On the Parameterized and Approximation Hardness of Metric Dimension

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Abstract

The NP-hard METRIC DIMENSION problem is to decide for a given graph G and a positive integer k whether there is a vertex subset of size at most k that *separates* all vertex pairs in G. Herein, a vertex v separates a pair $\{u, w\}$ if the distance (length of a shortest path) between v and u is different from the distance of v and w. We give a polynomial-time computable reduction from the BIPARTITE DOMINATING SET problem to METRIC DIMENSION on maximum degree three graphs such that there is a one-to-one correspondence between the solution sets of both problems. There are two main consequences of this: First, it proves that METRIC DIMENSION on maximum degree three graphs is W[2]-complete with respect to the parameter k. This answers an open question concerning the parameterized complexity of METRIC DIMENSION posed by Díaz et al. [ESA'12] and already mentioned by Lokshtanov [Dagstuhl seminar, 2009]. Additionally, it implies that METRIC DIMENSION cannot be solved in $n^{o(k)}$ time, unless the assumption FPT $\neq$ W[1] fails. This proves that a trivial $n^{O(k)}$ algorithm is probably asymptotically optimal.

Second, as DOMINATING SET is inapproximable within $o(\log n)$, it follows that METRIC DIMENSION on maximum degree three graphs is also inapproximable by a factor of $o(\log n)$, unless NP = P. This strengthens the result of Hauptmann et al. [JDA'12] who proved APX-hardness on bounded-degree graphs.

1 Introduction

Given an undirected graph $G = (V, E)$ a *metric basis* of G is a vertex subset $L \subseteq V$ such that each pair of vertices $\{u, w\} \subseteq V$ is *separated* by L, meaning that there is at least one $v \in L$ such that $\text{dist}(v, u) \neq \text{dist}(v, w)$. Herein, "dist$(v, u)$" denotes the length of a shortest path between v and u. The corresponding METRIC DIMENSION problem has been independently introduced by Harary and Melter [11] and Slater [19]:

METRIC DIMENSION [10, GT61]
Input: An undirected graph $G = (V, E)$ and an integer $k \geq 1$.
Question: Is there a metric basis of size at most k?

The metric dimension of graphs (the size of a minimum-cardinality metric basis) finds applications in various areas including network discovery &

verification [3], metric geometry [11], robot navigation, coin weighing problems, connected joins in graphs, and strategies for the Mastermind game. We refer to Cáceres et al. [4], Hernando et al. [13], and Bailey and Cameron [2] for a more comprehensive list and a more complete bibliography on the extensive study on metric dimension.

There is a rich literature about the metric dimension of graphs, but little is known about the computational complexity of METRIC DIMENSION. It is known to be NP-hard and there is a linear-time algorithm for trees [14]. From a polynomial-time approximation point of view, it has been shown to admit a $2 \log n$-approximation [14] and that it cannot be approximated within $o(\ln n)$, unless P=NP [3]. Hauptmann et al. [12] showed that, unless NP $\subseteq$ DTIME($n^{\log \log n}$), there is no $(1 - \epsilon) \ln n$ approximation algorithm for any $\epsilon > 0$. Furthermore, they proved APX-hardness on bounded-

degree graphs. Díaz et al. [6] showed that METRIC DIMENSION remains NP-hard on bipartite graphs but becomes polynomial-time solvable on outerplanar graphs. Additionally, Epstein et al. [8] examined the complexity of the vertex-weighted variant of METRIC DIMENSION on various graph classes.

Our Contribution. We provide a polynomial-time computable reduction that maps an instance of BIPARTITE DOMINATING SET, consisting of a bipartite graph and an integer h, to an equivalent METRIC DIMENSION instance (G, k) with $k = h + 4$ and G having maximum degree three.

Since BIPARTITE DOMINATING SET is W[2]-hard [18], our reduction proves that METRIC DIMENSION is W[2]-hard with respect to k even on graphs with maximum degree three. Additionally, we prove W[2]-completeness. The question on the parameterized complexity of METRIC DIMENSION (on general graphs) was posed by Lokshtanov [16]; also Díaz et al. [6] pointed to this question. The W[2]-hardness of METRIC DIMENSION showes that, unless the widely believed conjecture $FPT \neq W[2]$ fails, METRIC DIMENSION is not *fixed-parameter tractable*, that is, it cannot be solved within $f(k) \cdot |G|^{O(1)}$ time for any computable function f. On the other hand, an algorithm that tests each size-k subset being a metric basis runs in $O(n^{k+2})$ time. However, our reduction together with the result that BIPARTITE DOMINATING SET cannot be solved in $n^{o(k)}$ time [5, 18], implies that METRIC DIMENSION on an n-vertex graph cannot be solved in $n^{o(k)}$ time, unless FPT=W[1]. Thus the trivial $n^{O(k)}$-algorithm is (probably) asymptotically optimal.

Furthermore, as DOMINATING SET cannot be approximated within a factor of $o(\log n)$ unless NP = P [1], it also follows that there cannot be an $o(\log n)$-factor approximation for METRIC DIMENSION. This strengthens the APX-hardness result for bounded-degree graphs [12] and it shows that the $2 \log n$-approximation on general graphs [14] is up to constant factors also optimal on bounded-degree graphs.

Preliminaries. A problem that is shown to be *W[1]- or W[2]-hard* is not fixed-parameter tractable, unless W[1] or W[2] is equal to the class FPT which consists of all fixed-parameter tractable problems. One can prove W[1]- or W[2]-hardness by means of a *parameterized reduction* from a W[1]- or W[2]-hard problem. This is a mapping of an instance (I, k) of a problem $\mathcal{A}$ in $h(k) \cdot |I|^{O(1)}$ time (for any computable h) into an instance (I', k') for $\mathcal{B}$ such that $(I, k) \in \mathcal{A} \Leftrightarrow (I', k') \in \mathcal{B}$ and $k' \leq g(k)$ for some g. Our reduction is indeed a *polynomial-time* computable parameterized reduction. We refer to the monographs of Downey and Fellows [7], Flum and Grohe [9], and Niedermeier [17] for a detailed introduction to parameterized complexity analysis.

We use standard graph-theoretic notations. All the graphs are undirected and unweighted without self-loops. For a graph $G = (V, E)$ with vertex set V and edge set E we set $n := |V|$. A path P in G is a sequence of vertices $v_1 - v_2 - \ldots - v_s$ such that for $1 \leq i < s$ all $\{v_i, v_{i+1}\} \in E$. If there is a unique shortest path between two vertices u and v, then we write just $u - v$ for this path without listing the intermediate vertices. We write $\mathrm{dist}(v, u)$ for the length of a shortest path between v and u. Moreover, we set $\mathrm{dist}(v_1, v_2, \ldots, v_i) = \mathrm{dist}(v_1, v_2) + \mathrm{dist}(v_2, v_3) + \ldots + \mathrm{dist}(v_{i-1}, v_i)$.

Organization. In the next section we describe our reduction and prove its correctness in Section 3. We proceed by proving W[2]-completeness (Section 4) and, finally, in Section 5 we prove the running time as well as the approximation lower bound. Some proofs are defered to the appendix.

2 Construction of the Reduction

In this section we give a reduction from the W[2]-complete BIPARTITE DOMINATING SET problem [18] to METRIC DIMENSION.

BIPARTITE DOMINATING SET
Input: A bipartite graph $G = (V_1 \cup V_2, E)$ and an integer $h \geq 1$.
Question: Is there a *dominating set* of size at most h, that is, a vertex subset $V' \subseteq V_1 \cup V_2$ such that $N[v] \cap V' \neq \emptyset$ for all $v \in V_1 \cup V_2$?

Since the distances from a vertex to all the vertices in a closed neighborhood of another vertex differ by at most three, $\log_3 \Delta$ is a lower bound on the metric dimension, where Δ is the maximum degree. Avoiding large degrees was the main obstacle in the reduction below.
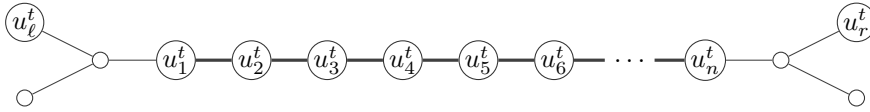
Figure 1: Schematic illustration of the top-line of the skeletal structure. Bold edges indicate y-paths.

Let $(G = (V_1 \cup V_2, E), h)$ be a BIPARTITE DOMINATING SET-instance and let $n := |V_1 \cup V_2|$. We set $V := V_1 \cup V_2$ and fix a numbering $V_1 = \{v_1, v_2, \ldots, v_s\}$ and $V_2 = \{v_{s+1}, \ldots, v_n\}$ such that for all $\{v_i, v_j\} \in E$ it holds that $j \geq i + 3$. (Clearly, increasing h by two, introducing two isolated vertices to V_1, and numbering them by v_{s+1} and v_{s+2} ensures this.) We construct an equivalent instance $(G' = (V', E'), k)$ of METRIC DIMENSION with $k := h + 4$.

