}_p(Q) = ((p+1)/2)^e$.

5. Modular characters

Let G be a group of automorphisms of a graph Γ. Let A be the adjacency matrix of Γ, and let V_θ be the θ-eigenspace of A, a subspace of $V := \langle I \rangle_{\mathbb{R}}$. Each $g \in G$ may be represented as a permutation matrix M_g; the fact that it is an automorphism means that $AM_g = M_g A$. It follows that V_θ is a $\mathbb{R}[G]$-module for each eigenvalue θ of A. If the corresponding character is χ_θ, then we have $\pi = \sum_\theta \chi_\theta$, where π is the permutation character: $\pi(g)$ is the number of vertices of Γ fixed by $g \in G$. Also $M := \langle A + bJ + cI \rangle_{\mathbb{R}}$ is an $\mathbb{R}[G]$-module, a submodule of V.

Now, let p be a prime, and consider for $F = \mathbb{F}_p$ the $F[G]$-module $\overline{V} := \langle I \rangle_F$ and its submodule $\overline{M} := \langle A + bJ + cI \rangle_F$. Our aim was to find the p-rank of $A + bJ + cI$, and in this context it may be expressed as $\mathrm{rk}_p(A + bJ + cI) = \dim_F \overline{M} = \overline{\eta}(1)$, if $\overline{\eta}$ is the p-modular character of G on $\overline{M}$.

If θ is an integral eigenvalue of Γ, then let $Z_\theta := V_\theta \cap \langle I \rangle_{\mathbb{Z}}$ and let $\overline{Z_\theta}$ be its image in $\overline{V}$ under reduction mod p. Then $\dim_F \overline{Z_\theta} = \dim V_\theta$, and, more generally, the p-modular character of G on $\overline{Z_\theta}$ is the restriction of the character of G on V_θ to the elements of G of order prime to p. Moreover, for each θ with $p | \theta + c$, we have $\overline{M} \subseteq \overline{Z_\theta}^\perp$, where $\perp$ is taken w.r.t. the natural inner product.

In particular, when Γ is a strongly regular graph with integral eigenvalues k, r, s and $r + c \equiv s + c \equiv 0 \pmod{p}$, then since $\overline{Z_r} \subseteq \overline{Z_s}^\perp$ and $\dim(\overline{Z_r}) = f$ and $\dim(\overline{Z_s}^\perp) = f + 1$, we find $\dim \overline{M}/(\overline{M} \cap \overline{Z_r}) \leq 1$ and similarly $\dim \overline{M}/(\overline{M} \cap \overline{Z_s}) \leq 1$. Thus, if we decompose the p-modular characters $\overline{\eta}, \overline{\chi_r}$ and $\overline{\chi_s}$ of G on $\overline{M}, \overline{Z_r}$ and $\overline{Z_s}$ into irreducibles: $\overline{\eta} = \sum m_\psi \psi, \overline{\chi_r} = \sum n_\psi \psi$ and $\overline{\chi_s} = \sum n'_\psi \psi$, we find that $m_\psi \leq \min(n_\psi, n'_\psi)$ for all p-modular irreducible characters ψ, except for the trivial character 1 of degree 1, where $m_1 \le \min(n_1, n_1') + 1$. Moreover, since $\overline{\eta}$ has values in $\mathbb{F}_p$, the multiplicities m_ψ of algebraically conjugate p-modular characters ψ are equal.

In some cases this suffices to determine $\mathrm{rk}_p(A + cI) = \overline{\eta}(1) = \sum m_\psi \psi(1)$, or to find a short list of possible values for this rank. Examples are given below.

6. Examples

In the table below we give for a few strongly regular graphs for each prime p dividing $r - s$ the p-rank of $A - sI$ and the unique b_0 such that $\mathrm{rk}_p(A - sI - b_0 J) = \mathrm{rk}_p(A - sI - bJ) - 1$ for all $b \neq b_0$, or '-' in case $\mathrm{rk}_p(A - sI - bJ)$ is independent of b. (When $p \not| \nu$ we are in the former case, and b_0 follows from the parameters. When $p | \nu$ and $p \not| \mu$, we are in the latter case.) For a description of most of these graphs, see Brouwer and van Lint [5]. A discussion of these graphs and a few infinite families follows.

1. The $n \times n$ grid graph. Interesting are the primes dividing n. We have $\mathrm{rk}_p(A + 2I + bJ) = 2n - 2$ for $p | n$.

   [Indeed, for $b = 0$ this follows from the Smith normal form given in Section 1. But $\mathbf{1}$ is the mod p sum of the n columns corresponding to an n-clique, so $\mathbf{1} \in \langle A + 2I + bJ \rangle$ for all b.]
2. The triangular graph $T(n)$. Interesting are the primes dividing $n - 2$. For $p | n - 2$, p odd, $n \ge 3$ we have $\mathrm{rk}_p(A + 2I + bJ) = n$ if $b \neq -2$, and $\mathrm{rk}_p(A + 2I - 2J) = n - 1$.

