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On the parabolic equation method for water wave propagation.

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Note. The present report contains the preliminary text, submitted for publication.

Abstract

A parabolic approximation to the reduced wave equation is investigated for the propagation of periodic surface waves in shoaling water.
The approximation is derived from splitting the wave field into transmitted and reflected components.
In the case of an area with straight and parallel bottom contourlines, the asymptotic form of the solution for high frequencies is compared with the geometrical optics approximation.
Two numerical solution techniques are applied to the propagation of an incident plane wave over a circular shoal.

1. Introduction

The propagation of periodic, small amplitude surface gravity waves over a seabed of mild slope can be described by the solution of the reduced wave equation

$$\nabla\cdot(cc_g\ \nabla\Phi) + \frac{c_g}{c}\ \omega^2\ \Phi = 0 \qquad (1)$$

with appropriate boundary conditions. Here $\Phi(x,y)$ is the complex two-dimensional potential function, $\nabla \equiv (\frac{\partial}{\partial x}, \frac{\partial}{\partial y})$ the horizontal gradient operator, ω the angular frequency, and c and c_g are the corresponding local phase and group velocities of the wave field. This reduced wave equation accounts for the combined effects of refraction and diffraction, while the influences of bottom friction, current and wind have been neglected.

The wave equation (1) has been derived by several authors, for the first time by Berkhoff (1972), and by Schönfeld (1972) in a different form.

Svendsen (1967) derived the equation for one horizontal dimension, as is pointed out by Jonsson and Brink-Kjaer (1973). Smith and Sprinks (1975) gave a formal derivation of (1). Booij (1978) has proposed a new wave equation, which includes the effect of a current, and which reduces to (1) in the current-free case.

The equation (1) is essentially of elliptic type, and therefore defines a problem which is in general properly posed only when a boundary condition along a closed curve is given. In order to obtain a numerical solution for short waves over a large area in the horizontal plane, a great amount of computing time and storage is thus needed. However, in many water wave problems involving a gently sloping bottom, wave energy is propagated without appreciable reflection into a preferred direction, and it should be natural to consider methods which make use of this property. In the classification of Lundgren (1976), such methods can be distinguished as R-methods (refraction methods) and P-methods (propagation methods), both of which represent an approximation to the mild-slope equation (1).

Refraction methods are based on the geometrical optics approximation, which fails to give a reliable solution near caustics and crossing wave rays, where diffraction effects become important.

Propagation methods should be able to account for such situations. Methods of this type have been proposed by Biésel (1972), Lundgren (1976) and Radder (1977), but these are lacking, among other things, in the possibility of making systematic corrections which are needed if one wants to recover the complete wave field.

In the present work, a parabolic approximation to the reduced wave equation (1) is derived from splitting the wave field into transmitted and reflected components. The result is a pair of coupled equations for the transmitted and reflected fields. By assuming that the reflected field is negligible, a parabolic equation is obtained for the transmitted field. This procedure has been applied to optics by Corones (1975), and to acoustics by McDaniel (1975). The derivation is based on the Helmholtz-equation; therefore, in § 2, a reduction of the mild-slope equation (1) to the Helmholtz-equation is given, and in § 3 a parabolic approximation is derived. An asymptotic form of the solution for high frequencies is presented in § 4, in the case of an area with straight and parallel bottom contourlines.

Finally, in § 5 and § 6 numerical solutions to the parabolic equation are obtained in the form of two finite-difference schemes, with application to plane wave propagation over a circular shoal with parabolic bottom profile.

2. Reduction of the mild-slope equation to the Helmholtz-equation

Although a parabolic approximation can be directly derived from equation (1), it is useful, to simplify the notation and applications, to reduce equation (1) to the Helmholtz-equation, without loss of generality.

A scaling factor is introduced

$$\phi = \Phi \sqrt{cc_g} \tag{2}$$

which turns (1) into the Helmholtz-equation

$$\nabla^2\phi + k_c^2\,\phi = 0 \tag{3}$$

Here the effective wave number k_c is defined by

$$k_c^2 = k^2 - \frac{\nabla^2 \sqrt{cc_g}}{\sqrt{cc_g}} \tag{4}$$

and the wave number k is the real root of the dispersion relation

$$\omega^2 = g\,k\,\tanh\,(kh) \tag{5}$$

with h the local water depth and g the gravitational acceleration. The phase and group velocities are then given by $c = \frac{\omega}{k}$, $c_g = \frac{\partial\omega}{\partial k}$. In shallow water, the difference $k_c^2 - k^2$ may become appreciable: in this case one has

$$k^2 \simeq \frac{\omega^2}{gh} \,,\; c = c_g \simeq \sqrt{gh} \,,\; k_c^2 \simeq \frac{\omega^2}{gh} - \frac{\nabla^2 h}{2h} + \frac{|\nabla h|^2}{4h^2} \tag{6}$$

It follows, that k_c may be approximated by k if

$$|\nabla^2 h| \ll 2\omega^2/g \tag{7a}$$

$$|\nabla h|^2 \ll 4\omega^2 h/g \tag{7b}$$

implying a slowly varying depth and a small bottom slope, or high frequency wave propagation.
Unless stated otherwise, k_c will be approximated by k in this paper, assuming that (7a) and (7b) are satisfied.

3. Derivation of the parabolic approximation

The Helmholtz-equation (3) can be written in the form

$$\frac{\partial^2\phi}{\partial x^2} = -\left(k^2 + \frac{\partial^2}{\partial y^2}\right)\phi \tag{8}$$

where x denotes the preferred direction of propagation, and the subscript c of the wavenumber k has been dropped (however, for the derivation of the parabolic approximation the restrictions (7) are not necessary).
The wave field ϕ should be split into a transmitted field ϕ^+ and a reflected field ϕ^-

$$\phi = \phi^+ + \phi^- \tag{9}$$

This can be achieved by the use of a splitting matrix T which defines the transmitted and reflected components by

$$\begin{pmatrix}\phi^+ \\ \phi^-\end{pmatrix} = T\begin{pmatrix}\phi \\ \frac{\partial\phi}{\partial x}\end{pmatrix} \equiv \begin{pmatrix}\alpha & \beta \\ \gamma & \delta\end{pmatrix}\begin{pmatrix}\phi \\ \frac{\partial\phi}{\partial x}\end{pmatrix} \tag{10}$$

The matrix T is formally arbitrary, but some general physical criteria limit the choice of T and lead to the governing parabolic equation in a natural way.