We start with a high-level description of the graph G': It consists of a *skeletal structure* in which we insert a vertex-gadget for each vertex of G and an edge-gadget for each edge of G. Furthermore, there are four additional vertices in the skeletal structure that are forced to be in any metric basis and these four vertices separate all but n vertex pairs in each vertex-gadget. Then, choosing a vertex in a vertex-gadget separates all of the n vertex pairs in its own gadget plus all the pairs in the vertex gadgets that are "adjacent" by an edge-gadget to the chosen vertex-gadget.

Throughout the construction, several times we will connect two vertices $\{u, v\}$ by a so-called *y-path*, meaning that we insert a path of length y from u to v. In all cases we make sure that the y-path is the unique shortest path between the endpoints and thus $u - v$ denotes this y-path. We set $y := 10n^2$ as we will assume that $\frac{1}{4}y > 2n + 2$. Intuitively, y-paths can be viewed as edges of weight y.

We now describe the construction of G' in detail: First, the skeletal structure is formed by $2n$ vertices $u_1^t, \ldots, u_n^t$ and $u_1^b, \ldots, u_n^b$ where all consecutive vertex pairs $\{u_i^t, u_{i+1}^t\}$ and $\{u_i^b, u_{i+1}^b\}$ are connected by a y-path. We call the vertices $u_1^t, u_n^t, u_1^b, u_n^b$ *endpoints*. For each endpoint add a length-three path, a so-called P_3, and make the endpoint adjacent to the middle vertex. We call the first path $u_1^t - u_2^t - \ldots - u_n^t$ the *top-line* and the second path $u_1^b - u_2^b - \ldots - u_n^b$ the *bottom-line* both including the $P_3's$. Additionally, let u_ℓ^t be any degree-one vertex in the P_3 attached to u_1^t and correspondingly let u_r^t, u_ℓ^b, u_r^b be degree-

one vertices in the P_3's attached to u_n^t, u_1^b, and u_n^b, respectively (see Figure 1). (These are the four vertices that separate all but n pairs in each vertex gadget.)

For each vertex $v_i \in V$ we add the *vertex-gadget* g_i^V to G' (see Figure 2): Construct a cycle of length $2n + 2$ and call two vertices a_i^t, a_i^b on the cycle with distance exactly $n + 1$ the *anchors* of the vertex gadget. Connect the *top-anchor* a_i^t by a y-path to u_i^t and, symmetrically, connect the *bottom-anchor* a_i^b by a y-path to u_i^b. There are two paths, each consisting of n vertices between the anchors and we denote the vertices on these paths by $l_1^i, \ldots, l_n^i$ and $r_1^i, \ldots, r_n^i$, respectively. The *left-vertices* $l_1^i, \ldots, l_n^i$ remain degree-two vertices in G' whereas the *right-vertices* $r_1^i, \ldots, r_n^i$ will be used in the following to connect the edge-gadgets.

Finally, for all edges $\{v_i, v_j\} \in E$ with $i < j$ insert an edge-gadget $g_{i,j}^E$ into G' (see Figure 3): Add a path of length $(j - i + \frac{3}{2})y$ between the two right-vertices r_j^i and r_i^j. Denote with $w_1^{i,j}$ the vertex on the path having distance y to r_j^i and denote with $w_2^{i,j}$ the vertex on the path having distance y to r_i^j. Furthermore, denote with $u_{i,j,1}^t$ $(u_{i,j,2}^t)$ the vertex in the top-line that lies between u_i^t and u_j^t and has distance j to u_i^t (distance i to u_j^t). Then connect $w_1^{i,j}$ by a y-path to $u_{i,j,1}^t$ and also connect $w_2^{i,j}$ by a y-path to $u_{i,j,2}^t$. This completes the construction of G'.

3 Correctness of the Reduction

Let $(G = (V_1 \cup V_2, E), h)$ be an instance of BIPARTITE DOMINATING SET and let $(G' = (V', E'), k)$ with $k = h + 4$ be the corresponding instance of METRIC DIMENSION that is constructed by the reduction above. Clearly, the reduction is polynomial-time computable and thus it remains to show that G has a dominating set of size h iff G' has a metric basis
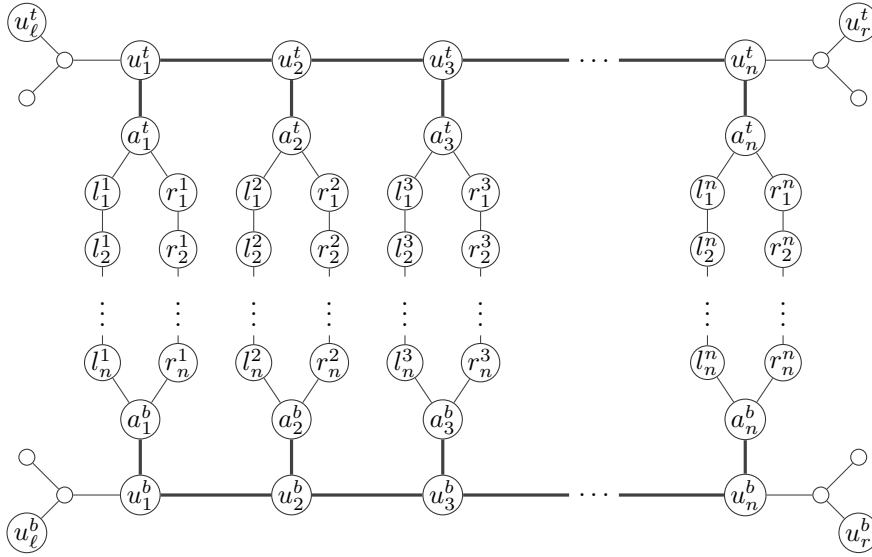
Figure 2: Schematic illustration of vertex-gadgets and their embedding into the skeletal structure. Bold edges indicate y-paths.

of size k. We first give an informal description of the basic ideas behind.

3.1 Basic Ideas and Intuition

First, observe that one has to choose at least one of the two degree-one vertices in the P_3's attached to each of the endpoints of the top- and bottom-line into any metric basis and a minimum-size metric basis would never take both. We shall show that $\{u_\ell^t, u_r^t, u_\ell^b, u_r^b\}$ separate each vertex pair in G' except the vertex pair $\{l_j^i, r_j^i\}$ for all $1 \leq i, j \leq n$. Towards this the main observation is that a shortest path from a vertex in the skeletal structure to a vertex that is either in a vertex gadget or also in the skeletal structure would never enter an edge-gadget. For example, traversing an edge-gadget $g_{i,j}^E$ by entering it at $u_{i,j,1}^t$ and leaving it at r_i^j gives a path of length $(|j - i| + \frac{3}{2})y$. However, the path $u_i^t - u_j^t - a_j^t - r_i^j$ that follows the top-line is of length at most $(|j - i| + 1)y + n$ and, thus, is shorter (recall that $\frac{1}{4}y > 2n + 2$). From this the separation of the vertices in the skeletal structure and most of the vertices in the vertex-gadgets can be deduced. The reason why $\{u_\ell^t, u_r^t, u_\ell^b, u_r^b\}$ cannot separate $\{l_j^i, r_j^i\}$ is that a shortest path starting in

one of them has to enter a vertex-gadget g_i^V always via the anchors $\{a_i^t, a_i^b\}$ and thus cannot distinguish between l_j^i and r_j^i.

The fact that $\{u_\ell^t, u_r^t, u_\ell^b, u_r^b\}$ separate each vertex pair consisting only of edge-gadget vertices is far from being obvious and proving it requires extensive case distinctions (see Lemma 2). Moreover, we prove that all vertices in a metric basis of G' except $\{u_\ell^t, u_r^t, u_\ell^b, u_r^b\}$ are chosen from the vertex-gadgets and that the corresponding vertices form a dominating set in G. Towards this it is crucial that the constant $\frac{3}{2}$ in the definition of edge-gadgets is between one and two: Clearly, taking r_1^i into a metric basis separates all pairs $\{l_j^i, r_j^i\}$ in its own gadget. The key point is that it separates also all pairs in g_j^V if the edge-gadget $g_{i,j}^E$ exists: A path from r_1^i to some r_s^j and also to l_s^j via traversing $g_{i,j}^E$ is of length at most $(|j - i| + \frac{3}{2})y + 3n$. The only "alternative path" from r_1^i to r_s^j is by leaving g_i^V via a_i^t following the top-line $u_i^t - u_j^t$, entering g_j^V via a_j^t and then taking the length-s path to r_s^j. In total this path has length at least $(|j - i| + 2)y$ and thus traversing $g_{i,j}^E$ is shorter. Hence, because $\frac{1}{4}y > 2n + 2$, the path traversing $g_{i,j}^E$ is a shortest path. The idea behind is that leaving and entering vertex-gadgets via the anchors costs $2y$ and traversing $g_{i,j}^E$ only costs $\frac{3}{2}y$
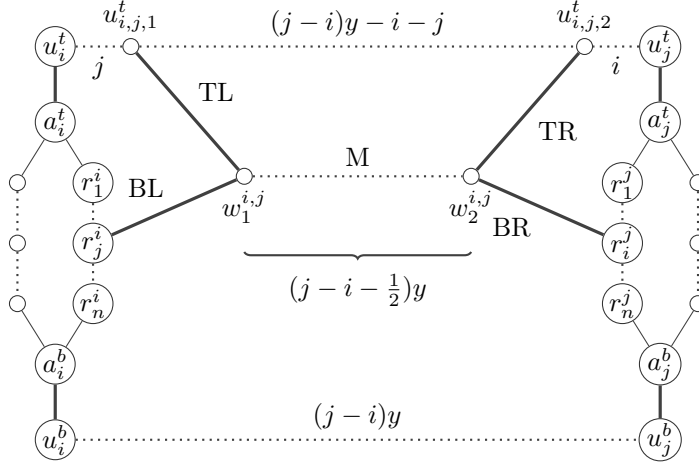
Figure 3: A schematic illustration of an edge-gadget $g_{i,j}^E$ for the edge $\{v_i, v_j\}$. Dotted edges indicate paths of length more than one. The concrete length of these paths is indicated by the labels next to the edge. Bold edges indicate y-paths. The edge-gadget consists of the five parts denoted by BL (bottom left), BR (bottom right), M (middle), TL (top right), and TR (top right).