   [Indeed, for $b = 0$ this follows from the Smith normal form given in Section 1, and since $\nu \equiv 1 \pmod{p}$ and $p | k$ we have $\mathbf{1} \in \langle A + 2I + bJ \rangle$ if and only if $b \not\equiv -2 \pmod{p}$.]

   For $p = 2$, n even, $n \ge 2$ we have $\mathrm{rk}_2(A) = n - 2$ and $\mathrm{rk}_2(A + J) = n - 1$.

   [Indeed, $\mathrm{rk}_2(A)$ is found from the Smith normal form. From $A + 2I = NN^T$, where N is the pair-point incidence matrix, and $\mathbf{1} \notin \langle N \rangle$ we find $\mathbf{1} \notin \langle A \rangle$. On the other hand, $\mathbf{1} \in \langle A + J \rangle$ as is seen by summing the columns corresponding to an $(n - 1)$-clique.]
3. Let Γ be the symplectic graph $Sp_{2m}(q)$, with as point set the set of 1-spaces (projective points) of $\mathbb{F}_q^{2m}$ provided with a nondegenerate symplectic form, where orthogonal points are adjacent. Then Γ is a strongly regular graph with parameters $\nu = (q^{2m} - 1)/(q - 1)$, $k = (q^{2m-1} - 1)/(q - 1) - 1$, $\lambda + 2 = \mu = (q^{2m-2} - 1)/(q - 1)$ and spectrum $k^1 (q^{m-1} - 1)^f (-q^{m-1} - 1)^g$, where $f = (\nu - 1 + q^m)/2$ and $g = (\nu - 1 - q^m)/2$. Thus, interesting primes are 2 and p, where $q = p^h$.

   Since $\nu \equiv 1 \pmod{p}$, and $J - I - A$ has row sums divisible by p, we have $\mathrm{rk}_p(J - I - A) + 1 = \mathrm{rk}_p(A + I + bJ)$ for all $b \neq -1$.

   If $q = 2$, we have $\mathrm{rk}_2(A + I) = 2m + 1$ and $\mathrm{rk}_2(J - I - A) = 2m$.

   [Indeed, if $q = 2$, then the mod 2 sum of any two distinct columns of

$J - I - A$ is again a column of $J - I - A$, so that $\langle J - I - A\rangle$ consists of the columns of $J - I - A$ together with $\mathbf{0}$.]

If q is odd and $m = 2$, then by Bagchi, Brouwer and Wilbrink [1] we have $\mathrm{rk}_2(A) = \mathrm{rk}_2(J - A) = \frac{1}{2}q(q^2 + 1) + 1$.

For the p-rank in case of $SP_n(q)$ where $q = p^e$, we have $\mathrm{rk}_p(J - I - A) = \binom{p+n-1}{n}^e$, since this is just the module spanned by the hyperplane complements (cf. MacWilliams and Mann [16]). S. Shpectorov remarks that it is easy to see that $\langle J - I - A\rangle$ is an irreducible $Sp_n(q)$ module.

4. Let Γ be the orthogonal graph $O_{2m+1}(q)$, with as point set the set of isotropic points of $PG(2m, q)$, provided with a nondegenerate quadratic form, where distinct points are adjacent when they are orthogonal. Then Γ is strongly regular with the same parameters as $Sp_{2m}(q)$ above. If q is odd and $m = 2$, then by Bagchi, Brouwer and Wilbrink [1] we have $\mathrm{rk}_2(A) = \mathrm{rk}_2(J - A) = q^2 + 1$.

   Concerning the 3-rank in case of $O_5(3)$, we find from Atlas and Modular Atlas: $\pi = \chi_1 + \chi_{15b} + \chi_{24}$ and $\overline{\chi}_{15b} = \psi_1 + \psi_{14}$ and $\overline{\chi}_{24} = \psi_{10} + \psi_{14}$, so $\mathrm{rk}_3(J - I - A) = 14$, and $\langle J - I - A\rangle_3$ is irreducible for $O_5(3)$.

5. Let Γ be the collinearity graph of any generalized quadrangle $GQ(q, q^2)$. Interesting primes are those dividing $q(q + 1)$. If q is odd, then by Bagchi, Brouwer and Wilbrink [1] we have $\mathrm{rk}_2(A) = \mathrm{rk}_2(J - A) = q^3 - q^2 + q + 1$. Adding the columns corresponding to a line, we find $\mathbf{1} \in \langle A + I + bJ\rangle_p$ for $p|q$ and $b \neq -1$, while clearly $\mathbf{1} \notin \langle J - I - A\rangle$.

   Concerning the 3-rank in case of the unique $GQ(3,9)$ (with parameters $(112, 30, 2, 10)$ and spectrum $30^1 2^{90} (-10)^{21}$), we have $\mathrm{rk}_3(J - I - A) = 19$.

   [Since the Gewirtz graph is an induced subgraph, we have $\mathrm{rk}_3(J - I - A) \geq 19$, and since our graph is the first subconstituent of the McLaughlin graph Λ with $\mathrm{rk}_3(J - I - A_\Lambda) = 21$ (see below), we have $\mathrm{rk}_3(J - I - A) \leq 19$.]