Firstly, equation (9) is valid for arbitrarily chosen ϕ^+ and ϕ^- only if T satisfies

$$\begin{aligned} \alpha + \gamma &= 1 \\ \beta + \delta &= 0 \end{aligned} \tag{11}$$

Using equation (8), it follows that

$$\frac{\partial\phi^+}{\partial x} = \left[- k^2\beta + \frac{\alpha\gamma}{\beta} + \frac{\partial\alpha}{\partial x} + \frac{\gamma}{\beta}\frac{\partial\beta}{\partial x} - \beta\frac{\partial^2}{\partial y^2}\right]\phi^+ + \\ + \left[- k^2\beta - \frac{\alpha^2}{\beta} + \frac{\partial\alpha}{\partial x} - \frac{\alpha}{\beta}\frac{\partial\beta}{\partial x} - \beta\frac{\partial^2}{\partial y^2}\right]\phi^- \tag{12a}$$

$$\frac{\partial\phi^-}{\partial x} = \left[k^2\beta + \frac{\gamma^2}{\beta} + \frac{\partial\gamma}{\partial x} - \frac{\gamma}{\beta}\frac{\partial\beta}{\partial x} + \beta\frac{\partial^2}{\partial y^2}\right]\phi^+ + \\ + \left[k^2\beta - \frac{\alpha\gamma}{\beta} + \frac{\partial\gamma}{\partial x} + \frac{\alpha}{\beta}\frac{\partial\beta}{\partial x} + \beta\frac{\partial^2}{\partial y^2}\right]\phi^- \tag{12b}$$

Further, when k is a constant, solutions of the form

$$\phi^+ \simeq e^{ikx} \, , \; \phi^- \simeq e^{-ikx} \tag{13}$$

should result, and the equations (12a) and (12b) should naturally decouple in this case. This can be achieved by choosing

$$k^2\beta + \frac{\alpha^2}{\beta} = 0 \quad , \quad k^2\beta + \frac{\gamma^2}{\beta} = 0 \tag{14}$$

and the resulting splitting matrix is (cf. Corones (1975))

$$T = \tfrac{1}{2}\begin{pmatrix} 1 & -i/k \\ 1 & i/k \end{pmatrix} \tag{15}$$

while (12) reduces to

$$\frac{\partial \phi^+}{\partial x} = \left(ik - \frac{1}{2k}\frac{\partial k}{\partial x} + \frac{i}{2k} \frac{\partial^2}{\partial y^2} \right) \phi^+ + \left(\frac{1}{2k} \frac{\partial k}{\partial x} + \frac{i}{2k} \frac{\partial^2}{\partial y^2} \right) \phi^- \qquad (16a)$$

$$\frac{\partial \phi^-}{\partial x} = \left(\frac{1}{2k}\frac{\partial k}{\partial x} - \frac{i}{2k} \frac{\partial^2}{\partial y^2} \right) \phi^+ + \left(-ik - \frac{1}{2k} \frac{\partial k}{\partial x} - \frac{i}{2k} \frac{\partial^2}{\partial y^2} \right) \phi^- \qquad (16b)$$

This pair of coupled equations is equivalent to equation (8). By neglecting the reflected field ϕ^-, a parabolic equation for the transmitted field ϕ^+ is obtained

$$\frac{\partial \phi^+}{\partial x} = \left(ik - \frac{1}{2k}\frac{\partial k}{\partial x} + \frac{i}{2k} \frac{\partial^2}{\partial y^2} \right) \phi^+ \qquad (17)$$

In a similar way, a parabolic approximation can be directly derived from equation (1), which yields for the transmitted field Φ^+

$$\frac{\partial \Phi^+}{\partial x} = \left[ik - \frac{1}{2kcc_g} \frac{\partial (kcc_g)}{\partial x} + \frac{i}{2kcc_g} \frac{\partial}{\partial y} cc_g \frac{\partial}{\partial y} \right] \Phi^+$$

Using (2) and (7), equation (17) is recovered.

By adding to the left hand sides of (14) the operator $\beta \frac{\partial^2}{\partial y^2}$, another splitting matrix is derived: $T_A = \frac{1}{2} \begin{pmatrix} 1 & -i/A \\ 1 & i/A \end{pmatrix}$, where $A = \sqrt{k^2 + \frac{\partial^2}{\partial y^2}}$, and a closer approximation to equation (8) may be obtained. Unfortunately, the square root operator A makes the resulting parabolic equation practically untractable, and a satisfactory approximation must be found for the operator A, in order to obtain numerical results (cf. McDaniel (1975)).

In the following, the parabolic equation (17) will be considered, in which the preferred direction of propagation x is defined through the direction of the incident plane wave.

4. Asymptotic analysis for the one-dimensional case

In order to test the validity of the parabolic equation (17) as an approximation to equation (8), solutions to both equations will be compared in the case of an area with straight and parallel bottom contourlines. The problem is equivalent to plane wave propagation in a plane stratified medium in optics and acoustics, and the asymptotic analysis of Seckler and Keller (1959) will be followed here.
Dropping the + superscript, equation (17) can be written as

$$\frac{\partial^2\phi}{\partial y^2} + 2ik_0 n \frac{\partial\phi}{\partial x} + \left(2k_0^2\, n^2 + ik_0 \frac{\partial n}{\partial x}\right)\phi = 0 \tag{18}$$

where k_0 denotes a constant wave number, and $n = k/k_0$ the index of refraction. By introducing a new coordinate system (ρ,σ)

$$\begin{aligned} \rho &= x \cos\alpha + y \sin\alpha \\ \sigma &= -x \sin\alpha + y \cos\alpha \end{aligned} \tag{19}$$

the bottom is defined through $h \equiv h(\rho)$,

$$n \equiv n(\rho); \quad \frac{\partial n}{\partial y} = \lambda \frac{\partial n}{\partial x}, \quad \lambda = \tan\alpha \tag{20}$$

where α is the angle of incidence, with $|\alpha| < \frac{\Pi}{2}$.
It will be assumed that $k(\rho)$ tends to the constant value k_0 (i.e. $n(\rho) \to 1$) as ρ tends to $-\infty$.

Now suppose a plane wave $e^{ik_0 x}$

is incident from $x = -\infty$. The field ϕ can then be written in the form

$$\phi = A(\rho) \exp [ik_0 (x - p \int n d\rho)] \tag{21}$$

with $p = \dfrac{\cos\alpha}{\sin^2\alpha}$. Upon inserting (21) into (18) one finds that A satisfies

$$A_{\rho\rho} + k_0^2 p^2 [n^2 + 2\lambda^2 n(n-1)] A = 0 \tag{22}$$

At $\rho = -\infty$, $A(\rho)$ is supposed to behave like

$$A(\rho) \simeq e^{ik_0 p\rho} + Re^{-ik_0 p\rho} \tag{23}$$

where the constant R denotes a complex reflection-coefficient. At $\rho = +\infty$, $A(\rho)$ should satisfy a radiation condition, i.e. no incoming wave from $+\infty$.

The equation (22) is in general not explicitly solvable, and the solution must be represented by an approximation, which usually takes on an asymptotic form for high frequencies, in the limit $k_0 \to \infty$.

A point at which the coefficient of A in (22) vanishes is called a turning point, where the character of the solution changes from oscillatory to exponential. In the geometrical optics approximation of the problem, a caustic line is formed at these turning points. If there is no turning point, and $n + 2\lambda^2(n-1) > 0$, the asymptotic form has an oscillatory character with R=0, and can be found by the WKB-method (Cf. Langer (1937)). Let

$$A_p = |n^2 + 2\lambda^2 n(n-1)|^{-\frac{1}{4}} \tag{24}$$

$$F_p = x + \frac{1}{\lambda^2} \int_{-\infty}^{x} [\sqrt{|n^2 + 2\lambda^2 n(n-1)|} - n] dx \tag{25}$$

then the WKB-approximation to ϕ is given by

$$\phi_p = A_p \; e^{ik_0 F_p} \tag{26}$$

A similar analysis for equation (8) results in the geometrical optics approximation. Let

$$A_g = \left| n^2 + \lambda^2 \; (n^2 - 1) \right|^{-\frac{1}{4}} \tag{27}$$

$$F_g = x + \frac{1}{1+\lambda^2} \int_{-\infty}^{x} \left[\sqrt{\left| n^2 + \lambda^2 \; (n^2 - 1) \right|} \; -1 \right] dx \tag{28}$$

then the asymptotic form is given by

$$\phi_g = A_g \; e^{ik_0 F_g} \tag{29}$$

In the special case $\alpha = \lambda = 0$, both ϕ_p and ϕ_g agree (if the scaling factor $\sqrt{cc_g}$ is taken into account) with the classical shoaling formula for a progressive wave

$$\phi_g \simeq \frac{1}{\sqrt{c_g}} \exp \left[i \int_{x_0}^{x} k \; dx \right] . \tag{30}$$

ϕ_p and ϕ_g are compared in table I, for some values of λ and n. It is assumed, that the incident wave is starting in deep water, $k_0 = \omega^2/g$, and a correction factor $c_A = \dfrac{n}{\sqrt{1 + k_0 h (n^2 - 1)}}$ should be applied for the wave amplitudes, according to equation (2).