more than the top- or (bottom-)line path $u_i^t - u_j^t$ ($u_i^b - u_j^b$). Moreover, r_1^i only separates pairs in "adjacent" vertex-gadgets since a shortest path starting in r_1^i never traverses two edge-gadgets. This would cause at least two times the additional cost of $\frac{3}{2}y$ whereas leaving and entering g_i^V to and from the top-line only costs $2y$.

We next give a formal proof of the correctness of the reduction.

3.2 General Observations and Additional Notation

We first introduce some additional notation for edge-gadgets.

Notation: For an edge-gadget $g_{i,j}^E$ the four vertices $\{r_j^i, r_i^j, u_{i,j,1}^t, u_{i,j,2}^t\}$ are called *entrance-vertices* (see Figure 3). Moreover, we partition $g_{i,j}^E$ into five parts: The y-path from $w_1^{i,j}$ to r_j^i is the BL- (bottom left) part, the TL- (top left) part is the y-path between $w_1^{i,j}$ and $u_{i,j,1}^t$, the TR- (top right) part is the y-path between $w_2^{i,j}$ and $u_{i,j,2}^t$, and the BR- (bottom right) part is the y-path between $w_2^{i,j}$ and r_i^j. Part M (middle) contains the remaining vertices, that is, the vertices between $w_1^{i,j}$ and $w_2^{i,j}$ including $w_1^{i,j}$ and $w_2^{i,j}$.

A path *enters* (*leaves*) an edge-gadget $g_{i,j}^E$ via a vertex v (u) if there are two consecutive vertices $v - u$ on it, such that both are contained in $g_{i,j}^E$ and v (u) is an entrance vertex of $g_{i,j}^E$. We say that an edge-gadget $g_{i,j}^E$ is *traversed* by a path P if it contains a subpath consisting only of vertices in $g_{i,j}^E$ that starts with entering $g_{i,j}^E$, contains the M-part, and ends with leaving $g_{i,j}^E$. Observe that without cycles there are only four different ways on how to traverse $g_{i,j}^E$ and each is of length $(|j - i| + \frac{3}{2})y$. The following observations are straightforward.

Observation 1. *A path that enters and afterwards leaves an edge-gadget without traversing it is more than $\frac{1}{4}y$ longer than a shortest path with the same endpoints.*

Proof. Let P be a path that starts with entering an edge-gadget $g_{i,j}^E$ and ends with leaving it but does not traverse it. If P does not contain any cycle, then it is equal either to $u_{i,j,1}^t - w_1^{i,j} - r_j^i$ or to $u_{i,j,2}^t - w_2^{i,j} - r_i^j$. Thus P is of length $2y$. However, the paths $u_{i,j,1}^t - u_i^t - a_i^t - r_j^i$ and $u_{i,j,2}^t - u_j^t - a_j^t - r_i^j$ are both of length at most $y + 2n$. Thus P is by $y - 2n > \frac{1}{4}y$ longer. $\square$

Next we prove that any shortest path traverses at most one edge-gadget.

Observation 2. *If a path is at most $\frac{1}{2}y$ longer than a shortest path with the same endpoints, then it traverses at most one edge-gadget.*

Proof. Assume that there is a path P in $G' = (V', E')$ that starts with traversing an edge-gadget $g_{i,j}^E$ and ends with traversing $g_{i',j'}^E$. Then P has length at least $(j-i+\frac{3}{2}+j'-i'+\frac{3}{2})y$. We assume that it starts traversing $g_{i,j}^E$ either in r_j^i or in $u_{i,j,1}^t$ (the other entrance vertices are completely symmetric). Then P ends either in $r_{j'}^{i'}$ or $u_{i',j',1}^t$. However, the path $r_j^i - a_i^t - u_i^t - \ldots - u_{i'}^t$ and also $u_{i,j,1}^t - u_{i'}^t$ both have length at most $j+(1+i'-i)y$ and $u_{i'}^t$ has distance at most $2y$ to $u_{i',j',1}^t$ and $r_{j'}^{i'}$. Hence, there is path that avoids traversing $g_{i,j}^E$ with distance at most $j+(1+i'-i)y < (j-i+\frac{3}{2}+j'-i'+\frac{3}{2})y$, implying that P is more than $\frac{1}{2}y$ longer than a shortest path. $\square$

We next prove that for all vertices in G' except for the vertices contained in an edge-gadget it holds that a shortest path to a vertex on the top- or bottom-line never contains a vertex in an edge-gadget.

Lemma 1. *In G' the following holds:*

i) *For all $1 \le i < j \le n$ the path along the top-line (bottom-line) is a shortest path from u_i^t to u_j^t (u_i^b to u_j^b). It has length $(j-i)y$.*

ii) *For all $1 \le i,j \le n$ the following paths of length $(|j-i|+2)y+n+1$ are for all $\min\{i,j\} \le s \le \max\{i,j\}$ the only shortest paths between u_i^t and u_j^b:*

$$u_i^t - u_{i+1}^t - \ldots - u_s^t - a_s^t - r_1^s - r_2^s - \ldots$$
$$-r_n^s - a_s^b - u_s^b - u_{s+1}^b - u_j^b$$
$$u_i^t - u_{i+1}^t - \ldots - u_s^t - a_s^t - l_1^s - l_2^s - \ldots$$
$$-l_n^s - a_s^b - u_s^b - u_{s+1}^b - u_j^b.$$

Proof. [Proof of i):] We prove the claim for $i=1$ and $j=n$ on the top-line. As the vertices for all other choices of i and j also lie on the top-line, this implies the correctness in all other cases. Lemma 1(i) can be analogously proven for the bottom-line.

Assume that there is a shortest path P from u_1^t to u_n^t that does not follow the top-line. We first show that P traverses at least one edge-gadget: Because the distances on the top- and the bottom-line are completely symmetric, a shortest path never starts on the top-line enters at some point the bottom-line and then later on re-enters the top-line. Hence, when leaving the top-line a shortest path enters an edge-gadget and, by Observation 1, it traverses it, say $g_{i,j}^E$.

Since P is a shortest path and traverses by Observation 2 only $g_{i,j}^E$, it follows that P enters $g_{i,j}^E$ via $u_{i,j,1}^t$ and leaves it via $u_{i,j,2}^t$. This subpath in P is of length $(j-i+\frac{3}{2})y$. Contradictorily, the path from $u_{i,j,1}^t$ to $u_{i,j,2}^t$ along the top-line is of length less than $(j-i)y$.

[Proof of ii):] As the other case can be proven completely analogously, we prove the claim only for the case $i \le j$. Furthermore, for any choice of $i \le s \le j$ the corresponding path has length $(|j-i|+2)y+n+1$. Thus it remains to prove that every other path is longer.

Suppose towards a contradiction that there is a path P' from u_i^t to u_j^b of length at most $(j-i+2)y+n+1$ that is different from the paths described in Lemma 1(ii). By Lemma 1(i) it holds that if there are two vertices in P' that lie on the top-line (bottom-line), then all vertices on the subpath between them also lie on the top-line (bottom-line, resp.). Thus, there are two vertices u_α^t and u_β^b that are both in P' but $u_{\alpha+1}^t, u_{\beta-1}^b \notin P'$. If on the subpath from u_α^t to u_β^b no edge-gadget is used, then $\beta = \alpha$ and the path P' is identical to the path described by Lemma 1(i) for $s = \alpha$.