6. The folded 5-cube is strongly regular with parameters $(\nu, k, \lambda, \mu) = (16, 5, 0, 2)$ and spectrum $5^1 1^{10} (-3)^5$. (Its complement, with parameters (16, 10, 6, 6), is called the Clebsch graph. It is the halved 5-cube.) We have $\mathrm{rk}_2(A + I) = \mathrm{rk}_2(J - I - A) = 6$.

   [Indeed, its second subconstituent is the Petersen graph, the complement of $T(5)$, which has $\mathrm{rk}_2(A_P) = 5$, so $6 \leq \mathrm{rk}_2(A + I) \leq g + 1 = 6$. Next, $\mathrm{rk}_2(J - I - A)$ differs from this by at most one, but is even, so equals 6 also.]

7. The Schläfli graph is strongly regular with parameters (27, 16, 10, 8) and spectrum $27^1 4^6 (-2)^{20}$. We have $\mathrm{rk}_2(A) = 6, \mathrm{rk}_2(J - A) = 7$, and $\mathrm{rk}_3(A + 2I + bJ) = 7$ for all b.

   [Indeed, we saw under 2. above that for the triangular graph $T(8)$ we have $\mathbf{1} \notin \langle A_{T(8)}\rangle$ and $\mathrm{rk}_2(A_{T(8)}) = 6$. Since the Schläfli graph is obtained from $T(8)$ by switching a point isolated, we find $\mathrm{rk}_2(A) = 6$. Since ν is odd and k is even, $\mathrm{rk}_2(A + J) = \mathrm{rk}_2(A) + 1$. For the 3-ranks, $J - 2A - I$ has the same Smith normal form as $J - 2A_{T(8)} - I$, and hence has 3-rank 7. The complementary graph is a generalized quadrangle $GQ(2, 4)$, and adding the

Name	v	k	λ	μ	r^f	s^g	p	$rk_p(A-sI)$	b_0
folded 5-cube	16	5	0	2	1^{10}	$(-3)^5$	2	6	-
Schläfli	27	16	10	8	4^6	$(-2)^{20}$	2	6	0
							3	7	-
$T(8)$	28	12	6	4	4^7	$(-2)^{20}$	2	6	0
							3	8	2
3 Chang graphs	28	12	6	4	4^7	$(-2)^{20}$	2	8	-
							3	8	2
$G_2(2)$	36	14	4	6	2^{21}	$(-4)^{14}$	2	8	-
							3	14	-
$Sp_4(3)$	40	12	2	4	2^{24}	$(-4)^{15}$	2	16	-
							3	11	1
$O_5(3)$	40	12	2	4	2^{24}	$(-4)^{15}$	2	10	-
							3	15	1
Hoffmann-Singleton	50	7	0	1	2^{28}	$(-3)^{21}$	5	21	-
Gewirtz	56	10	0	2	2^{35}	$(-4)^{20}$	2	20	-
							3	20	1
M_{22}	77	16	0	4	2^{55}	$(-6)^{21}$	2	20	0
sub $GQ(3,9)$	81	20	1	6	2^{60}	$(-7)^{20}$	3	19	-
Higman-Sims	100	22	0	6	2^{77}	$(-8)^{22}$	2	22	-
							5	23	-
Hall-Janko	100	36	14	12	6^{36}	$(-4)^{63}$	2	36	0
							5	23	-
$GQ(3,9)$	112	30	2	10	2^{90}	$(-10)^{21}$	2	22	-
							3	20	1
001 ...in $S(5,8,24)$	120	42	8	18	2^{99}	$(-12)^{20}$	2	20	-
							7	20	5
$Sp_4(5)$	156	30	4	6	4^{90}	$(-6)^{65}$	2	66	-
							5	36	1
sub McL	162	56	10	24	2^{140}	$(-16)^{21}$	2	20	0
							3	21	-
edges of Ho-Si	175	72	20	36	2^{153}	$(-18)^{21}$	2	20	0
							5	21	-
01 ...in $S(5,8,24)$	176	70	18	34	2^{154}	$(-18)^{21}$	2	22	-
							5	22	3
switched version	176	90	38	54	2^{153}	$(-18)^{22}$	2	22	-
of previous							5	22	3
Cameron	231	30	9	3	9^{55}	$(-3)^{175}$	2	55	1
							3	56	1
B-vL-S	243	22	1	2	4^{132}	$(-5)^{110}$	3	67	-
Delsarte	243	110	37	60	2^{220}	$(-25)^{22}$	3	22	-
$S(4,7,23)$	253	112	36	60	2^{230}	$(-26)^{22}$	2	22	0
							7	23	5
McLaughlin	275	112	30	56	2^{252}	$(-28)^{22}$	2	22	0
							3	22	1
							5	23	-
switched version	276	140	58	84	2^{252}	$(-28)^{23}$	2	24	-
of previous plus							3	23	2
isolated point							5	24	3
$G_2(4)$	416	100	36	20	20^{65}	$(-4)^{350}$	2	38	-
							3	65	1
dodecads mod 1	1288	792	476	504	8^{1035}	$(-36)^{252}$	2	22	0
							11	230	3

three columns corresponding to a line, we see that $\mathbf{1} \in \langle A + 2I + bJ \rangle$ for all b.]

