

et les conditions aux limites

$$(22) \quad x(\lambda_1) = 0, \quad x(\lambda_2) = 0.$$

La solution générale de (21) est

$$x(\lambda) = c_1 e^{i\lambda} + c_2 e^{-i\lambda},$$

où c_1 et c_2 sont des opérateurs arbitraires. Il s'ensuit que la solution satisfaisant aux conditions (22) a la forme

$$(23) \quad x(\lambda) = f \sin(\lambda - \lambda_1),$$

où f est un opérateur arbitraire. Pour que la fonction (23) soit paramétrique, il faut et il suffit que f soit une fonction ordinaire de la variable t . C'est ce qui implique la forme (20) pour toute solution du problème initial, concernant l'équation (18).

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On the Paley-Wiener theorem

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§ 1. Introduction. By the well known and important theorem of PALEY and WIENER¹⁾, every entire function $F(z)$ of exponential type which belongs to L_2 along an infinite axis can be represented as a Fourier integral. In this paper we shall formulate and prove an analogous theorem for analytic functions which are considered in a half-plane only. From such a theorem we can easily obtain the theorem for entire functions (§ 3), but not conversely.

PLANCHEREL and PÓLYA²⁾ have shown that the condition L_2 can be replaced by L_1 . However, they have given a new proof for both cases. In this paper, we shall prove our theorem by the hypothesis that $F(z)$ belongs to L_p ($1 \leq p \leq 2$) along the boundary of the considered half-plane.

The form of the Plancherel and Pólya theorem is slightly sharper than that of Paley and Wiener. This form and still sharper forms will be discussed in § 6.

The proof of Plancherel and Pólya is based on the properties of entire functions and cannot be applied to the half-plane. The proof given in the sequel is, in some points, analogous to that of Paley and Wiener; it leads, however, to an independent and more elementary argument for the particular case $p=1$.

§ 2. Theorem. We suppose throughout this paper that $F(z)$ is an analytic function in the half-plane $\Re z > 0$ and that

$$\lim_{x \rightarrow 0^+} F(x + iy) = F(iy) \quad \text{for almost every real } y.$$

Theorem. If $e^{-k|z|} F(z)$ is bounded in the half-plane $\Re z > 0$ and $F(iy) \in L_p(-\infty, \infty)$ ($1 \leq p \leq 2$), then $F(z)$ can be represented, for $\Re z > 0$, as an absolutely convergent integral.

¹⁾ Paley-Wiener [4], p. 12-13.

²⁾ Plancherel-Pólya [6].

$$(1) \quad F(z) = \int_{-k}^{\infty} e^{-zt} f(t) dt.$$

In the case $p=1$, the function $f(t)$ in (1) is continuous and bounded for $t \geq -k$ and is given in the form of the absolutely convergent integral

$$(2) \quad f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{iyt} F(iy) dy.$$

In the case $1 < p \leq 2$, the function $f(t)$ in (1) belongs to $L_q \left(\frac{1}{p} + \frac{1}{q} = 1 \right)$ and is given as the limit in mean with exponent q

$$(3) \quad f(t) = \text{l.i.m.}_{\omega \rightarrow \infty} \frac{1}{2\pi} \int_{-\omega}^{\omega} e^{iyt} F(iy) dy.$$

Moreover, the function $f(t)$ defined by (2) vanishes for $t \leq -k$ and that defined by (3) vanishes almost everywhere for $t \leq -k$.

§ 3. Case of entire functions. From the preceding Theorem we can easily deduce an analogous theorem for entire functions. In fact, if $F(z)$ is entire, $e^{-k|z|} F(z)$ is bounded (in the whole plane) and $F(iy) \in L_p(-\infty, \infty)$ ($1 \leq p \leq 2$), we can obviously apply the Theorem. On the other hand, we can apply the same Theorem to the function $G(z) = F(-z)$. We see, in case $p=1$, that the function

$$(4) \quad g(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{iyt} G(iy) dy$$

vanishes for $t \leq -k$. Replacing y by $-y$ in (4), we find that $g(t) = f(-t)$. Thus, $f(t) = 0$ for $t \geq k$ and the integral (1) gets reduced to

$$F(z) = \int_{-k}^k e^{-zt} f(t) dt.$$

This equality holds for $\Re z > 0$, but both its sides are entire functions and, therefore, it must hold in the whole plane.

An analogous argument can be used in case $1 < p \leq 2$.

§ 4. Two lemmas. In the proof of the theorem we shall need the following two lemmas:

Lemma 1. Let θ and k be any positive numbers. If $e^{-k|z|} F(z)$ is bounded in $\Re z > 0$ and

$$\int_y^{y+\theta} F(i\eta) d\eta$$

is bounded in $-\infty < y < \infty$, then the function

$$\Phi(z) = \frac{1}{\theta i} e^{-kz} \int_z^{z+i\theta} F(\zeta) d\zeta$$

is continuous and bounded in $\Re z > 0$.