Thus P' traverses exactly one edge-gadget (Observation 2), say $g_{\alpha,\beta}^E$. Hence the subpath in P' from u_α^t via $g_{\alpha,\beta}^E$ to u_β^b has length at least $(\beta - \alpha + \frac{3}{2} + 1)y$. Contradictorily, the path $u_\alpha^t - a_\alpha^t - a_\alpha^b - u_\alpha^b - u_{\alpha+1}^b - \ldots - u_\beta^b$ is of length at most $(2+\beta-\alpha)y+2n$ and thus is by at least $\frac{1}{4}y$ shorter. $\square$

Observation 1, Observation 2, and Lemma 1 together with the following proposition is all what we need to prove the correctness of our reduction.

Proposition 1. *The four vertices $\{u_\ell^t, u_r^t, u_\ell^b, u_r^b\}$ separate all vertices in G' except the pairs $\{l_j^i, r_j^i\}$ for all $1 \le i,j \le n$.*

The next subsection is dedicated to prove Proposition 1. Based on it, in Section 3.4 we give a formal correctness proof of the reduction.

3.3 Proof of Proposition 1

The major work in proving Proposition 1 is to show that the vertices in the edge-gadgets are separated. To this end, we first show which entrance vertices are used by shortest paths starting in $\{u_\ell^t, u_r^t, u_\ell^b, u_r^b\}$ and ending in an edge-gadget vertex.

Observation 3. *Let e be a vertex in an edge-gadget $g_{i,j}^E$. If e is contained in the* TL-*,* BL- *(*TR-*,* BR-*,) or* M-*part, then all shortest paths from u_ℓ^t or u_ℓ^b (u_r^t or u_r^b) to e enter $g_{i,j}^E$ either via $u_{i,j,1}^t$ ($u_{i,j,2}^t$) or r_j^i (r_i^j).*

Proof. We prove the claim for shortest paths starting in u_ℓ^t or u_ℓ^b. The other case is completely symmetric. A path from u_ℓ^t or from u_ℓ^b to e that neither enters $g_{i,j}^E$ via r_j^i nor via $u_{i,j,1}^t$ has to traverse either the TR or BR-part till $w_2^{i,j}$. With this requirement the shortest paths are $u_\ell^t - u_{j-1}^t - u_{i,j,2}^t - w_2^{i,j}$ with length $2 + jy - i$ and $u_\ell^b - u_j^b - a_j^b - r_i^j - w_2^{i,j}$ with length $2 + (j+1)y + (n-i+1)$. However, each of the paths $u_\ell^t - u_i^t - u_{i,j,1}^t - w_1^{i,j} - w_2^{i,j}$ with length $2 + j + (j - \frac{1}{2})y$ and $u_\ell^b - u_i^b - a_i^b - r_j^i - w_1^{i,j} - w_2^{i,j}$ with length $2 + (n - j + 1) + (j + \frac{1}{2})y$ are at least by $\frac{1}{4}y$ shorter, respectively. $\square$

Lemma 2. *The four vertices $\{u_\ell^t, u_r^t, u_\ell^b, u_r^b\}$ separate each vertex-pair consisting of two edge-gadget vertices.*

We now have all the ingredients to prove Proposition 1.

Proof of Proposition 1. We will show that for each pair of vertices in G' except for a pair $\{l_j^i, r_j^i\}$ for some $1 \le i, j \le n$ there is a vertex in $\{u_\ell^t, u_r^t, u_\ell^b, u_r^b\}$ that separates it. We have three groups of vertices in G' namely vertex-gadget-vertices, edge-gadget vertices, and vertices in the skeletal structure (consisting of top- and bottom-line). Next, we shall show that each of them is separated from all others.

Skeletal vertices: We prove that all vertices on the top- and bottom-line vertices are separated from all others. First, for each vertex pair from the skeletal structure by Lemma 1 there is a shortest path between two vertices from $\{u_\ell^t, u_r^t, u_\ell^b, u_r^b\}$ that contains both vertices, implying that they are separated.

Next, consider a vertex pair $\{u, v\}$ where u is contained in the skeletal structure and v is contained

in a vertex-gadget g_s^V. More specifically, let u be on the top-line between u_i^t and u_{i+1}^t (the proof is completely analogous for the bottom-line). If $i \le s$, then by Lemma 1(i) the following path is a shortest path $u_\ell^t - u_{i+1}^t - u_s^t - a_s^t - v - a_s^b - u_s^b - u_{s+1}^b - u_r^b$. Symmetrically, if $i > s$, then the following is a shortest path $u_\ell^b - u_s^b - a_s^b - v - a_s^t - u_s^t - u_i^t - u_r^t$. In both cases the vertex pair lie on a shortest path starting in $\{u_\ell^t, u_r^t, u_\ell^b, u_r^b\}$ and thus is separated.

Finally, by Lemma 1 for each vertex u in the skeletal structure there is a path from u_ℓ^t to u_r^b that contains u and there is no shortest path containing any edge-gadget vertex, implying that u is separated by u_ℓ^t or u_r^b from all edge-gadget vertices.

Vertex-gadget: By the argument above, vertex-gadget vertices are separated from vertices in the skeletal structure. Furthermore, for each vertex-gadget vertex v there is a shortest path from u_ℓ^t to u_r^b via v and no shortest path between them contains any edge-gadget vertex, implying that u_ℓ^t or u_r^b separate v from all edge-gadget vertices.

It remains to prove that any vertex-gadget vertex v is separated from any other vertex-gadget vertex v' except in the case that they correspond to a pair $\{l_j^i, r_j^i\}$. Consider first the subcase where v and v' are contained in the same vertex-gadget, say g_i^v. Then, by Lemma 1(ii) the following is a shortest path: $u_\ell^t - u_s^t - a_s^t - r_1^s - r_2^s - \ldots - r_n^s - a_s^b - u_s^b - u_r^b$. Clearly, the subpath $r_1^s - r_2^s - \ldots - r_n^s$ can be exchanged by $l_1^s - l_2^s - \ldots - l_n^s$. This implies that v and v' are separated.

Consider the subcase where v and v' are in different vertex-gadgets, say $v \in g_i^V$ and $v' \in g_j^V$ with $i < j$. By Lemma 1(ii) the following paths are shortest paths: $u_\ell^t - u_i^t - a_i^t - v$ and $u_\ell^t - u_j^t - a_j^t - v'$. Thus $\text{dist}(u_\ell^t, v) \le 2 + (i-1)y + y + n$ and $\text{dist}(u_\ell^t, v') > (j-1)y + y$, implying that $\text{dist}(u_\ell^t, v) \ne \text{dist}(u_\ell^t, v')$.

Edge-gadget: Because of the above considerations it is enough to prove that edge-gadget vertices are separated from other edge-gadget vertices. This is done by Lemma 2. $\square$

3.4 Correctness of the Reduction

Based on Observation 1, Observation 2, and Proposition 1 we next prove the correctness of our reduction (see Section 2). For the sake of readability the proof is separated into two implications.

Proposition 2. *If (G, h) is a yes-instance of* Bipartite Dominating Set, *then* (G', k) *is a yes-instance of* Metric Dimension.

Proof. For a yes-instance $(G = (V, E), h)$ with $V = \{v_1, \ldots, v_n\}$ of Bipartite Dominating Set denote by $K \subseteq V$ a dominating set of size at most h. We prove that the corresponding Metric Dimension instance $(G' = (V', E'), k)$ with $k = h + 4$ is also a yes-instance. More specifically, we prove that the set $L \subseteq V'$ that contains $\{u_\ell^t, u_r^t, u_\ell^b, u_r^b\}$ and for each vertex $v_i \in K$ the vertex r_1^i is a metric basis.

By Proposition 1 the vertices $\{u_\ell^t, u_r^t, u_\ell^b, u_r^b\} \subseteq L$ separate all pairs of vertices in V' except the vertex pair $\{l_j^i, r_j^i\}$ for all $1 \leq i, j \leq n$. Clearly, all pairs $\{l_j^i, r_j^i\}$ are separated if $r_1^i \in L$. Thus, consider the case where $r_1^i \notin L$. As K is a dominating set there is a vertex $r_1^\alpha \in L$ such that $\{v_i, v_\alpha\} \in E$, implying that there is an edge-gadget $g_{i,\alpha}^E$. Next, we prove that r_1^α separates the pair $\{l_j^i, r_j^i\}$ for all $1 \leq j \leq n$. This is done by proving that $P^l := r_1^\alpha - r_i^\alpha - w_2^{i,\alpha} - w_1^{i,\alpha} - r_\alpha^i - r_j^i$ (if $\alpha < i$ then interchange $w_1^{i,\alpha}$ and $w_2^{i,\alpha}$) is a shortest path and all other paths between r_1^α and r_j^i are more than $\frac{1}{4}y$ longer. Having proved this it follows that P^l extended by a shortest path within g_i^V is also a shortest path for l_j^i. Thus $\{l_j^i, r_j^i\}$ is separated by r_1^α.