8. $G_2(2)$ has a rank 3 representation on 36 points, giving rise to a strongly regular graph Γ with parameters (36, 14, 4, 6) and spectrum $14^1 2^{21} (-4)^{14}$. (For $U_3(3)$, the rank is 4, with suborbits $1 + 7 + 7 + 21$.) We have $\mathrm{rk}_2(A + bJ) = 8$ and $\mathrm{rk}_3(A - 2I + bJ) = 14$, for all b.

   [The permutation character of $U_3(3)$ is the sum of four irreducibles: $\pi = \chi_1 + \chi_{7b} + \chi_{7c} + \chi_{21}$, where the subscript numbers denote the degree, and the subscript letters indicate the sequence as in the Atlas [6]. Restricting these to the elements of order prime to 2 and decomposing into 2-modular irreducibles (using the Atlas and the Modular Atlas [17]) we find $\overline{\chi}_{7b} = \overline{\chi}_{7c} = \psi_1 + \psi_6$ and $\overline{\chi}_{21} = \psi_1 + \psi_6 + \psi_{14}$. Thus $\overline{\eta} = m\psi_1 + \psi_6$ with $m \in \{0, 1, 2\}$. Now Γ is locally bipartite, and summing the 8 columns corresponding to a $K_{1,7}$ yields 1, showing that $\mathbf{1} \in \langle A + bJ \rangle^+$, so that $m = 2$ and $\mathrm{rk}_2(A + bJ) = 8$. In fact the permutation module splits as $1 + 6 + 1 + (6 \oplus 14) + 1 + 6 + 1$. Similarly, we can decompose into 3-modular irreducibles, and now find $\overline{\chi}_{7b,c} = \psi_1 + \psi_{6a,b}$ and $\overline{\chi}_{21} = 2\psi_1 + \psi_{6a} + \psi_{6b} + \psi_7$. Since ψ_{6a} and ψ_{6b} are conjugate, it follows that $\overline{\eta} = m\psi_1 + \psi_{6a} + \psi_{6b}$ with $m \in \{0, 1, 2\}$. Now Γ is a subgraph of the Hall-Janko graph, and the μ-graphs of the latter graph yield subgraphs M isomorphic to the 2-coclique extension of $\overline{C}_6$ (hence regular of valency 6 on 12 points) with the property that each vertex of Γ outside M is adjacent to precisely 4 vertices of M. This shows that $\mathbf{1} \in \langle A + I + bJ \rangle^+$, so that $m = 2$ and $\mathrm{rk}_3(A + I + bJ) = 14$. In fact the permutation module splits as $1 + 12 + 1 + (1 \oplus 7) + 1 + 12 + 1$.]

9. The Hoffman-Singleton graph Γ is strongly regular with parameters (50, 7, 0, 1) and spectrum $7^1 2^{28} (-3)^{21}$.

   An explicit construction different from the usual ones is as follows: Take as vertices the 20 ternary vectors of weight 1 and the 30 ternary vectors of length 10 and weight 4 obtained by taking in the extended ternary Golay code all vectors of weight six starting with 11... or 12... and deleting the first two coordinates. Join two weight 1 vectors when they have distance 1; join a weight 1 and a weight 4 vector when they have distance 3; join two weight 4 vectors when they have distance 8. This yields the Hoffman-Singleton graph and shows that it has a partition into a subgraph $10K_2$ and two 15-cocliques. (In the usual 15 + 35 representation, we find a 15 + 15 + 20 by fixing a symplectic polarity on PG (3, 2).)

   We have $\mathrm{rk}_5(A - 2I + bJ) = 21$ for all b.

   [Indeed, summing the columns corresponding to a Petersen subgraph we find $\mathbf{1} \in \langle A - 2I + bJ \rangle$ for all b. As we saw, the graph Γ contains subgraphs $10K_2$ and no further column of $A - 2I$ is dependent on the $10K_2$, so $\mathrm{rk}_5(A - 2I) \geq 21$. The permutation character of $U_3(5)$ is the sum of three irreducibles: $\pi = \chi_1 + \chi_{21} + \chi_{28}$, where the subscripts denote the degree, and restricting these to the elements of order prime to 5 and decomposing into 5-modular irreducibles (using the Atlas [6] and the Modular Atlas [17]) we find $\overline{\chi}_{21} = 2\psi_1 + \psi_{19}$ and $\overline{\chi}_{28} = \psi_1 + \psi_8 + \psi_{19}$. We must conclude that $\overline{\eta}(1) \leq 21$. In fact the permutation module splits as $1+19+1+8+1+19+1$, while $\langle A-2I\rangle$ splits as $1+19+1$.]