Lemma 2. Let a be any positive number. If $\Phi(z)$ is analytic in $\Re z > 0$, continuous and bounded in $\Re z \geq 0$, then the function

$$(5) \quad \varphi(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{iyt} \frac{\Phi(iy)}{(1+aiy)^2} dy$$

is continuous and bounded in $-\infty < t < \infty$, and vanishing for $t \leq 0$; moreover, we have, in $\Re z > 0$,

$$\frac{\Phi(z)}{(1+az)^2} = \int_0^{\infty} e^{-zt} \varphi(t) dt.$$

Proof of Lemma 1. The continuity of $\Phi(z)$ in $\Re z > 0$ and that of $\Phi(iy)$ for real y are obvious. But $F(z)$ is bounded in any bounded region and this implies, by the Lebesgue theorem,

$$\lim_{x \rightarrow 0+} \int_{x+iy}^{x+i(y+\theta)} F(\zeta) d\zeta = \int_{iy}^{i(y+\theta)} F(\zeta) d\zeta$$

and, consequently,

$$\lim_{x \rightarrow 0+} \Phi(x+iy) = \Phi(iy).$$

This suffices to ensure the continuity of $\Phi(z)$ in the closed half-plane $\Re z \geq 0$.

Now, we have, for a positive number M ,

$$|\Phi(z)| \leq \frac{M}{\theta} \left| \int_z^{z+i\theta} e^{k|z|} dz \right| = \frac{M}{\theta} \int_y^{y+\theta} e^{k|x+iy|} d\eta \quad \text{for } \Re z > 0;$$

since in the last integral we have $|x+iy| \leq |x+iy| + \theta$, it follows that

$$|\Phi(z)| \leq M e^{k\theta} \cdot e^{k|z|} \quad \text{for } \Re z > 0.$$

We have, further,

$$|\Phi(x)| \leq \frac{1}{\theta} e^{-kx} \int_0^{\theta} |F(x+i\eta)| d\eta \leq \frac{M}{\theta} e^{-kx} \int_0^{\theta} e^{k|x+i\eta|} d\eta \leq \frac{M}{\theta} \int_0^{\theta} e^{k\theta} d\eta = M e^{k\theta}$$

for positive x . Thus $\Phi(x)$ is bounded on the imaginary axis and on the real positive axis, and is of exponential type in the half-plane $\Re z > 0$. By the Phragmén-Lindelöf theorem, it must be bounded in the whole half-plane $\Re z \geq 0$.

Proof of Lemma 2. Let z be arbitrarily fixed in $\Re z > 0$. We have

$$(6) \quad \frac{\Phi(z)}{(1+az)^2} = \frac{1}{2\pi i} \int_C \frac{\Phi(s)}{(1+as)^2} \cdot \frac{ds}{s-z}$$

where the contour C_r ($r > |z|$) is composed of the semi-circle

$$(7) \quad |s|=r, \quad \Re s \geq 0$$

and of the segment of the imaginary axis embraced by this semi-circle. On the other hand, given any real u , we have

$$(8) \quad 0 = \frac{1}{2\pi i} \int_{C_r} \frac{\Phi(s)}{(1+as)^2} \frac{e^{-(s-z)u} - 1}{s-z} ds,$$

for the integrand is analytic inside the contour C_r and continuous on it. Adding (6) and (8) we get

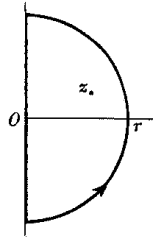
$$\frac{\Phi(z)}{(1+az)^2} = \frac{1}{2\pi i} \int_{C_r} \frac{\Phi(s)}{(1+as)^2} e^{-(s-z)u} \frac{ds}{s-z}$$

If $u \geq 0$, the part of that integral which belongs to the semi-circle (7) tends to 0, as $r \rightarrow \infty$, and so we get

$$(9) \quad \frac{\Phi(z)}{(1+az)^2} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\Phi(iy)}{(1+aiy)^2} \cdot \frac{e^{-(i y - z)u}}{z - iy} dy.$$

But

$$\frac{e^{-(s-z)u}}{z-s} = \int_{-u}^{\infty} e^{t(s-z)} dt \quad \text{for } \Re s < \Re z;$$



substituting this into (9) and interchanging the order of integration we get

$$\frac{\Phi(z)}{(1+az)^2} = \int_{-u}^{\infty} e^{-zt} dt \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\Phi(iy)}{(1+aiy)^2} dy.$$

This formula is true for $\Re z > 0$ and any $u \geq 0$; but its left side does not depend of u , which implies that the interior integral, equal to the function $\varphi(t)$ in (5), must vanish almost everywhere for $t \leq 0$. On the other hand

$$|\varphi(t)| \leq \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{|\Phi(iy)|}{1+a^2y^2} dy,$$

which proves the boundedness of $\varphi(t)$ and the uniform convergence of the integral in (5). Thus $\varphi(t)$ is continuous and our proof is complete.