The length of P^l is $(i-1)+(\alpha-1)+(|\alpha-i|+\frac{3}{2})y$. By Observation 2 each path that is at most $\frac{1}{2}y$ longer than a shortest path from r_1^α to r_j^i traverses at most one edge-gadget and each path that traverses an edge-gadget different from $g_{i,\alpha}^E$ is at least by $2y$ longer than P^l. Thus it remains to consider the paths from r_1^α to r_j^i that do not traverse any edge-gadget. There are only two of them, one following the top-line, $r_1^\alpha - a_\alpha^t - u_\alpha^t - u_i^t - a_i^t - r_j^i$, and the other following the bottom-line, $r_1^\alpha - a_\alpha^b - u_\alpha^b - u_i^b - a_i^b - r_j^i$. Both are of length more than $(|\alpha-i|+2)y$ and, thus, are at least $\frac{1}{4}y$ longer than P^l. Thus P^l is a shortest path, implying that r_1^α separates $\{l_j^i, r_j^i\}$. $\qquad\square$

Proposition 3. *If (G', k) is a yes-instance of* Metric Dimension, *then* (G, h) *is a yes-instance of* Bipartite Dominating Set.

Proof. Let $(G' = (V', E'), k)$ by a yes-instance of Metric Dimension where G' is constructed from the Bipartite Dominating Set instance $(G = (V, E), h)$ with $k = h + 4$ and $V = \{v_1, \ldots, v_n\}$. Furthermore, let L be a metric basis of G' of size at most k. As already argued, L contains at least one degree-one neighbor of each of the endpoints $\{u_1^t, u_n^t, u_1^b, u_n^b\}$ (otherwise the degree-one neighbors would not be separated). Then Proposition 1 proves that these degree-one neighbors separate all vertices in G' except the vertex pairs $\{l_j^i, r_j^i\}$ for all $1 \leq i, j \leq n$.

We now form a vertex subset $K \subseteq V$ and prove that it is a dominating set of size at most h: For each vertex $v \in L$ that is contained in a vertex-gadget g_i^V add $v_i \in V$ to K. Additionally, for each vertex $v \in L$ contained in an edge-gadget $g_{i,j}^E$ with $i < j$ add v_i to L if v is contained on the TL- or BL-part of $g_{i,j}^E$ and add v_j to L in all other cases.

We next prove that K is a dominating set for G. Suppose towards a contradiction that there is a vertex $v_i \in V$ that is not dominated by K. By definition of K none of the vertices in g_i^V is contained in the metric basis L. However, there is one vertex $u \in L$ that separates $\{l_1^i, r_1^i\}$. Denote by $\mathcal{P}^l$ the set of all shortest paths from u to l_1^i and by $\mathcal{P}^r$ the set of all shortest paths from u to r_1^i. Observe that l_1^i and r_1^i both have the same distance to a_i^t and a_i^b and that each path in $\mathcal{P}^l$ either contains a_i^t or a_i^b. Thus all paths in $\mathcal{P}^r$ neither contain a_i^t nor a_i^b, since otherwise l_1^i and r_1^i would not have been separated by u. Hence, each path in $\mathcal{P}^r$ enters g_i^V via an entrance vertex r_j^i of an edge-gadget $g_{i,j}^E$. If u is contained either directly in one of these edge-gadgets or it is contained in g_j^V, then by the construction of K this implies that either v_i or v_j is contained in K. This yields a contradiction since $\{v_i, v_j\} \in E$ and thus v_i is dominated.

Towards a contradiction, consider a shortest path in $P \in \mathcal{P}^r$ entering g_i^V via r_j^i but u is neither contained in $g_{i,j}^E$ nor in g_j^V. Clearly, by Observation 1 it follows that P traverses $g_{i,j}^E$. Thus, P enters $g_{i,j}^E$ via $u_{i,j,2}^t$ ($u_{i,j,1}^t$ if $i > j$) or r_i^j. However, by Lemma 1 the shortest path from $u_{i,j,2}^t$ ($u_{i,j,1}^t$) to r_j^i contains a_i^t (a_i^b), implying a contradiction in the first case. Hence, we can assume that P enters $g_{i,j}^E$ via r_i^j. By Observation 2 it traverses only $g_{i,j}^E$, implying that it enters g_j^V either via an anchor or via some r_α^j. If P enters g_j^V via the anchor a_j^t (a_j^b) this implies that the path from u to r_1^i contains u_j^t (u_j^b). However, by Lemma 1 the shortest path from u_j^t (u_j^b) to r_1^i contains a_i^t (a_i^b), yielding a contradiction. In the remaining case the path from u to r_1^i en-

ters g_j^V via r_α^j and since it traverses only $g_{i,j}^E$, this implies that u is contained in $g_{j,\alpha}^E$. In addition, by the construction of K it follows that u has distance greater than y to r_α^j and, hence, P contains $w_1^{j,\alpha}$ or $w_2^{j,\alpha}$. The subpath from $w_1^{j,\alpha}$ or from $w_2^{j,\alpha}$ to r_1^i is of length at least $(1 + |j - i| + \frac{3}{2})y$. However, either $w_1^{i,j}$ or $w_2^{i,j}$ has distance at most $y + n$ to u_j^t and $\text{dist}(u_j^t, r_1^i) = (|j - i| + 1)y + 1$, implying that P is not a shortest path. □

[Proposition 2](#) together with [Proposition 3](#) imply that our reduction given in [Section 2](#) is correct. Additionally, observe that the maximum degree in any graph constructed by our reduction is three. In the remaining part we discuss the computation lower bounds that are implied by it.

4 W[2]-Completeness

In the previous section we proved the correctness of our reduction which maps an instance (G, h) of BI-PARTITE DOMINATING SET into an instance (G', k) of METRIC DIMENSION with $k = h+4$. Since BIPARTITE DOMINATING SET is W[2]-hard with respect to h [18], this implies that METRIC DIMENSION is W[2]-hard with respect to k on maximum degree three graphs. Note that this classification is tight in the sense that METRIC DIMENSION is (trivially) polynomial-time solvable on graphs with maximum degree two. We prove in this section that METRIC DIMENSION is indeed W[2]-complete.

Theorem 1. METRIC DIMENSION *on graphs with bounded degree three is W[2]-complete with respect to the parameter size of a metric basis.*

Proof. The W[2]-hardness follows from the discussion above. Hence, it remains to show containment in W[2]. This is done by giving a parameterized reduction to the W[2]-complete RED-BLUE DOMINATING SET problem [7]: Given a bipartite graph $(R \cup B, E)$ and an integer $h \geq 1$ it is asked whether there is a size at most h vertex subset $D \subseteq R$ that dominates all vertices in B.

For an instance $(G = (V, E), k)$ of METRIC DI-MENSION we construct an equivalent RED-BLUE DOMINATING SET instance $(G' = (R \cup B, E'), k)$ as follows: First the vertex set B is formed by inserting for each vertex pair $\{u, w\} \subseteq V$ a vertex $\alpha_{u,w}$. Then R is a copy of V and there is an edge between

$v \in R$ and $\alpha_{u,w} \in B$ if $\text{dist}(v, u) \neq \text{dist}(v, w)$. It is straightforward to argue that there is a one-to-one correspondence between the vertices in a metric basis for G and a red-blue dominating set in G'. □

5 Running Time and Approximation Lower Bounds

We next show a running time as well as an approximation lower bound for METRIC DIMENSION.

Chen et al. [5] proved that DOMINATING SET (given an n-vertex graph, decide whether it has a size-h dominating set) cannot be solved in $n^{o(h)}$ time, unless FPT = W[1]. By the details of the reduction in [18] (there is a one-to-one correspondence between the solution sets) this also holds for BI-PARTITE DOMINATING SET. This implies together with the observation that the parameter k in our reduction (see [Section 2](#)) is linearly upper-bounded by the parameter h from the BIPARTITE DOMINAT-ING SET instance where we reduce from, the same running-time lower bound for METRIC DIMENSION.

Theorem 2. *Unless FPT = W[1],* METRIC DI-MENSION *cannot be solved in $n^{o(k)}$ time, even on maximum degree three graphs.*

Note that the lower bound provided by [Theorem 2](#) is asymptotically tight in the sense that a trivial brute-force algorithm that tests each size-k vertex subset whether it is a metric basis achieves a running time of $O(n^{k+2})$.

Additionally, observe that the proof of [Proposi-tion 3](#) also provides a one-to-one correspondence between a metric basis and a dominating set in the instance where we reduce from. Moreover, our reduction can be computed in polynomial time. The reduction from DOMINATING SET to BIPARTITE DOMINATING SET [18] also admits these two proper-ties. Thus, the result that DOMINATING SET cannot be approximated within $o(\log n)$, unless NP = P [1], transfers to METRIC DIMENSION.