Table I Comparison of wave amplitudes A, wave numbers $|\nabla F|$ and wave directions θ (no turning point).

α	λ	n	$c_A^* A_p$	$c_A^* A_g$	$\|\nabla F_p\|$	$\|\nabla F_g\|$	θ_p	θ_g
45°	1	2	0.88	0.91	2.01	2	24.4°	24.3°
45°	1	3	1.01	1.07	3.03	3	31.5°	31.4°
63.4°	2	2	0.70	0.74	2.04	2	37.4°	36.9°
63.4°	2	3	0.79	0.85	3.12	3	46.8°	46.1°

The agreement is rather close, even for comparatively large values of λ.

Now suppose there is just one turning point at $\rho=\rho_0$, a point where the coefficient of $A(\rho)$ in (22) vanishes.

This will occur when n takes on the value n_p

$$n_p = \frac{2\lambda^2}{1+2\lambda^2} \tag{31}$$

In case of equation (8), the corresponding value is given by n_g

$$n_g = \frac{|\lambda|}{\sqrt{1+\lambda^2}} \tag{32}$$

An analysis of the turning point problem can be found in the article of Langer (1937):

for $\rho>\rho_0$, $A(\rho)$ takes on an exponentially decreasing form

$$A(\rho) \simeq |Q|^{-\frac{1}{2}} \exp\left[-\int_{\rho_0}^{\rho} |Q| d\rho\right] , \tag{33a}$$

and for $\rho < \rho_0$ an oscillatory form

$$A(\rho) \simeq 2|Q|^{-\frac{1}{2}} \cos[\int_{\rho}^{\rho_0} Q\, d\rho - \frac{\Pi}{4}] \quad , \tag{33b}$$

where $Q = k_0 p \sqrt{|n^2 + 2\lambda^2 n(n-1)|}$ (34)

Near the turning point, the asymptotic form of the solution can be represented by Airy functions.

Upon inserting (33) into (21) one obtains the asymptotic form ϕ_p. Here, only the behaviour of ϕ_p at $-\infty$ will be given explicitly

$$\phi_p\,(-\infty) \simeq \exp[ik_0 x] + R_p \exp[ik_0(x(\lambda^2-2) - 2y\lambda)/\lambda^2] \tag{35}$$

with $|R_p| = 1$, i.e. a fully reflected plane wave arises, with wave number k_p and wave direction α_p given by

$$k_p = k_0 \sqrt{1 + 4/\lambda^4} \quad , \qquad \tan\alpha_p = \frac{2\lambda}{2-\lambda^2} \tag{36}$$

For the geometrical optics solution, the corresponding formulas are given by

$$\phi_g\,(-\infty) \simeq \exp[ik_0 x] + R_g \exp[ik_0(x(\lambda^2-1) - 2y\lambda)/(1+\lambda^2)]$$

$$|R_g| = 1 \quad , \quad k_g = k_0 \quad , \quad \tan\alpha_g = \frac{2\lambda}{1-\lambda^2} \tag{37}$$

For some values of λ, a comparison is presented in table II.

Table II Comparison of reflected plane waves at a turning point.

α	λ	n_p	n_g	k_p/k_0	k_g/k_0	α_p	α_g
63.4°	2	0.89	0.89	1.1	1	-63.4°	-53.1°
45°	1	0.67	0.71	2.2	1	-117°	-90°

When λ approaches zero, the reflected waves in the solutions ϕ_p and ϕ_g deviate more and more from each other, as would be expected. Actually, the parabolic approximation is valid provided $\lambda^2 >> 1$, otherwise the coupling between the transmitted and reflected wave fields in equations (16) must be taken into account, if one wants to recover the complete wave field. For systematic corrections to the parabolic approximation, see Corones (1975).

5. Numerical solutions for the general case

The parabolic equation (17) may be solved by using finite-difference techniques. In this section, two alternatives will be dealt with. Assuming plane wave incidence

$$\phi = \Psi \; e^{ik_0 x} \tag{38}$$

then equation (18) yields for the complex potential function Ψ

$$\frac{\partial^2 \Psi}{\partial y^2} + 2\, i\, k_0 n \,\frac{\partial \Psi}{\partial x} + f\, \Psi = 0 \tag{39}$$

where $$f = k_0^2 n \,[2(n - 1) + \frac{i}{k_0}\frac{\partial \ell n(n)}{\partial x}] \tag{40}$$

A Crank-Nicholson finite-difference equation is used for the numerical solution to equation (39), cf. Richtmyer and Morton (1967): let a rectangular grid be given with grid spacings Δx and Δy, and let the approximation to $\Psi(\ell\Delta x,\, j\Delta y)$ be denoted by Ψ_j^ℓ , $\ell, j=0,1,2,\ldots$ The scheme I is then defined by

$$\begin{aligned}\Psi_{j+1}^{\ell+1} + \Psi_{j-1}^{\ell+1} + \Psi_{j+1}^{\ell} + \Psi_{j-1}^{\ell} + [-2 + (\Delta y)^2\, f_j^{\ell+\frac{1}{2}}].(\Psi_j^{\ell+1} + \Psi_j^{\ell})+ \\ + 4\, ik_0\frac{(\Delta\, y)^2}{\Delta x}\; n_j^{\ell+\frac{1}{2}}\; .(\Psi_j^{\ell+1} - \Psi_j^{\ell}) = 0\end{aligned} \tag{41}$$

where

$$n_j^{\ell+\frac{1}{2}} = (n_j^{\ell+1} + n_j^{\ell}) /2 \ , \quad f_j^{\ell+\frac{1}{2}} = k_0^2 \, n_j^{\ell+\frac{1}{2}} \, [2(n_j^{\ell+\frac{1}{2}} -1) + \frac{i}{k_0} \ln(\frac{n_j^{\ell+1}}{n_j^{\ell}})/\Delta x] \qquad (42)$$

and with initial condition

$$\Psi_j^{\circ} = 1 \ , \quad j = 0,1,2,\ldots \qquad (43)$$

and appropriate boundary conditions, to be specified later on. Another solution technique, which may be preferable, is based on the change of variable

$$\Psi = e^{\zeta} \qquad (44)$$

which turns (39) into

$$\frac{\partial^2\zeta}{\partial y^2} + \frac{\partial\zeta}{\partial y} \cdot \frac{\partial\zeta}{\partial y} + 2ik_0 n \frac{\partial\zeta}{\partial x} + f = 0 \qquad (45)$$

It may be expected that the solution ζ is a less rapidly varying function than Ψ, thus providing a more accurate approximation on the same grid.