10. The Gewirtz graph Γ is strongly regular with parameters (56, 10, 0, 2) and spectrum $10^1 2^{35}(-4)^{20}$. We have $\mathrm{rk}_2(A+bJ) = 20$ for all b, and $\mathrm{rk}_3(J-I-A) = 19, \mathrm{rk}_3(A+I+bJ) = 20$ for $b \neq -1$.

    [Indeed, Γ contains a subgraph $10K_2$, and summing the corresponding columns we find $\mathbf{1} \in \langle A+bJ\rangle_2$ for all b. This same subgraph shows $\mathrm{rk}_2(A) \geq 20$, and since $\mathrm{rk}(7A-14I-J) = \mathrm{rk}(E_2) = 20$, we have $\mathrm{rk}_2(A-J) \leq 20$. The value of $\mathrm{rk}_3(A+I)$ was determined in Brouwer and Haemers [4].]

11. The 77-point strongly graph Γ has parameters $(77, 16, 0, 4)$ and spectrum $16^1 2^{55}(-6)^{21}$. We have $\mathrm{rk}_2(A) = 20$ and $\mathrm{rk}_2(J-A) = 21$.

    [Since Γ contains the Gewirtz graph as an induced subgraph, we have $20 \leq \mathrm{rk}_2(A) = \mathrm{rk}_2(22I-11A+2J) \leq \mathrm{rk}(22I-11A+2J) = \mathrm{rk}(E_2) = 21$. But $\mathrm{rk}_2(A)$ is even.]

12. The strongly regular graph with parameters $(81, 20, 1, 6)$ has spectrum $20^1 2^{60}(-7)^{20}$. We have $\mathrm{rk}_3(A-2I+bJ) = 19$ for all b.

    [Since it is the second subconstituent of $GQ(3,9)$, we find $\mathrm{rk}_3(J-I-A) = 19$ and $\mathbf{1} \in \langle bJ-I-A\rangle$ for all b.]

13. The Higman-Sims graph Γ is strongly regular with parameters (100, 22, 0, 6) and spectrum $22^1 2^{77}(-8)^{22}$. We have $\mathrm{rk}_2(A+bJ) = 22$ and $\mathrm{rk}_5(A-2I+bJ) = 23$ for all b.

    [Since Γ contains the 77-point strongly regular graph as second subconstituent, we have $\mathrm{rk}_2(A) \geq 22$ and $\mathrm{rk}_5(A-2I) \geq 23$. This settles $\mathrm{rk}_2(A)$ and $\mathrm{rk}_5(A-2I+bJ)$ since $g+1 = 23$ and $\mathrm{rk}_2(A)$ is even and $5|\nu, 5\nmid\mu$. Furthermore, $\mathrm{rk}_2(J-A) \leq \mathrm{rk}(A-2I-\frac{1}{5}J) = \mathrm{rk}E_2 = 22$. Finally, using Lemma B and the fact that for any two adjacent vertices x, y the subgraph of Γ induced by the points nonadjacent to both x and y is isomorphic to the Gewirtz graph, we find $\mathrm{rk}_2(J-A) \geq 22$.]

14. The Hall-Janko graph Γ is strongly regular with parameters $(100, 36, 14, 12)$ and spectrum $36^1 6^{36}(-4)^{63}$. We have $\mathrm{rk}_2(A) = 36, \mathrm{rk}_2(A+J) = 37$ and $\mathrm{rk}_5(A-I+bJ) = 23$ for all b.

    [Using the fact that $\Gamma_2(x)$ is the distance-2 graph of a generalized hexagon of order (2, 2), and that the λ-graphs are isomorphic to the point-block incidence graph of the unique biplane 2-$(7,4,2)$, one sees that if $\{x,y,z\}$ induces $K_1 + K_2$, then $\{x,y,z\}^\perp \cong C_4$. It follows that $\mathbf{0}$ is the sum of the 9 columns in A indexed by two adjacent vertices y, z and one bipartite half of the set of their common neighbours. Hence $\mathbf{1} \in \langle A+J\rangle_2$. Looking at the 2-modular characters for the group HJ we find $\overline{\chi}_{36} = \psi_{36}$, and since $\mathrm{rk}_2 A$ is even, this settles the 2-ranks. In fact the permutation module (for $p = 2$) splits as $(1 \oplus 36) + 12 + 1 + 1 + 12 + (1 \oplus 36)$. Similarly, for $p = 5$, we find 5-modular characters $\overline{\chi}_{36} = \psi_1 + \psi_{14} + \psi_{21}$ and $\overline{\chi}_{63} = \psi_1 + \psi_{21} + \psi_{41}$, so that $\overline{\eta} = m\psi_1 + \psi_{21}$ with $m \in \{0,1,2\}$. But $\mathbf{1} \in \langle A-I+bJ\rangle^+$, so $m = 2$. In fact the permutation module (for $p = 5$) splits as $1 + 21 + (1 \oplus 14 \oplus 41) + 21 + 1$.]