Theorem 3. *Unless NP = P,* METRIC DIMEN-SION *on maximum degree three graphs cannot be approximated within a factor of $o(\log n)$.*

6 Conclusion

We have shown that METRIC DIMENSION is W[2]-complete even on graphs with maximum degree three. By modifying our construction appropriately we conjecture that it is possible to show that METRIC DIMENSION is W[2]-complete even on bipartite graphs with maximum degree three.

We performed a first step towards a systematic study of the parameterized complexity of METRIC DIMENSION. From our perspective, the most interesting questions that arise is whether METRIC DIMENSION is fixed-parameter tractable on planar graphs or with respect to the treewidth of the input graph. By simple observations on vertices with the same neighborhood, it is straightforward to argue that METRIC DIMENSION is fixed-parameter tractable with respect to the size of a vertex cover. This motivates a systematic study of "stronger parameterizations" [15], for instance the size of a feedback vertex set. Finally, we would like to mention the open question whether the $2^{o(n)}$ lower bound for DOMINATING SET (unless the exponential time hypothesis fails) can be transfered to METRIC DIMENSION.

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A Proofs

A.1 Proof 1 (Lemma 2)

Proof. Let u be a vertex in the edge-gadget $g_{i,j}^E$ with $i < j$, and let v be a vertex in the edge-gadget $g_{i',j'}^E$ with $i' < j'$. By our vertex numbering it follows that $i < j'$ and $i' < j$. We shall show that u and v are separated by u_1^t, u_n^t, u_1^b, or u_r^b, implying that they are also separated by the corresponding degree-one vertices $\{u_\ell^t, u_r^t, u_\ell^b, u_r^b\}$.

Recall that Lemma 1 provides the distance between any vertex on the top- or bottom-line to any other vertex either contained in a vertex-gadget or on top- or bottom-line. The following distances will be frequently used:

$$\text{dist}(r_j^i, u_1^t) = iy + j \tag{1}$$

$$\text{dist}(r_j^i, u_n^t) = (n - i + 1)y + j \tag{2}$$

$$\text{dist}(r_j^i, u_1^b) = iy + n - j + 1 \tag{3}$$

$$\text{dist}(r_j^i, u_n^b) = (n - i + 1)y + n - j + 1 \tag{4}$$

$$\text{dist}(u_{i,j,1}^t, u_1^t) = (i - 1)y + j \tag{5}$$

$$\text{dist}(u_{i,j,1}^t, u_n^t) = (n - i)y - j \tag{6}$$

$$\text{dist}(u_{i,j,1}^t, u_n^b) = (n - i + 2)y - j + n + 1 \tag{7}$$

$$\text{dist}(u_{i,j,2}^t, u_1^t) = (j - 1)y - i \tag{8}$$

$$\text{dist}(u_{i,j,2}^t, u_n^t) = (n - j)y + i \tag{9}$$

$$\text{dist}(u_{i,j,2}^t, u_n^b) = (n - j + 2)y + i + n + 1 \tag{10}$$

We prove Lemma 2 by several case distinctions. Therein, the following five claims are helpful to simplify the argumentation (the proofs are separate subsections in the appendix).

Claim 1: $\{u, v\}$ are separated if $i = i'$ and $j = j'$.

Claim 2: $\{u, v\}$ are separated if $u \in \text{TL} \cup \text{BL}$ and $i < i'$. Symmetrically, $\{u, v\}$ are separated if $u \in \text{TR} \cup \text{BR}$ and $j > j'$.

Claim 3: $\{u, v\}$ are separated if $i + j \neq i' + j'$ and either

i) $\text{dist}(u_{i,j,1}^t, u, u_{i,j,2}^t) = (j - i + \frac{3}{2})y$ and $\text{dist}(u_{i',j',1}^t, v, u_{i',j',2}^t) = (j' - i' + \frac{3}{2})y$, or

ii) $\text{dist}(r_j^i, u, r_i^j) = (j - i + \frac{3}{2})y$ and $\text{dist}(r_{j'}^{i'}, v, r_{i'}^{j'}) = (j' - i' + \frac{3}{2})y$.

Claim 4: $\{u, v\}$ are separated if $u \in \text{M}$ and $v \in \text{TL}' \cup \text{TR}'$.

Claim 5: $\{u, v\}$ are separated if $u \in \text{M}$ and $v \in \text{BL}' \cup \text{BR}'$.

We now prove Lemma 2 by a case distinction on how the indices i, i', j, and j' are related to each other. Without loss of generality, we assume that $i \leq i'$. Moreover, by Claim 1 either $i \neq i'$ or $j \neq j'$. We first prove the case with $j = j'$ and $i \neq i'$ (Case 1). The case where $i = i'$ and $j \neq j'$ is omitted because it can be proven completely analogously. Hence, the remaining cases are $i < i' < j < j'$ (Case 2) and $i < i' < j' < j$ (Case 3). Note that in all these cases, by Claim 2 we may assume that $u \notin \text{TL} \cup \text{BL}$.

Case 1 $i < i' < j = j'$:

If $u \in M$, then Claim 3, 4, and 5 prove that $\{u, v\}$ are separated. It remains to consider $u \in \text{TR} \cup \text{BR}$.

Subcase 1: $u \in \text{TR}$.

If $\text{dist}(u_n^t, u) = \text{dist}(u_n^t, v)$, then $\text{dist}(u_n^t, u) = \text{dist}(u_n^t, u_{i,j,2}^t, u) \overset{(9)}{<} (n - j)y + i + y$. Since $i < i'$ and it follows from Equations 2, 6, and 9 that $v \in \text{TR}'$. Thus, $\text{dist}(u, u_{i,j,2}^t) = i' - i + \text{dist}(v, u_{i',j',2}^t)$. Let $x = \text{dist}(u, u_{i,j,2}^t)$, then $\text{dist}(v, u_{i',j',2}^t) = x + i - i'$.

Subcase 1.1: $x < \frac{2}{3}y$.

Then $\text{dist}(u_1^t, u) = \text{dist}(u_1^t, u_{i,j,2}^t) + x$ and $\text{dist}(u_1^t, v) = \text{dist}(u_1^t, u_{i',j',2}^t) + x + i - i'$. Since $i < i'$ it follows from Equation 8 that $\text{dist}(u_1^t, u_{i,j,2}^t) > \text{dist}(u_1^t, u_{i',j',2}^t)$. Hence $\text{dist}(u_1^t, u) > \text{dist}(u_1^t, v)$.

Subcase 1.2: $x \geq \frac{2}{3}y$.

Then $\text{dist}(u, u_1^b) = \text{dist}(u, r_j^i, u_1^b) \overset{(3)}{=} iy + n - j + 1 + (j - i + \frac{3}{2})y - x = (j + \frac{3}{2})y + n - j + 1 - x$. Furthermore, $\text{dist}(v, u_1^b) = \text{dist}(v, r_{j'}^{i'}, u_1^b) \overset{(3)}{=} i'y + n - j' + 1 + (j' - i' + \frac{3}{2})y - x - i + i' = (j + \frac{3}{2})y + n - j + 1 - x - i + i'$. Since $i \neq i'$ it follows that u and v are separated.

Subcase 2: $u \in \text{BR}$.

This subcase is analogous to the previous one: Assume that $\text{dist}(u_n^b, u) = \text{dist}(u_n^b, v)$. Then $\text{dist}(u_n^b, u) = \text{dist}(u_n^b, r_i^j, u) \overset{(4)}{<} (n - j + 1)y + n - i + 1 + y$. Since $i < i'$ it follows from Equations 4, 7, and 10 that $v \in \text{BR}'$. Thus, $\text{dist}(u, r_i^j) = i - i' + \text{dist}(v, r_{i'}^{j'})$. Denoting with $x = \text{dist}(u, r_i^j)$ it follows that $\text{dist}(v, r_{i'}^{j'}) = x - i + i'$.

Subcase 2.1: $x < \frac{2}{3}y$.

Then $\text{dist}(u_n^t, u) = \text{dist}(u_n^t, r_i^j) + x$ and $\text{dist}(u_n^t, v) =$

$\text{dist}(u_n^t, r_{i'}^{j'}) + x - i + i'$. Since $i < i'$ it follows from Equation 2 that $\text{dist}(u_n^t, r_i^j) < \text{dist}(u_n^t, r_{i'}^{j'})$. Hence $\text{dist}(u_n^t, u) < \text{dist}(u_n^t, v)$.

Subcase 2.2: $x \geq \frac{2}{3}y$.

Then $\text{dist}(u, u_1^t) = \text{dist}(u, u_{i,j,1}^t, u_1^t) \overset{(5)}{=} (i-1)y + j + (j - i + \frac{3}{2})y - x = (j + \frac{1}{2})y + j - x$. Furthermore, $\text{dist}(v, u_1^t) = \text{dist}(v, u_{i',j',1}^t, u_1^t) \overset{(5)}{=} (i'-1)y + j' + (j' - i' + \frac{3}{2})y - x + i - i' = (j + \frac{1}{2})y + j - x + i - i'$. Since $i \neq i'$ it follows that u and v are separated.