However, the transformation (44) is singular at points where Ψ=0 (branch-points, or: amphidromic points), and a direct application of a scheme like (41) is not possible. In order to prevent the non-linear instabilities involved, it appears to be useful to add to the left hand side of (45) an artificial viscosity term of the form

$$- i\beta \frac{(\Delta y)^2}{4} \cdot \left|\frac{\partial^2\zeta}{\partial y^2}\right| \cdot \frac{\partial^2\zeta}{\partial y^2} \qquad (46)$$

where β is a dimensionless constant of the order of 1 (There is some resemblance with the Lax-Wendroff treatment of shocks, where an analogous dissipative term has been introduced to insure stability; see Richtmyer and Morton (1967), chapter 12).

Let

$$g_j^\ell = 2 - i\beta.\left|\ \zeta_{j+1}^\ell - 2\ \zeta_j^\ell + \zeta_{j-1}^\ell\right| \qquad (47)$$

then the scheme II is defined by

$$\zeta_{j+1}^{\ell+1}.[g_j^\ell + (\zeta_{j+1}^\ell - \zeta_{j-1}^\ell)] + \zeta_{j-1}^{\ell+1}\ [g_j^\ell - (\zeta_{j+1}^\ell - \zeta_{j-1}^\ell)] +$$

$$+ \zeta_j^{\ell+1}.[-\ 2\ g_j^\ell + 8\ ik_0\frac{(\Delta y)^2}{\Delta x} n_j^{\ell+\frac{1}{2}}] + 2\ (\zeta_{j+1}^\ell + \zeta_{j-1}^\ell) +$$

$$+ \zeta_j^\ell\ .\ [-\ 4 - 8\ ik_0\frac{(\Delta y)^2}{\Delta x} n_j^{\ell+\frac{1}{2}}\ ] + 4\ (\Delta y)^2.f_j^{\ell+\frac{1}{2}} = 0 \qquad (48)$$

with initial condition

$$\zeta_j^\circ = 0 \quad , \quad j = 0,1,2,... \qquad (49)$$

and appropriate boundary conditions.

Both schemes being implicit, a system of simultaneous linear equations has to be solved. For systems like (41) or (48) very efficient methods are available.

The rate of convergence will be exemplified in the next section, where numerical solutions are obtained for the case of a circular shoal.

6. Application to circular shoal

As an example, the propagation of an incident plane wave will be considered over a circular symmetric shoal with parabolic bottom profile. Calculations for this severe test case have been made by Berkhoff (1976), Bettess and Zienkiewicz (1977), Flokstra and Berkhoff (1977), and Ito and Tanimoto (1972), who additionally conducted some laboratory experiments.
The shoal is represented by the depth profile

$$\begin{aligned} h &= h_m + e_0 r^2 \qquad &&\text{for } r<R \, , \\ h &= h_0 &&\text{for } r \geqslant R \, , \end{aligned} \tag{50}$$

where $r^2 = (x - x_m)^2 + (y - y_m)^2$,

and $e_0 = (h_0 - h_m)/R^2$.

To be definite, short wave propagation is considered, and the assumptions (7) should apply, which amount to:

$$e_0 \ll \omega^2/g \tag{51}$$

This implies that the curvature of the bottom is much less than the wave number, regardless of the value of the minimum depth.
The value of the angular frequency ω follows from the dispersion relation (5): $\omega^2 = g\ k_0 \tanh\ (k_0 h_0)$; denoting the corresponding wavelength by $L_0 = 2\Pi/k_0$, the problem is then defined through the parameters h_m/R , h_0/R and L_0/R.
In order to specify the boundary conditions, it is useful to analyse the asymptotic character of the solution for large distance x (see appendix). The governing equation stands for the Schrödinger-equation of a free particle, which is represented by a one-dimensional

wave packet. The behaviour of this wave packet for large x is a well known problem in wave mechanics: the spreading of the packet increases linearly with the distance x, and the magnitude approaches zero, as $1/\sqrt{x}$.

It follows, that the required boundary conditions for schemes I and II, in case of a shoal, can be given by the undisturbed initial values of the solution, $\Psi=1$ and $\zeta=0$, provided these boundaries are taken sufficiently far away from the area of interest.

In this way, the artificial reflections which may occur at the boundaries, can be avoided.

Some calculations with the numerical schemes have been performed, for two configurations of the shoal:

- configuration I, defined through:

  $h_m/R = 0.0625 \quad ; h_0/R = 0.1875 \quad ; L_0/R = 0.5 \quad ;$

- configuration II, defined through:

  $h_m/R = 0.016 \quad ; h_0/R = 0.116 \quad ; L_0/R = 0.288 \quad .$

The parameter $e_0 g/\omega^2$ takes on the value 0.01 for configuration I, and the value 0.005 for configuration II, so the inequality (51) is valid in both cases.

The constant β in (46) is chosen to be 1, and the grid spacings have been varied according to

$\Delta y/\Delta x = \frac{1}{2}$; $\Delta x/L_0 = 1, \frac{1}{2}, \frac{1}{4}$ and 1/8 .

Configuration I has been studied by Ito and Tanimoto (1972), who use a finite difference timestep method, and by Flokstra and Berkhoff (1977), who use a finite element elliptic method. Table III demonstrates the agreement between the various methods for the maximum relative wave amplitudes, and gives some impression of the rate of convergence of the numerical schemes I and II.

A detailed view of the solution is given in figures 1-5, which show a comparison of wave amplitudes between the mentioned methods, contourlines of the amplitude and of the phase, and energy flux lines (wave orthogonals). A grid is used with 281 * 449 grid points; the centre of the shoal is located at $x_m = 33$, $y_m = 113$ (in grid units), and the radius of the shoal is $R=16\ \Delta x$.

Energy flux lines are defined through the energy streamfunction G:

$$G = k_0 y - \int_0^x A^2 \frac{\partial F}{\partial y} dx \quad , \tag{52}$$

Table III — Comparison of maximum wave amplitudes for a circular shoal.

$\Delta y/\Delta x = 1/2$	$\Delta x/L_0$	Configuration I Maximum A	Configuration I location x/L_0	Configuration II Maximum A	Configuration II location x/L_0
Scheme I	1	2.17	8.5	2.57	9.0
	1/2	2.08	7.0	3.18	7.0
	1/4	2.05	6.8	3.01	6.0
	1/8	2.05	6.6	2.96	5.7
Scheme II ($\beta = 1$)	1	2.10	7.0	2.85	6.0
	1/2	2.03	6.5	2.96	5.8
	1/4	2.04	6.6	2.92	5.7
	1/8	2.05	6.6	2.97	5.7
Ito and Tanimoto (1972)		2.1	6.3	-	-
Flokstra and Berkhoff (1977)		2.04	6.4	3.1	5.7
Bettess and Zienkiewicz (1977)		-	-	2.9	5.5

where amplitude A and phase F are given by $\phi = Ae^{iF}$.
(If the field ϕ satisfies the Helmholtz-equation (8), it follows that $\nabla F . \nabla G=0$, i.e. orthogonality of F and G, which provides another test of validity for the parabolic approximation).
Configuration II has been studied by Flokstra and Berkhoff (1977), and Bettess and Zienkiewicz (1977), using a finite element elliptic method. In figures 6 and 7, the relative wave amplitudes on the line of symmetry, $y=y_m$, are presented.
It appears that the minimum near the rear end of the shoal cannot be represented properly by the solution of scheme II. This is caused by the occurrence of branchpoints, for which A=0.
In the vicinity of such points, the phase is a multiple valued function, and the energy flux lines are closed. So, the application of scheme II then results in a smoothed solution, which has better convergence properties, and which is preferable when the accuracy requirements are not too high.