§ 5. Proof of Theorem. Consider the function

$$(10) \quad \Phi_{\theta}(z) = \frac{1}{\theta i} e^{-kz} \int_0^{z+i\theta} F(\zeta) d\zeta,$$

where θ is any given positive number. By the Hölder inequality, we have

$$|\Phi_{\theta}(iy)| \leq \frac{1}{\theta} \int_y^{y+\theta} |F(iy)| dy \leq \frac{1}{\theta} \left(\int_y^{y+\theta} d\eta \right)^{1/q} \cdot \left(\int_y^{y+\theta} |F(i\eta)|^p d\eta \right)^{1/p} \leq \theta^{1/q-1} \left(\int_{-\infty}^{\infty} |F(i\eta)|^p d\eta \right)^{1/p}$$

for any real y and $1/p + 1/q = 1$. Thus $\Phi_{\theta}(z)$ is bounded on the imaginary axis. Since $e^{-k|z|} F(z)$ is bounded in $\Re z > 0$, we can apply Lemma 1. This asserts the continuity and boundedness of $\Phi_{\theta}(z)$ in $\Re z \geq 0$.

By Lemma 2 we have

$$(11) \quad \frac{\Phi_{\theta}(z)}{(1+az)^2} = \int_0^{\infty} e^{-zt} \varphi_{\alpha, \theta}(t) dt$$

where the function

$$\varphi_{\alpha,\theta}(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{iyt} \frac{\Phi_{\theta}(iy)}{(1+aiy)^2} dy,$$

is continuous and bounded in $-\infty < t < \infty$, and vanishing for $t \leq 0$. We shall show that, as $\theta \rightarrow 0$, the function $\varphi_{\alpha,\theta}(t)$ tends to

$$(12) \quad \varphi_{\alpha}(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{iy(t-k)} \frac{F(iy)}{(1+aiy)^2} dy$$

uniformly in the interval $-\infty < t < \infty$. In fact, we have

$$(13) \quad \begin{aligned} |\varphi_{\alpha,\theta}(t) - \varphi_{\alpha}(t)| &\leq \frac{1}{2\pi} \int_{-\infty}^{\infty} \left| \frac{1}{\theta} \int_y^{y+\theta} F(i\eta) d\eta - F(iy) \right| \frac{dy}{1+a^2y^2} \\ &= \frac{1}{2\pi\theta} \int_{-\infty}^{\infty} \left| \int_0^{\theta} [F(iy+i\eta) - F(iy)] d\eta \right| \frac{dy}{1+a^2y^2} \\ &\leq \frac{1}{2\pi\theta} \int_0^{\theta} \psi_{\alpha}(\eta) d\eta, \end{aligned}$$

where

$$\psi_{\alpha}(\eta) = \int_{-\infty}^{\infty} |F(iy+i\eta) - F(iy)| \frac{dy}{1+a^2y^2}.$$

If $F(iy)$ belongs to L_1 , we may write

$$(14) \quad \psi_{\alpha}(\eta) \leq \int_{-\infty}^{\infty} |F(iy+i\eta) - F(iy)| dy;$$

if $F(iy)$ belongs to L_p ($1 < p \leq 2$), then by the Hölder inequality

$$(15) \quad \psi_{\alpha}(\eta) \leq \left(\int_{-\infty}^{\infty} |F(iy+i\eta) - F(iy)|^p dy \right)^{1/p} \cdot \left(\int_{-\infty}^{\infty} \frac{dy}{(1+a^2y^2)^q} \right)^{1/q}.$$

Both (14) and (15) prove that

$$\lim_{\mu \rightarrow 0} \psi_{\alpha}(\eta) = 0.$$

Thus, by (13), $\varphi_{\alpha,\theta}(t)$ tends to $\varphi_{\alpha}(t)$ uniformly in $-\infty < t < \infty$. Moreover, $\varphi_{\alpha}(t)$ is a continuous function, vanishing for $t \leq 0$.

Letting θ tend to 0, we obtain from (10) and (11)

$$\frac{e^{-kz} F(z)}{(1+az)^2} = \int_0^{\infty} e^{-zt} \varphi_{\alpha}(t) dt$$

or, which is equivalent,

$$(16) \quad \frac{F(z)}{(1+az)^2} = \int_0^{\infty} e^{-zt} \varphi_{\alpha}(t+k) dt.$$

Now we have to pass to the limit with α . If $F(iy)$ belongs to L_1 , we get from (12)

$$|\varphi_{\alpha}(t+k)| \leq \frac{1}{2\pi} \int_{-\infty}^{\infty} |F(iy)| dy$$

and, by (2),

$$\lim_{\alpha \rightarrow 0} \varphi_{\alpha}(t+k) = f(t).$$

Thus, as $\alpha \rightarrow 0$, we obtain from (16), by the Lebesgue theorem, the formula (1). Moreover, one sees from (2), that $f(t)$ is continuous.