Case 2 $i < i' < j < j'$:

If $u \in M$, then Claim 3, 4, and 5 prove that $\{u, v\}$ are separated. It remains to consider $u \in \text{TR} \cup \text{BR}$.

Subcase 1: $u \in \text{TR}$.

It follows from the Claims 2, 3, 4, and 5 that $v \in \text{TL}' \cup \text{BL}'$. If $\text{dist}(u_n^t, u) = \text{dist}(u_n^t, v)$, then $\text{dist}(u_n^t, u) \overset{(9)}{=} (n - j)y + i + x$ where $x = \text{dist}(u, u_{i,j,2}^t) < y$. It follows that $v \in \text{TL}'$ and $\text{dist}(u_n^t, v) = \text{dist}(u_n^t, u_{i',j',1}^t, v) \overset{(6)}{=} (n - i')y - j' + \text{dist}(u_{i',j',1}^t, v)$. Assuming $\text{dist}(u_n^t, u) = \text{dist}(u_n^t, v)$, we have $j = i' + 1$. Hence,

$$(n - j)y + i + x = (n - j + 1)y - j' + \text{dist}(u_{i',j',1}^t, v)$$
$$\text{dist}(u_{i',j',1}^t, v) = x + i + j' - y.$$

Since $x < y$ it follows that $\text{dist}(u_{i',j',1}^t, v) < i + j'$. Hence,

$$\text{dist}(u_1^t, v) = \text{dist}(u_1^t, u_{i',j',1}^t, v)$$
$$\overset{(5)}{=} (i' - 1)y + j' + x + i + j' - y$$
$$= (j - 3)y + 2j' + i + x < (j - 1)y - i$$
$$\overset{(8)}{=} \text{dist}(u_1^t, u_{i,j,2}^t) < \text{dist}(u_1^t, u).$$

Thus, u and v are separated.

Subcase 2: $u \in \text{BR}$.

Again, it remains to consider $v \in \text{TL}' \cup \text{BL}'$. Hence, $\text{dist}(u_n^b, u) = \text{dist}(u_n^b, r_i^j, u) \overset{(4)}{=} (n-j+1)y + n - i + 1 + x$ where $x = \text{dist}(u, u_{i,j,2}^t) < y$. It follows that $v \in \text{BL}'$ and $\text{dist}(u_n^b, v) = \text{dist}(u_n^b, r_{j'}^{i'}, v) \overset{(4)}{=} (n - i' + 1)y + n - j' + 1 + \text{dist}(r_{j'}^{i'}, v)$. Assuming $\text{dist}(u_n^b, u) = \text{dist}(u_n^b, v)$, we have $j = i' + 1$. Hence,

$$(n - j + 1)y + n - i + 1 + x = (n - j + 2)y + n$$
$$- j' + 1 + \text{dist}(r_{j'}^{i'}, v)$$
$$\text{dist}(r_{j'}^{i'}, v) = x - i + j' - y.$$

Since $x < y$ it follows that $\text{dist}(r_{j'}^{i'}, v) < j' - i$. Hence,

$$\text{dist}(u_1^b, v) = \text{dist}(u_1^b, r_{j'}^{i'}, v)$$
$$\overset{(3)}{=} i'y + n - j' + 1 + x - i + j' - y$$
$$= (j - 2)y + n - i + 1 + x$$
$$< jy + n - i + 1$$
$$\overset{(3)}{=} \text{dist}(u_1^b, r_i^j) < \text{dist}(u_1^b, u).$$

Thus, u and v are separated.

Case 3 $i < i' < j' < j$:

If $u \in \text{TR} \cup \text{BR}$, then since $j' < j$ by Claim 2 it follows that $\{u, v\}$ are separated. It thus remains the case where $u \in M$. If $v \in \text{TL}' \cup \text{TR}' \cup \text{BL}' \cup \text{BR}'$, then Claim 4 & 5 prove that $\{u, v\}$ are separated. Thus let $v \in M'$ and $i + j = i' + j'$ (otherwise they would be separated by Claim 2).

Assume that $\text{dist}(u, u_1^t) = \text{dist}(v, u_1^t)$ and let $x = \text{dist}(w_1^{i,j}, u)$. From $\text{dist}(u_1^t, w_1^{i',j'}) - \text{dist}(u_1^t, w_1^{i,j}) = (i'y + j') - (iy + j) = (i' - i)y + j' - j$ it follows that $\text{dist}(w_1^{i',j'}, v) = x - j' + j - (i' - i)y$.

From Equation 3 follows $\text{dist}(u_1^b, u) = \text{dist}(u_1^b, r_j^i) + y + x = iy + n - j + 1 + y + x$. Furthermore, it follows that

$$\text{dist}(u_1^b, v) = \text{dist}(u_1^b, r_{j'}^{i'}) + y + x - j' + j - (i' - i)y$$
$$= (i + 1)y + n + j - 2j' + 1 + x.$$

Thus, $\text{dist}(u_1^b, u) - \text{dist}(u_1^b, v) = (i+1)y + n - j + 1 + x - ((i+1)y + n + j - 2j' + 1 + x) = 2j' - 2j$. Since $j \neq j'$ it follows that u and v are separated. $\square$

A.2 Proof 2 (Claim 1 in Lemma 2)

Claim 1: It follows that u and v are contained in $g_{i,j}^E$ and we make a case distinction on in which part they lie.

Case 1: *At least one of $\{u, v\}$ is contained in the M-part.* Suppose u is contained in the M-part, then by Observation 3 a shortest path from u_ℓ^t (u_ℓ^b) to u contains the subpath $u_{i,j,1}^t - w_1^{i,j}$ ($r_j^i - w_1^{i,j}$) and from there traverses the M-part till u. Clearly, if v is also contained in the M-part, then the same holds for v and thus $\text{dist}(u_\ell^t, v) \neq \text{dist}(u_\ell^t, u)$. If v is either contained in the TL- or BL-part, then v is contained either on a shortest path from u_ℓ^t to u or on a shortest path from u_ℓ^b to u. The remaining

subcase where v is contained in the TR- or BR-part can be proven analogously by interchanging the role of $\{u_\ell^t, u_\ell^b\}$ by $\{u_r^t, u_r^b\}$.

Case 2: u, v *are both either in the* TL-, BL-, TR-, *or* BR-*part.* Observe that if u and v are both on the TL- (BL-, TR-, BR-) part, then they are separated by u_ℓ^t (u_ℓ^b, u_r^t, u_r^b) since there is a shortest path from u_ℓ^t or u_ℓ^b (u_r^t, u_r^b) to $w_1^{i,j}$ ($w_2^{i,j}$) that contains v and u.

Case 3: u *is contained in the* TL- *or* BL-*part and* v *is contained in the* TR- *or* BR-*part.* Recall that by our vertex numbering it holds that $j \geq i+3$. In this case, since u_i^t has distance less than $2y$ to all vertices on the TL- and BL-part, it follows that $\mathrm{dist}(u_1^t, u) < (i+1)y$. Additionally, a shortest path from u_1^t to v has to contain either $u_{i,j,2}^t$, r_i^j, or $w_2^{i,j}$. Since $\mathrm{dist}(u_1^t, u_{i,j,2}^t) \stackrel{(8)}{=} (j-1)y - i \geq (i+2)y$, $\mathrm{dist}(u_1^t, w_2^{i,j}) > (i-1+j-i+\frac{1}{2})y \geq (i+2+\frac{1}{2})y$, and $\mathrm{dist}(u_1^t, r_i^j) > \mathrm{dist}(u_1^t, u_{i,j,2}^t)$ it follows that u and v are separated by u_1^t.

Case 4: u *is contained in the* TL- *(*TR-*) part and* v *is contained in the* BL- *(*BR-*) part.* We prove the claim in case of $u \in$ TL and $v \in$ BL. The other case is completely symmetric. If the shortest path from v to u_1^t goes via $u_{i,j,1}^t$, then u lies on this shortest path and, thus, u and v are separated. If the shortest path from v to u_1^t goes via r_j^i, then observe that $\mathrm{dist}(u_1^t, v) > \mathrm{dist}(u_1^t, r_j^i) \stackrel{(1)}{=} iy + j$. Furthermore, from Equation 5 it follows that $\mathrm{dist}(u_1^t, u) \leq (i-1)y + j + y = iy + j$. $\qquad\square$

A.3 Proof 3 (Claim 2 in Lemma 2)

Claim 2: We prove that if $u \in$ TL $\cup$ BL and $i < i'$, then the vertex u_1^t or u_1^b separate $\{u, v\}$: It follows that $\mathrm{dist}(u_1^t, u, u_1^b)$

$$\leq \mathrm{dist}(u_1^t, u_{i,j,1}^t, u) + \mathrm{dist}(u_1^b, r_j^i, u)$$
$$\leq \mathrm{dist}(u_1^t, u_{i,j,1}^t) + \mathrm{dist}(u_1^b, r_j^i) + 2y$$
$$\stackrel{(3,5)}{=} (iy + n - j + 1) + ((i-1)y + j) + 2y$$
$$= (2i+1)y + n + 1.$$