7. Summary and conclusions

For the propagation of periodic surface waves in shoaling water, a parabolic wave equation (18) has been derived, based on the splitting technique of Corones (1975). This method yields a pair of coupled equations for the transmitted and reflected fields, and the parabolic equation results from neglecting the reflected field.
In the case of an area with straight and parallel bottom contour-lines, the asymptotic form of the solution for high frequencies is compared with the geometrical optics approximation. There is a close agreement, if there is no caustic line.

In the presence of a caustic, there is a reasonable agreement provided the angle of incidence is close enough to 90°. Otherwise the coupling between the transmitted and reflected wave fields cannot be neglected, and systematic corrections should be applied, if one wants to recover the complete wave field. Finally, two numerical solution techniques are presented in the form of finite difference schemes, each based on a different form of the parabolic equation. As an example, wave propagation over a circular shoal is considered, where the geometrical optics approximation predicts a cusped caustic line. For two bottom configurations, the results are compared with similar calculations in literature, showing a reasonable agreement. Which solution technique is preferable depends upon the required accuracy and the available computer capacity.
The parabolic equation method may be applied to short wave propagation in large coastal areas of complex bottom topography.

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Appendix Motion and spreading of a wave packet

The asymptotic character of the wave field behind a shoal of finite extension, in water of constant depth, is governed in the parabolic approximation by equation (39), with n=1. Let $\Psi = 1 + \varepsilon$, $x' = k_0 x$, $y' = k_0 y$; then the disturbance ε satisfies the Schrödinger-equation (omitting'):

$$\frac{\partial^2 \varepsilon}{\partial y^2} + 2i \frac{\partial \varepsilon}{\partial x} = 0 \qquad \text{(A1)}$$

Assuming ε (and derivatives) sufficiently square integrable, the following quantities are defined:

- the norm of the wave packet

$$N = \int_{-\infty}^{\infty} |\varepsilon|^2 \, dy \qquad \text{(A2)}$$

- the norm of the derivative

$$M = \int_{-\infty}^{\infty} \left| \frac{\partial \varepsilon}{\partial y} \right|^2 dy \qquad \text{(A3)}$$

- the mean position of the packet

$$\langle y \rangle = \frac{1}{N} \int_{-\infty}^{\infty} y |\varepsilon|^2 \, dy \qquad \text{(A4)}$$

- the mean velocity

$$\langle v \rangle = \frac{d}{dx} \langle y \rangle \qquad \text{(A5)}$$

- the spreading

$$S = \sqrt{\langle (y - \langle y \rangle)^2 \rangle} = \sqrt{\langle y^2 \rangle - \langle y \rangle^2} \qquad \text{(A6)}$$

Using (A1) and integration by parts, it follows that N, M and $\langle v \rangle$ are constants of the motion:

$$\frac{dN}{dx} \equiv 0 \quad , \quad \frac{dM}{dx} \equiv 0 \quad , \quad \frac{d}{dx} \langle v \rangle \equiv 0 \tag{A7}$$

For the spreading S, one finds

$$\frac{d^2}{dx^2} S^2 = 2\left(\frac{M}{N} - \langle v \rangle^2\right) \tag{A8}$$

Integrating (A8) twice, one obtains the result, that for large x, S increases linearly:

$$S\ (x \rightarrow \infty) \simeq \sqrt{\frac{M}{N} - \langle v \rangle^2}\ .x \tag{A9}$$

It follows then from (A2) and (A9), that the magnitude $|\varepsilon|$ of the wave packet approaches zero, as $1/\sqrt{x}$.

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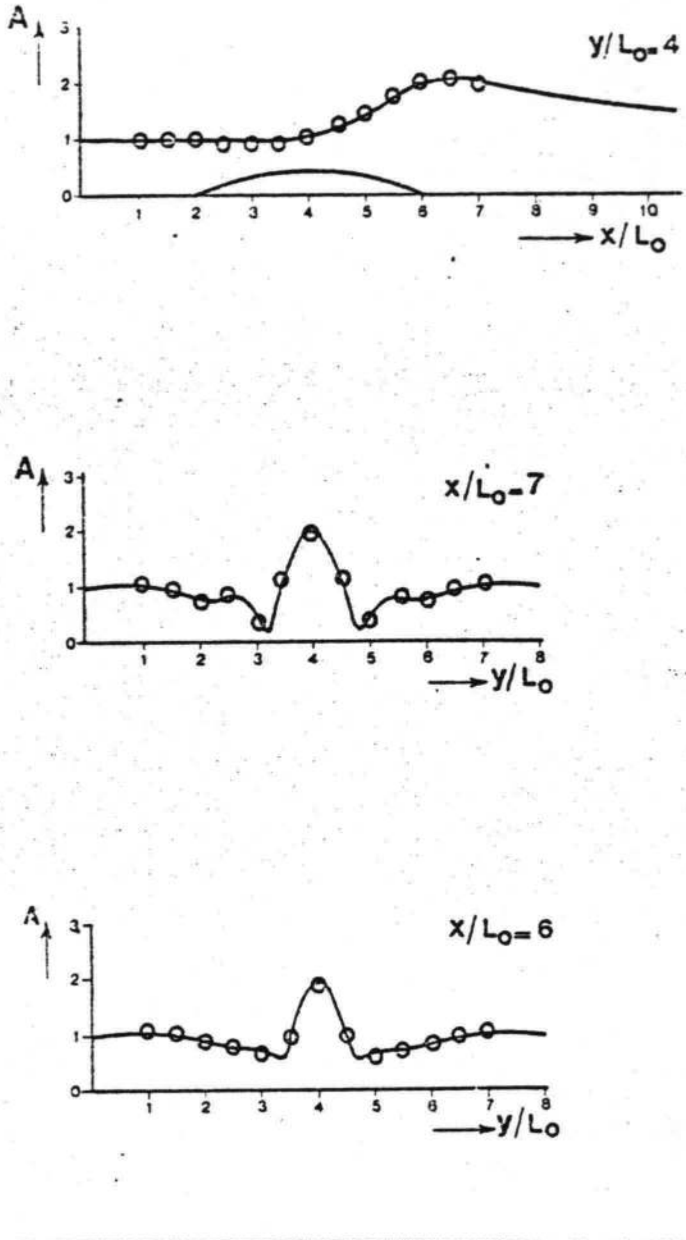

Fig. 1 Comparison of relative wave amplitudes for bottom configuration I, between results of the schemes I and II (continuous curves, $\Delta x/L_0=1/8$) and results of Ito and Tanimoto (1972) and of Flokstra and Berkhoff (1977) (circles).