15. Let $\Sigma = \{\sigma_1, \cdots, \sigma_{23}\}$ be a set of size 23, and let $\mathbf{D} = (\Sigma, \mathbf{B})$ be a Steiner system $S(4,7,23)$ on it. Define a graph Γ on $\mathbf{B}$ by letting $B \sim C$ whenever $|B \cap C| = 1$. For $i = 0,1,2$, let $\mathbf{D}^{(i)} = (\Sigma^{(i)}, \mathbf{B}^{(i)})$ be the design obtained from $\mathbf{D}$ by throwing away the i symbols $\sigma_1, \cdots, \sigma_i$ and all blocks containing at least one of these symbols. Let $\Gamma^{(i)}$ be the subgraph of Γ induced by $\mathbf{B}^{(i)}$. Then $\Gamma^{(0)}, \Gamma^{(1)}$, and $\Gamma^{(2)}$ are strongly regular, with parameters (253,112,36,60), (176,70,18,34) and (120,42,8,18) and spectra $112^1 2^{230}(-26)^{22}, 70^1 2^{154}(-18)^{21}$ and $42^1 2^{99}(-12)^{20}$, respectively. We have $\mathrm{rk}_7(2J-2I+A) = 22, \mathrm{rk}_5(2J-2I+A^{(1)}) = 21$ and $\mathrm{rk}_7(2J-2I+A^{(2)}) = 19$.
   [Let $N^{(i)}$ be the point-block incidence matrix of $\mathbf{D}^{(i)}$. Then $(N^{(i)})^T N^{(i)} = 7I + A^{(i)} + 3(J - I - A^{(i)})$. We have $\langle N \rangle_7 = \langle e_{23} - e_j | 1 \le j \le 22 \rangle_7$ so that $\mathrm{rk}_7 N^T N = \mathrm{rk}_7(I_{22} + J_{22}) = 22$. Similarly, $\langle N^{(2)} \rangle_7 = \langle e_{21} - e_j | 1 \le j \le 20 \rangle_7$ so that $\mathrm{rk}_7 (N^{(2)})^T N^{(2)} = \mathrm{rk}_7(I_{20} + J_{20}) = 19$. Finally, $\langle N^{(1)} \rangle_5 = \langle e_j | 1 \le j \le 22 \rangle_5$ so that $\mathrm{rk}_5 (N^{(1)})^T N^{(1)} = \mathrm{rk}_5(I_{22}) = 22$.]
   We have $\mathrm{rk}_2 A = \mathrm{rk}_2 A^{(1)} = \mathrm{rk}_2(A^{(1)} + J) = 22$ and $\mathrm{rk}_2 A^{(2)} = \mathrm{rk}_2(A^{(2)} + J) = 20$.
   [Indeed, $\mathrm{rk}_2 A = \mathrm{rk}_2 A^{(1)}$, since the five rows of A corresponding to the five blocks containing three fixed symbols sum to zero, so that the 77 rows and columns indexed by $\mathbf{B} \backslash \mathbf{B}^{(1)}$ can be expressed in terms of those indexed by $\mathbf{B}^{(1)}$. (More generally, $Ax = 0 \pmod 2$ if $Nx = c\mathbf{1} \pmod 4$ for some c.) Looking at the 2-modular characters we find for M_{23} (acting rank 3 on $\mathbf{B}$): $\overline{\chi}_{22} = \psi_{11a} + \psi_{11b}$ so $\overline{\eta} = m\psi_1 + \psi_{11a} + \psi_{11b} (m \le 1)$, but $\mathrm{rk}_2 A$ is even, so $m = 0$ and $\mathrm{rk}_2 A = 22$ (and then $\mathrm{rk}_2(A + J) = 23$). Next, for M_{22} (rank 3 on $\mathbf{B}^{(1)}$) : $\overline{\chi}_{21} = \psi_1 + \psi_{10a} + \psi_{10b}$ so $\overline{\eta} = m\psi_1 + \psi_{10a} + \psi_{10b} (m \le 2)$; it follows that $\mathrm{rk}_2(A^{(1)} + J) = 22$ and that $\mathbf{1} \in \langle A^{(1)} \rangle \cap \langle A^{(1)} + J \rangle$.
   Finally, for $L_3(4)$ (acting rank 4 on $\mathbf{B}^{(2)}$): $\overline{\chi}_{20} = 2\psi_1 + \psi_{9a} + \psi_{9b}$ and $\overline{\chi}_{35a} + \overline{\chi}_{64} = (\psi_1 + \psi_{8a} + \psi_{8b} + \psi_{9a} + \psi_{9b}) + \psi_{64}$ so $\overline{\eta} = m\psi_1 + \psi_{9a} + \psi_{9b} (m \le 2)$. Using the linear dependence that we found above twice, we see that the difference of two rows or columns indexed by $\mathbf{B}^{(1)} \backslash \mathbf{B}^{(2)}$ is a linear combination of six rows or columns indexed by $\mathbf{B}^{(2)}$. Hence $\mathrm{rk}_2(A^{(2)} + bJ) \ge \mathrm{rk}_2(A^{(1)} + bJ) - 2 = 20$, and therefore $m = 2$ and we have equality everywhere.]
   It is possible to obtain a strongly regular graph Δ with parameters (176,90,38,54) by suitably switching $\Gamma^{(1)}$. The 5-ranks for Δ are necessarily the same as for $\Gamma^{(1)}$. We do not know whether the same holds for the 2-ranks, but for at least one choice of switching set the 2-ranks remain unchanged (as follows by explicit computation).
16. The Hoffman-Singleton graph has 175 edges, and if we join two of these when they are disjoint and lie together in a pentagon, then we get a strongly regular graph Γ with parameters (175,72,20,36). We have $\mathrm{rk}_2(A) = 20$ and $\mathrm{rk}_5(A - 2I) = 21$.
   [Indeed, Γ may be obtained from the (176, 70, 18, 34) graph Δ discussed above, so that $\mathrm{rk}_5(J - 2A + 4I) = \mathrm{rk}_5(J - 2B + 4I) = 21$, where B is the adjacency matrix of Δ. For the same reason $\mathrm{rk}_2(A) \in \{20, 22\}$, but since v is odd $\mathrm{rk}_2(A) + 1 = \mathrm{rk}_2(A + J) \le g + 1 = 22$.]