If $F(iy)$ belongs to L_p ($1 < p \leq 2$), then, as $\alpha \rightarrow 0$, the function

$$f_{\omega}(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{iyt} F(iy) dy$$

converges, by a theorem of TITCHMARSH³⁾, in mean with exponent q to a function $f(t)$ such that

$$(17) \quad \int_{-\infty}^{\infty} |f(t)|^q dt \leq (2\pi)^{1-q} \left(\int_{-\infty}^{\infty} |F(iy)|^p dy \right)^{\frac{1}{p-1}}.$$

Similarly, by applying the same theorem of Titchmarsh to the difference

$$\frac{F(z)}{(1+az)^2} - F(z),$$

we are led to the inequality

$$(18) \quad \int_{-\infty}^{\infty} |\varphi_{\alpha}(t+k) - f(t)|^q dt \leq (2\pi)^{1-q} \left(\int_{-\infty}^{\infty} \left| \frac{F(iy)}{(1+aiy)^2} - F(iy) \right|^p dy \right)^{\frac{1}{p-1}}.$$

³⁾ Titchmarsh [7], p. 96, Theorem 74.

But the last integral tends to 0, as $\alpha \rightarrow 0$, which implies that $\varphi_\alpha(t+k)$ converges in mean with exponent q to $f(t)$. Since $\varphi_\alpha(t+k)$ vanishes for $t \leq -k$, the function $f(t)$ vanishes almost everywhere for $t \leq -k$.

Now, it is easy to see that, as $\alpha \rightarrow 0$, the formula (16) takes the form (1) and that the integral in (1) is absolutely convergent for $\Re z > 0$ (because f belongs to L_q).

§ 6. Sharper forms of Theorem. Plancherel and Pólya have given a slightly sharper form to the theorem of Paley and Wiener. Namely, suppose that

(α) $e^{-k|z|} F(z)$ is bounded in the whole plane of z ,

(β) $F(iy)$ belongs to L_2 .

Then the Paley-Wiener theorem asserts that $F(z)$ can be represented in the form

$$(19) \quad F(z) = \int_{-k}^k e^{-zt} f(t) dt.$$

Plancherel and Pólya have shown that if (α) and (β) hold and, moreover, if for some k' and k'' ($-k \leq -k' < k'' \leq k$)

(γ) $e^{-k'x} F(x)$ and $e^{-k''x} F(-x)$ are bounded for $x > 0$,

then the formula (19) can be improved by introducing narrower bounds of integration:

$$(20) \quad F(z) = \int_{-k'}^{k''} e^{-zt} f(t) dt.$$

It is easy to show that (20) follows, by (γ), from (19). In fact, when $|F(-x)| < M e^{k''x}$ for $x > 0$, then it follows from (19) that, for $x > 0$,

$$\left| \int_{-k}^k e^{(t-k'')x} f(t) dt \right| < M$$

or, which is equivalent,

$$\left| \int_{-k-k''}^{-k-k'} e^{tx} f(t+k'') dt \right| < M.$$

Hence

$$\left| \int_0^{k-k''} e^{tx} f(t+k'') dt \right| < M_1 = M + \int_{-k-k''}^0 |f(t+k'')| dt,$$

and by an elementary theorem of PICONE⁴⁾ we have $f(t+k'') = 0$

⁴⁾ Picone [5]; see also Mikusiński [2].

almost everywhere for $-0 \leq t \leq k-k''$ or $f(t) = 0$ almost everywhere for $k'' \leq t \leq k$.

Hence, the upper bound in the integral (19) is to be replaced by k'' . Similarly, one can show that the effective bound in (19) is $-k'$.

If we use a stronger theorem than that of Picone, we can easily relax the condition (γ) and obtain in this way still sharper forms of the Paley-Wiener theorem. For instance, if $F(z)$ satisfies (α), (β) and

(δ) $e^{-k'x} F(x)$ is bounded for a sequence of positive numbers x_1, x_2, \dots such that $x_{n+1} - x_n > \delta > 0$ and $\sum_{n=1}^{\infty} 1/x_n = \infty$,

then by a theorem of LEVINSON⁵⁾, $e^{-k'x} F(x)$ will be bounded everywhere for $x > 0$. Consequently, we can replace in (19) the lower bound of integration by $-k'$. We can proceed similarly with the upper bound.

The same argument holds, of course, in case of a half-plane, as in our Theorem.

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⁵⁾ Levinson [1], p. 241, Theorem VII; see also Mikusiński-Nardzewski [3].