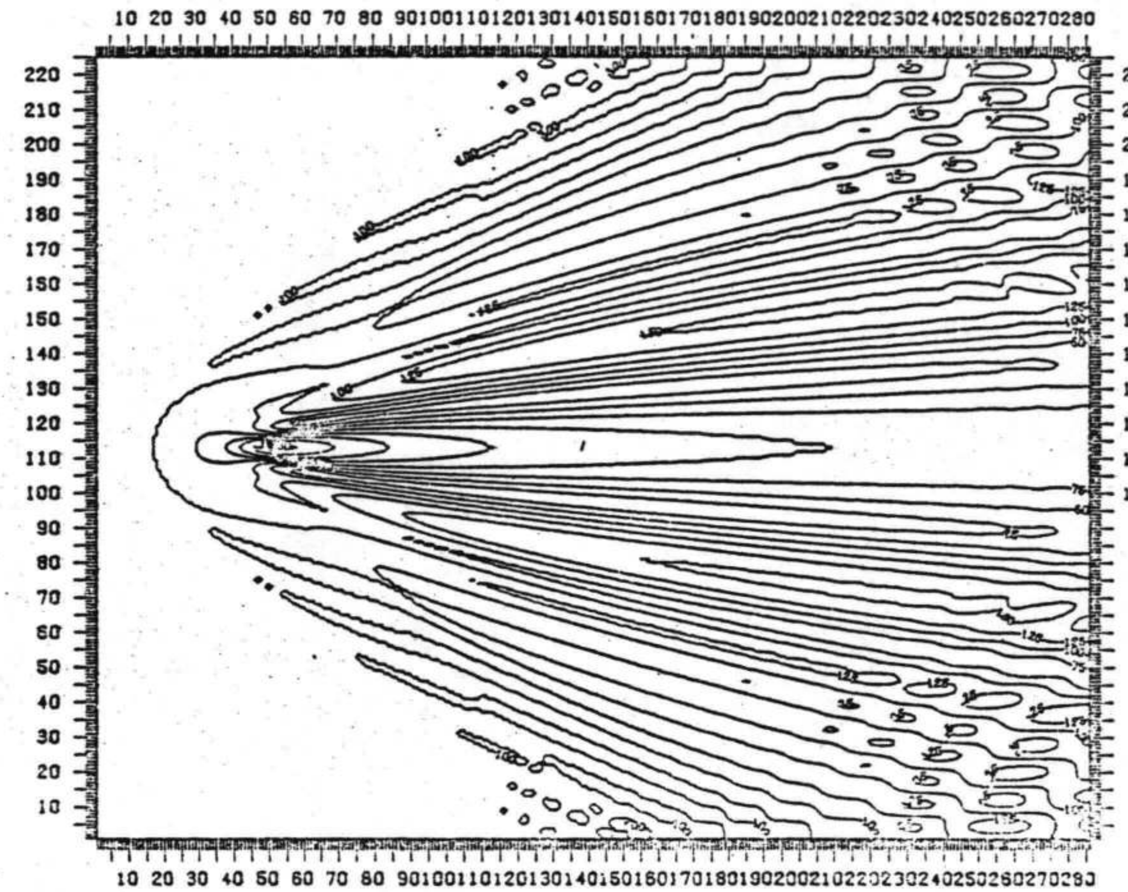

Fig. 2 Contourlines of the amplitude for configuration I, according to scheme I $(\Delta x/L_0 = 1/8)$.

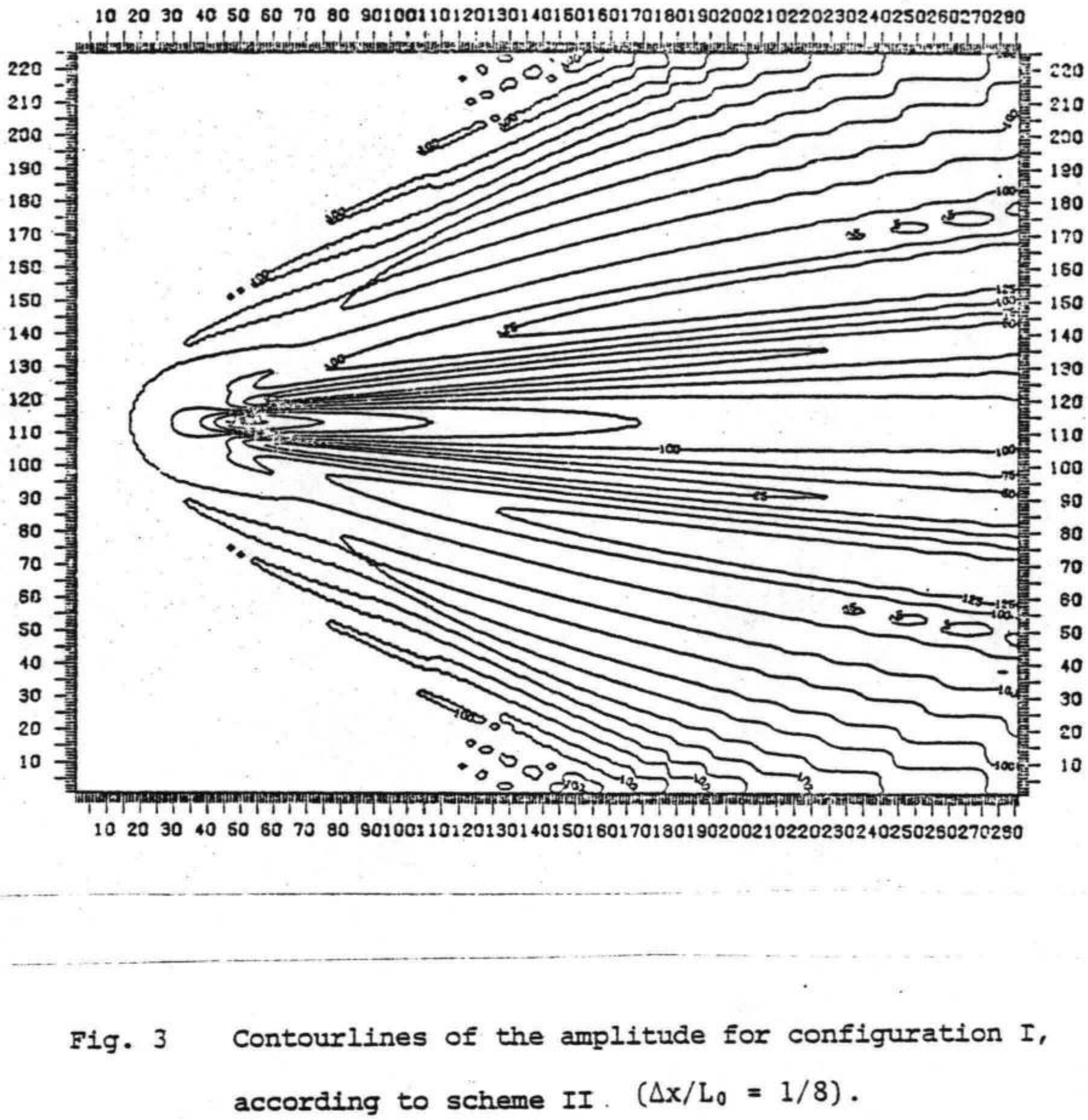

Fig. 3 Contourlines of the amplitude for configuration I, according to scheme II. $(\Delta x/L_0 = 1/8)$.