17. The Cameron graph is the strongly regular graph with parameters (231, 30, 9, 3) constructed from $S(3,6,22)$ by taking the $\binom{22}{2}$ pairs of symbols as vertices, and joining two pairs when they are disjoint and their union is contained in a block. We have $\mathrm{rk}_2(J-I-A) = 54$ and $\mathrm{rk}_3(J-A) = 55, \mathrm{rk}_3(A) = 56$.

    [Indeed, these numbers are upper bounds since $\mathrm{rk}_p(A + 3I - \frac{1}{7}J) \leq \mathrm{rk}E_1 = 55$ for $p \neq 7$ (and $\mathrm{rk}_2(J-I-A)$ is even). M_{22} has permutation character $\pi = \chi_1 + \chi_{55} + (\chi_{21} + \chi_{154})$. Looking at 3-modular characters we find $\overline{\chi}_{21} = \psi_{21}$ and $\overline{\chi}_{55} = \psi_{55}$ and $\overline{\chi}_{154} = \psi_1 + (\psi_{49a} + \psi_{49b}) + \psi_{55}$. It follows that $\overline{\eta} = m\psi_1 + \psi_{55}$ with $m \leq 1$. Consequently, $\langle J - A\rangle_3$ is irreducible and hence $\mathrm{rk}_3(A) = \mathrm{rk}_3(J-A) + 1$. Direct computation yields $\mathrm{rk}_2(J-I-A) = 54$. (Looking at 2-modular characters is not so successful: we find $\overline{\chi}_{21} = \psi_1 + (\psi_{10a} + \psi_{10b}), \overline{\chi}_{55} = \overline{\chi}_{21} + \psi_{34}$ and $\overline{\chi}_{154} = \overline{\chi}_{55} + \psi_1 + \psi_{98}$. Thus, $\overline{\eta} = m\psi_1 + \delta(\psi_{10a} + \psi_{10b}) + \varepsilon\psi_{34}$ with $m \leq 2$ and $\delta, \varepsilon \in \{0,1\}$.)]
18. The Berlekamp-van Lint-Seidel graph is the strongly regular graph with parameters (243, 22, 1, 2) obtained by taking the cosets of the perfect ternary Golay code and joining them when they differ by a weight 1 vector. Direct computation shows that $\mathrm{rk}_3(A) = 67$. Its dual is the Delsarte graph, with parameters (243, 110, 37, 60), and can be described as the graph on the shortened extended ternary Golay code where two vectors are joined when they have Hamming distance 9. Direct computation shows that $\mathrm{rk}_3(A) = 22$.
19. The McLaughlin graph Λ is strongly regular with parameters (275, 112, 30, 56) and spectrum $112^1 2^{252}(-28)^{22}$. Haemers e.a. [10] show that $\mathrm{rk}_2(A) = 22$. We have $\mathrm{rk}_3(J-I-A) = 21, \mathrm{rk}_5(A - 2I + bJ) = 23$.

    [Indeed, Λ contains a 22-coclique C, so $\mathrm{rk}_p(A + cI) \geq 22$ for $c \neq 0$. But for every vertex $x \notin C$, column x of A is the mod 3 sum of the (7 or 16) columns of $A + I$ indexed by the neighbours of x in C. Thus $\mathrm{rk}_3(A+I) = 22$. The second subconstituent of Λ is strongly regular, and for it we have $\mathrm{rk}_5(A'' - 2I) = 22$, so $\mathrm{rk}_5(A - 2I) \geq 23$, and since $g = 22$ we have equality. (And $\mathrm{rk}_2(A) = 22$ follows since the first subconstituent of Λ is the collinearity graph of $GQ(3,9)$, and we saw above that $\mathrm{rk}_2(A') = 22$.)]

    The second subconstituent Σ of Λ is strongly regular with parameters (162, 56, 10, 24). We have $\mathrm{rk}_2(A'') = 20, \mathrm{rk}_2(A''+J) = 21, \mathrm{rk}_3(A''-2I+bJ) = 21$.