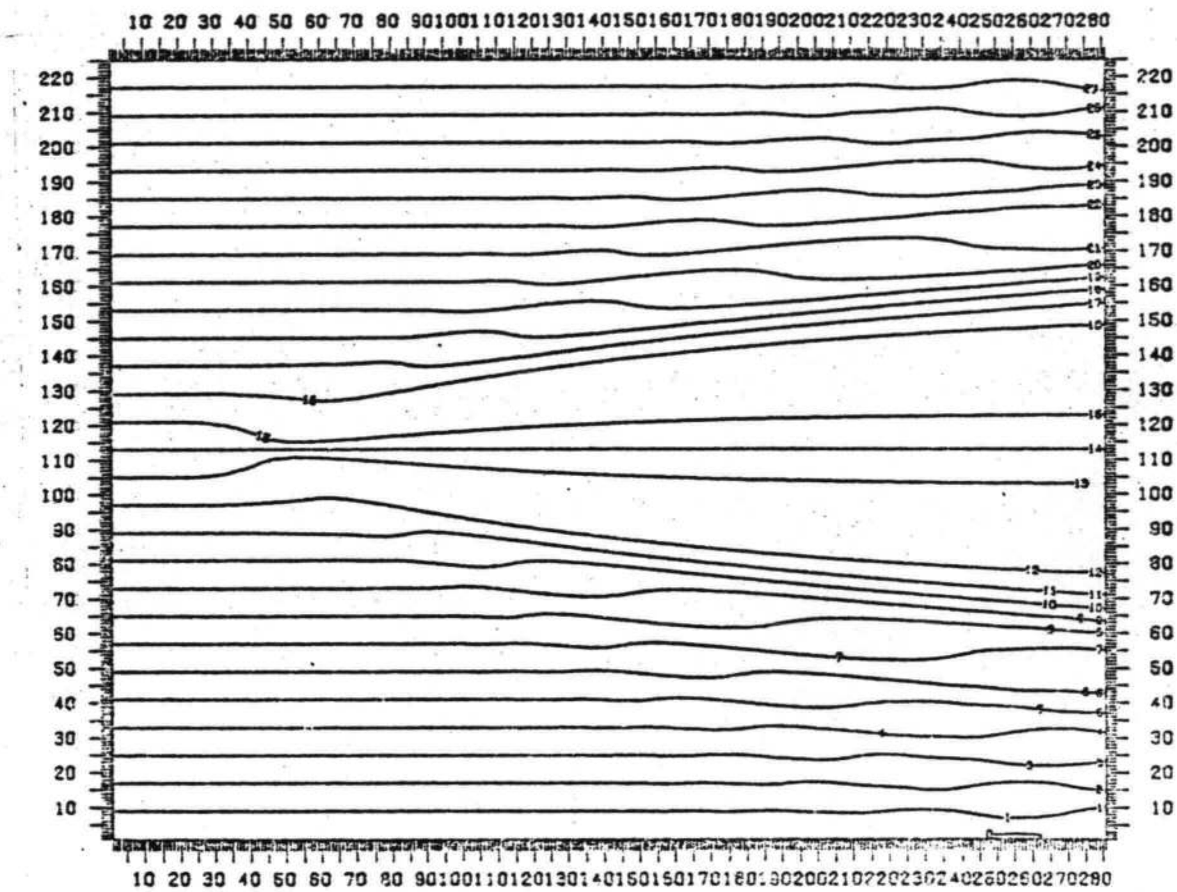

Fig. 4 Energy flux lines for configuration I, according to scheme I ($\Delta x/L_0 = 1/8$).

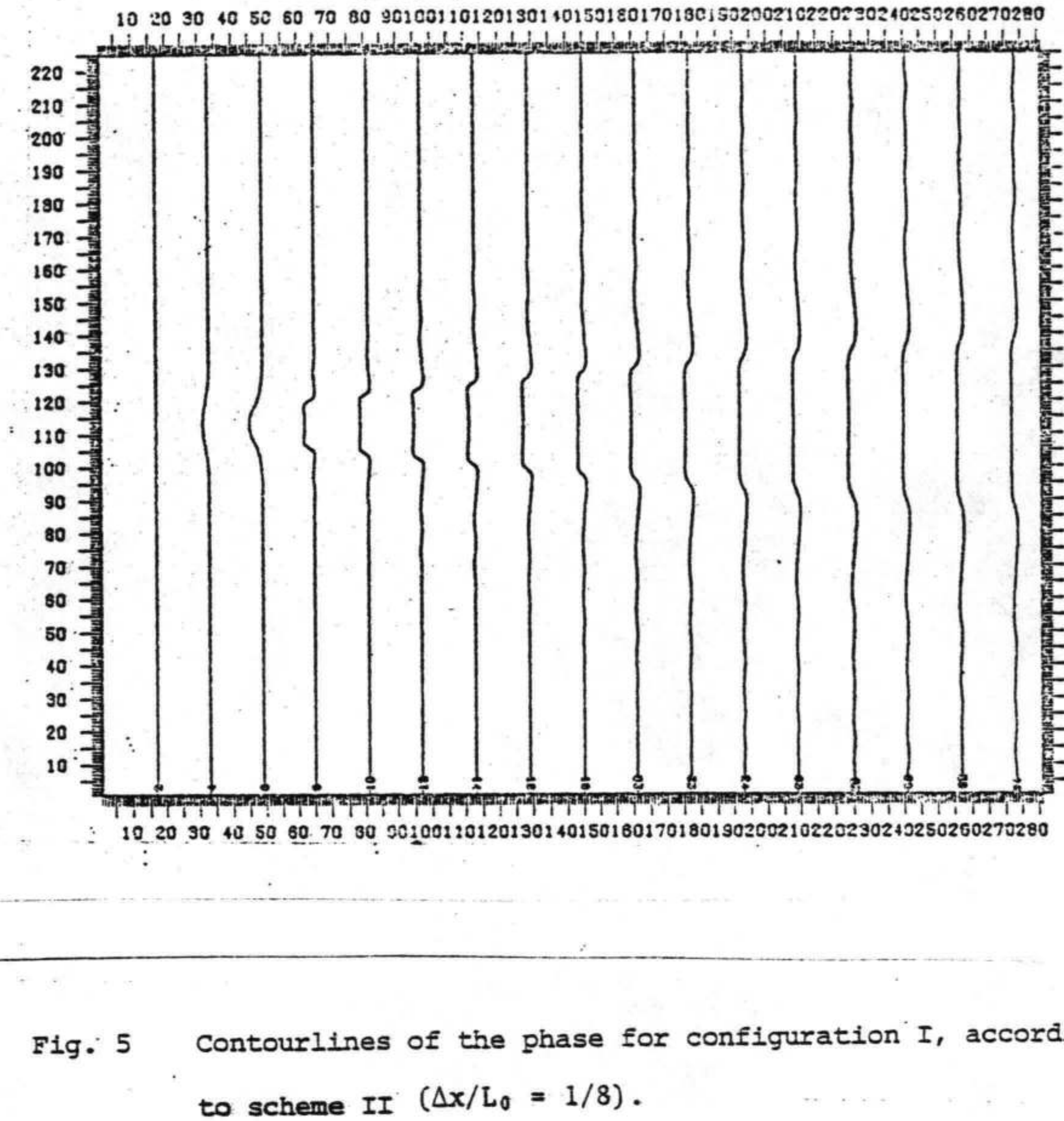

Fig. 5 Contourlines of the phase for configuration I, according to scheme II ($\Delta x/L_0 = 1/8$).

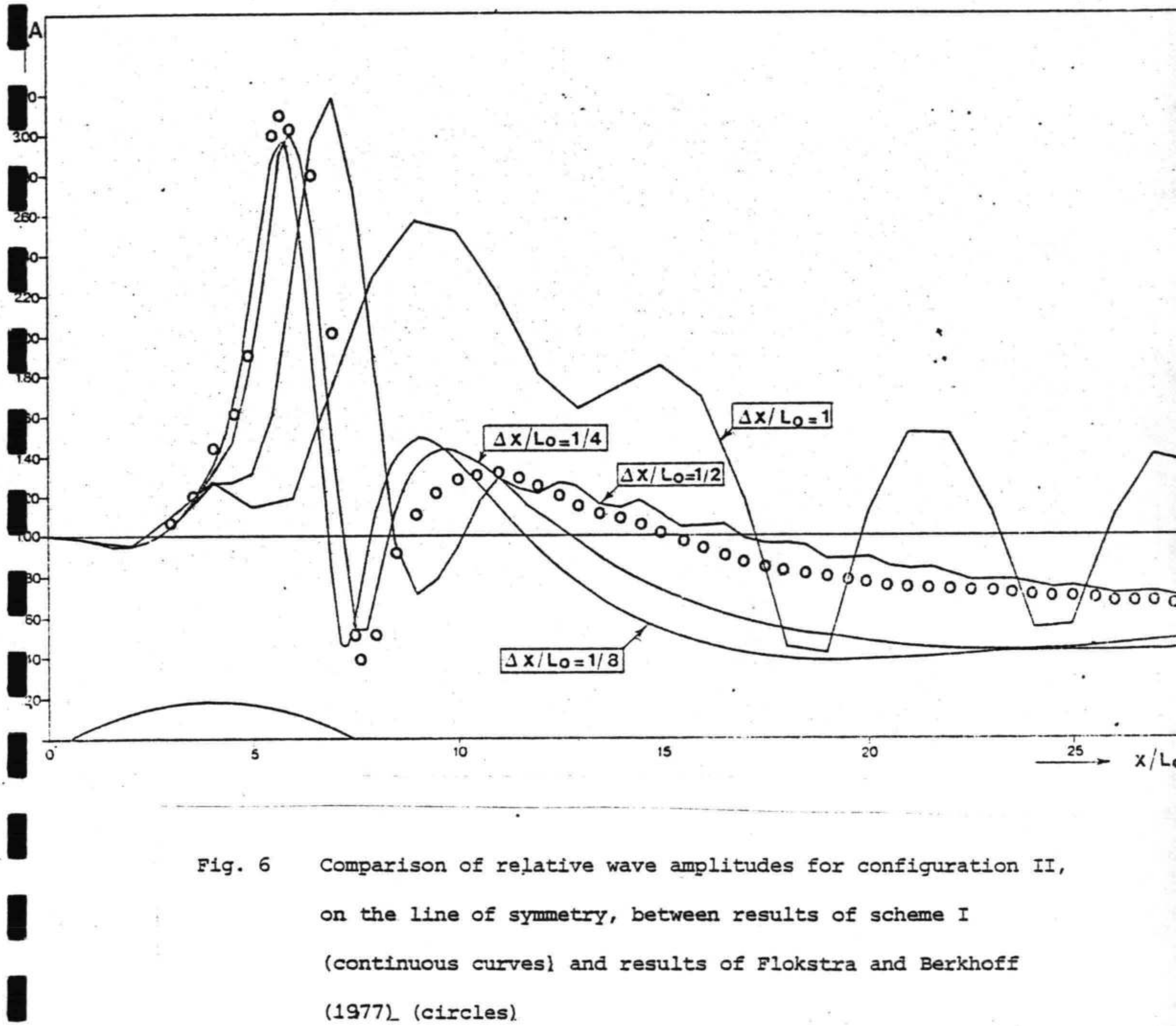

Fig. 6 Comparison of relative wave amplitudes for configuration II, on the line of symmetry, between results of scheme I (continuous curves) and results of Flokstra and Berkhoff (1977) (circles)

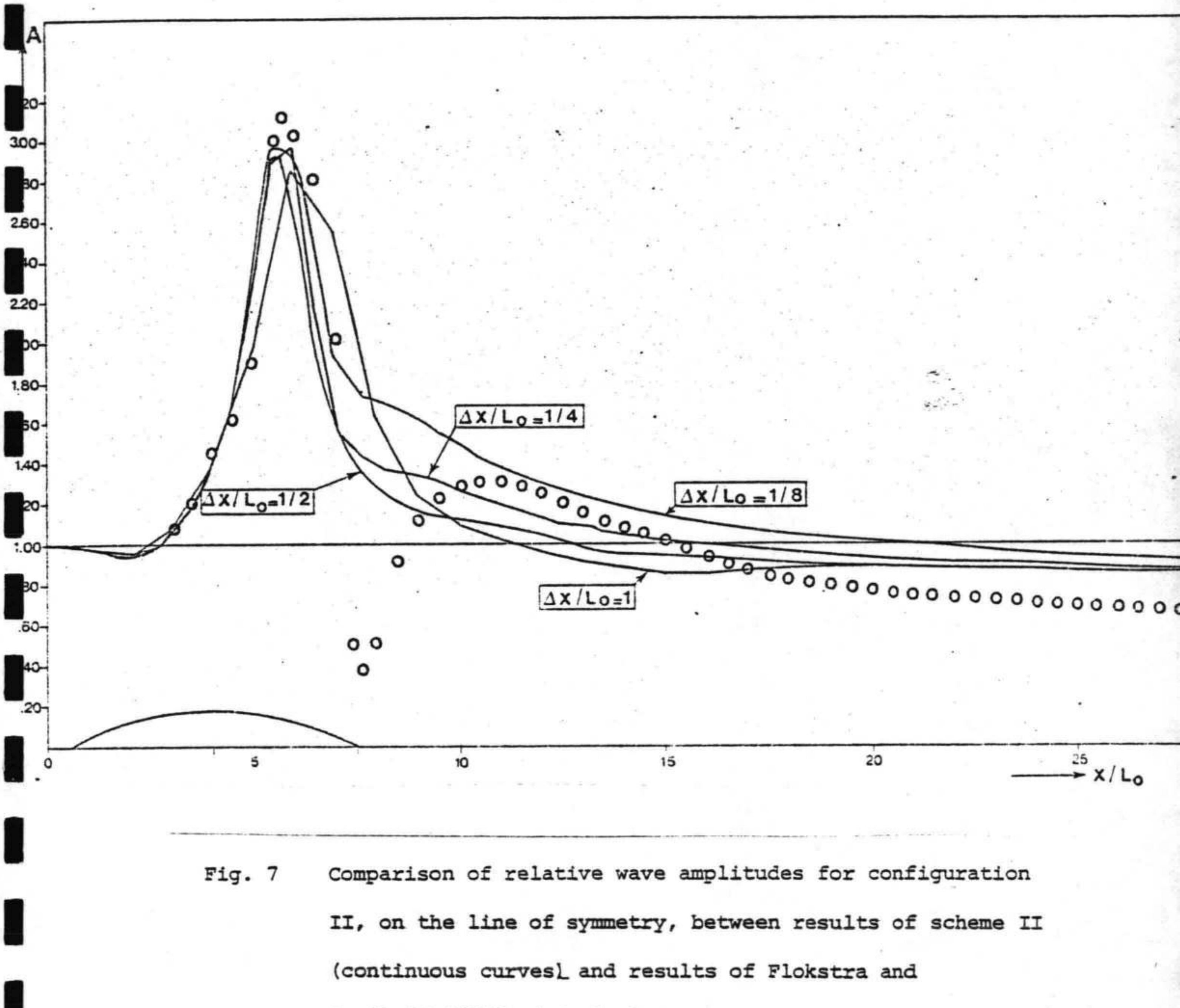

Fig. 7 Comparison of relative wave amplitudes for configuration II, on the line of symmetry, between results of scheme II (continuous curves) and results of Flokstra and Berkhoff (1977) (circles).