    [Indeed, Σ contains a 21-coclique such that each point outside has 7 or 16 neighbours in it. It follows that $\mathbf{1} \in \langle A'' + I\rangle_3$. Consequently, by the values found above for Λ, the values claimed are upper bounds, and in the case of 3-ranks also lower bounds. Since the (81, 20, 1, 6) strongly regular graph is a subgraph of $\Sigma, \mathrm{rk}_2(A'') \geq 20$. This same subgraph shows that $\mathbf{1} \in \langle A'' + J\rangle$. On the other hand, looking at the 2-modular character of $U_4(3)$ we find $\overline{\chi}_{21} = \psi_1 + \psi_{20}$ and $\overline{\chi}_{140} = \psi_{20} + \psi_{140}$, so that $\langle A''\rangle$ is irreducible and, in particular, $\mathbf{1} \notin \langle A''\rangle$. Thus $\mathrm{rk}_2(A'' + J) = 21$.]
20. Goethals and Seidel [9] showed that the switching class of the regular two-graph on 276 points contains a strongly regular graph Γ with parameters (276, 140, 58, 84) possessing a 6-clique C. For such a graph we have

$\mathrm{rk}_2(A) = \mathrm{rk}_2(A+J) = 24, \mathrm{rk}_3(A-2I+J) = 22, \mathrm{rk}_5(A-2I) = 24$. As W. Haemers remarks, such a graph may be obtained from the McLaughlin graph plus an isolated point, by switching with respect to the disjoint union of 28 5-cliques, so probably there are many nonisomorphic such graphs. One particularly nice construction is the following: Take the Delsarte graph as described above, and add 33 vertices (i,j), $1 \le i \le 11, j \in \mathbb{F}_3$. Join a ternary vector ν to (i,j) when $\nu_i = j$. Join (i,j) to $(i,k)(j \ne k)$. This yields a graph in the switching class of the regular 2-graph on 276 vertices. Now switch with respect to the set $\{(i,j)|i=1\} \cup \{\nu|\nu_1 \ne \nu_2\}$. This yields Γ, and we have $\mathrm{rk}_3(A-2I) = \mathrm{rk}_3(A-2I-J) = 23$.

[Indeed, Γ is switching equivalent to the graph Λ^* (with adjacency matrix B) obtained from Λ by adding an isolated point. Each vertex outside C has 3 neighbours in C, so by adding the six columns of A or $A+J$ corresponding to the vertices in C, we see that $\mathbf{1} \in \langle A+bJ\rangle_2$, settling the 2-ranks. Next, $\mathrm{rk}_5(A-2I-\frac{1}{2}J) = \mathrm{rk}_5(B-2I-\frac{1}{2}J) = 23$, settling the 5-ranks. Finally, $\mathrm{rk}_3(A-2I-\frac{1}{2}J) = \mathrm{rk}_3(B-2I-\frac{1}{2}J) = 22$, and the other 3-ranks follow by direct computation.]

21. $G_2(4)$ has a rank 3 representation on the cosets of HJ, giving rise to a strongly regular graph with parameters (416, 100, 36, 20). According to CAYLEY, the permutation module for $p=2$ splits as $1+36+(1\oplus 28)+12+1+1+12+(36\oplus 196)+12+1+1+12+(1\oplus 28)+36+1$, and we have $\mathrm{rk}_2(A) = \mathrm{rk}_2(A+J) = 38$. The permutation module for $p=3$ splits as $(1\oplus 64)+286+(1\oplus 64)$, and we have $\mathrm{rk}_3(J-I-A) = 64$.
22. If we join the dodecads in the perfect binary Golay code when their distance is 12, we get a strongly regular graph with parameters (1288, 792, 476, 504). We have $\mathrm{rk}_2(A) = 22, \mathrm{rk}_2(A+J) = 23$.

[Indeed, let N be the 1288×23 dodecad-symbol incidence matrix, then $A \equiv \frac{1}{2}NN^T \pmod 2$. Since $\mathrm{rk}_2(A)$ is even, it follows that $\mathrm{rk}_2(A) \le 22$. Since 22 is the smallest possible dimension of a nontrivial irreducible M_{23}-module, $\mathrm{rk}_2(A) = 22$, and $\langle A\rangle^+ = \langle A\rangle$. Consequently $\mathrm{rk}_2(A+J) = 23$.]

Moreover, $\mathrm{rk}_{11}(A-8I) = 230, \mathrm{rk}_{11}(A-8I-3J) = 229$.

[Indeed, looking at the 11-modular characters of M_{23} (acting rank 4 on the 1288 dodecads) we find $\overline{\chi}_{252} = \overline{\chi}_{22} + \overline{\chi}_{230} = \psi_{22} + (\psi_1 + \psi_{229})$ and $\overline{\chi}_{1035} = \psi_{229} + \psi_{806}$ so that $\overline{\eta} = m\psi_1 + \psi_{229}$ with $m \le 1$.]

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