Furthermore, from Observation 3 it follows that $\mathrm{dist}(u_1^t, v, u_1^b)$

$$> \mathrm{dist}(u_1^t, u_{i',j',1}^t) + \mathrm{dist}(u_1^b, r_{j'}^{i'})$$
$$\stackrel{(3,5)}{=} ((i'-1)y + j') + (i'y + n - j' + 1)$$
$$= (2i'-1)y + n + 1 \geq (2i+1)y + n + 1.$$

Hence, u_1^b and u_1^t separate $\{u, v\}$. The symmetric case can be proven analogously be interchanging the role of $\{u_1^t, u_1^b\}$ by $\{u_n^t, u_n^b\}$. $\qquad\square$

A.4 Proof 4 (Claim 3 in Lemma 2)

Claim 3: By the requirements it is ensured that a path from

i) u_1^t to u_n^t via u (v) enters $g_{i,j}^E$ ($g_{i',j'}^E$) via $u_{i,j,1}^t$ ($u_{i',j',1}^t$), traverses it, and leaves it via $u_{i,j,2}^t$ ($u_{i',j',2}^t$), or

ii) u_1^b to u_n^b via u (v) enters $g_{i,j}^E$ ($g_{i',j'}^E$) via r_j^i ($r_{j'}^{i'}$), traverses it, and leaves it via r_i^j ($r_{i'}^{j'}$).

In case of i) it holds that $\mathrm{dist}(u_1^t, u, u_n^t)$

$$= \mathrm{dist}(u_1^t, u_{i,j,1}^t, u) + \mathrm{dist}(u, u_{i,j,2}^t, u_n^t)$$
$$\stackrel{(5,9)}{=} ((i-1)y + j) + ((n-j)y + i) + (j - i + \frac{3}{2})y$$
$$= i + j + (n + \frac{3}{2} - 1)y.$$

Symmetrically, $\mathrm{dist}(u_1^t, v, u_n^t) = i' + j' + (n + \frac{3}{2} - 1)y$. Since $i + j \neq i' + j'$ it follows that u and v are separated by u_1^t and u_n^t.

Now, assume that ii) holds, then $\mathrm{dist}(u_1^b, u, u_n^b)$

$$= \mathrm{dist}(u_1^b, r_j^i, u) + \mathrm{dist}(u, r_i^j, u_n^b)$$
$$\stackrel{(3,4)}{=} iy + n - j + 1 + n - i + 1$$
$$\quad + (n - j + 1)y + (j - i + \frac{3}{2})y$$
$$= (n + 1 + \frac{3}{2})y + 2n - j - i + 2$$

and, symmetrically, $\mathrm{dist}(u_1^b, v, u_n^b) = (n + 1 + \frac{3}{2})y + 2n - j' - i' + 2$. Since $i + j \neq i' + j'$ it follows that u and v are separated. $\qquad\square$

A.5 Proof 5 (Claim 4 in Lemma 2)

Claim 4: We prove the claim for $v \in \mathrm{TL}'$. The case with $v \in \mathrm{TR}'$ follows from the symmetry of the construction. From Observation 3 follows $\mathrm{dist}(u_1^t, v) = \mathrm{dist}(u_1^t, u_{i',j',1}^t, v) \overset{(5)}{=} (i'-1)y + j' + x$ with $x = \mathrm{dist}(u_{i',j',1}^t, v) < y$. Furthermore, $\mathrm{dist}(u_1^t, u) = \mathrm{dist}(u_1^t, u_{i,j,1}^t, w_1^{i,j}, u) = (i-1)y + j + y + \mathrm{dist}(w_1^{i,j}, u) = iy + j + \mathrm{dist}(w_1^{i,j}, u)$. Assuming that $\mathrm{dist}(u_1^t, v) = \mathrm{dist}(u_1^t, u)$ (otherwise u and v are separated) we have $(i'-1)y + j' + x = iy + j + \mathrm{dist}(w_1^{i,j}, u)$ and, hence, $\mathrm{dist}(w_1^{i,j}, u) = (i'-i-1)y + j' - j + x$. Thus, $i' \geq i$. From this together with Observation 3 it follows that $\mathrm{dist}(u_1^b, u)$

$$= \mathrm{dist}(u_1^b, r_j^i, w_1^{i,j}, u)$$
$$\overset{(3)}{=} iy + n - j + 1 + y + (i'-i-1)y + j' - j + x$$
$$= i'y + n + j' - 2j + 1 + x.$$

Furthermore, it follows that $\mathrm{dist}(u_1^b, v) = \mathrm{dist}(u_1^b, r_{j'}^{i'}, v) \overset{(3)}{=} i'y + n - j' + 1 + \mathrm{dist}(r_{j'}^{i'}, v)$. By Observation 3 $\mathrm{dist}(r_{j'}^{i'}, v)$

$$= \min\{\mathrm{dist}(r_{j'}^{i'}, u_{i',j',1}^t, v), \mathrm{dist}(r_{j'}^{i'}, w_1^{i',j'}, v)\}$$
$$= \min\{y + 2j' + x, 2y - x\}.$$

Hence, $\mathrm{dist}(u_1^b, v) \overset{(3)}{=} i'y + n - j' + 1 + \min\{y + 2j' + x, 2y - x\}$. Assuming $\mathrm{dist}(u_1^b, u) = \mathrm{dist}(u_1^b, v)$ it follows that

$$i'y + n + j' - 2j + 1 + x$$
$$= i'y + n - j' + 1 + \min\{y + 2j' + x, 2y - x\}$$

and thus $2j' - 2j + x = \min\{y + 2j' + x, 2y - x\}$.
Since the case $2j' - 2j + x = y + x + 2j'$ yields a contradiction ($y = -2j$), it follows that $2j' - 2j = 2y - 2x$. However, $x < y$ implies $j' > j$ and thus $i \leq i' < j < j'$. Since $x = y - j' + j$, $\mathrm{dist}(u_{i',j',1}^t, v, u_{i',j',2}^t) = \mathrm{dist}(u_{i',j',1}^t, v) + \mathrm{dist}(v, w_1^{i',j'}, w_2^{i',j'}, u_{i',j',2}^t)$, implying that all preconditions of Claim 3 are fulfilled and thus u and v are separated. □

A.6 Proof 6 (Claim 5 in Lemma 2)

Claim 5: We prove the claim for $v \in \mathrm{BL}'$. The case with $v \in \mathrm{BR}'$ follows from the symmetry of

the construction. From Observation 3 it follows that $\mathrm{dist}(u_1^b, v) = \mathrm{dist}(u_1^b, r_{j'}^{i'}, v) \overset{(3)}{=} i'y + n - j' + 1 + x$ with $x = \mathrm{dist}(r_{j'}^{i'}, v) < y$. Furthermore, $\mathrm{dist}(u_1^b, u) = \mathrm{dist}(u_1^b, r_j^i, w_1^{i,j}, u) \overset{(3)}{=} iy + n - j + 1 + y + \mathrm{dist}(w_1^{i,j}, u) = (i+1)y + n - j + 1 + \mathrm{dist}(w_1^{i,j}, u)$. Assuming that $\mathrm{dist}(u_1^b, v) = \mathrm{dist}(u_1^b, u)$ (otherwise u and v are separated) we have $i'y + n - j' + 1 + x = (i+1)y + n - j + 1 + \mathrm{dist}(w_1^{i,j}, u)$ and, hence, $\mathrm{dist}(w_1^{i,j}, u) = (i'-i-1)y + j - j' + x$. Thus, $i' \geq i$. This together with Observation 3 implies $\mathrm{dist}(u_1^t, u)$

$$= \mathrm{dist}(u_1^t, u_{i,j,1}^t, w_1^{i,j}) + \mathrm{dist}(w_1^{i,j}, u)$$
$$\overset{(5)}{=} (i-1)y + j + y + (i'-i-1)y + j - j' + x$$
$$= (i'-1)y - j' + 2j + x.$$

In addition, $\mathrm{dist}(u_1^t, v) = \mathrm{dist}(u_1^t, u_{i'}^t, v) = (i'-1)y + \mathrm{dist}(u_{i'}^t, v)$ and

$$\mathrm{dist}(u_{i'}^t, v) = \min\{\mathrm{dist}(u_{i'}^t, w_1^{i',j'}, v), \mathrm{dist}(u_{i'}^t, r_{j'}^{i'}, v)\}$$
$$= y + j' + \min\{y - x, x\}.$$

Hence, $\mathrm{dist}(u_1^t, v) = i'y + j' + \min\{x, y - x\}$. Assuming $\mathrm{dist}(u_1^t, u) = \mathrm{dist}(u_1^t, v)$ (otherwise u and v are separated) implies

$$(i'-1)y - j' + 2j + x = i'y + j' + \min\{x, y - x\}$$
$$2j - 2j' + x = y + \min\{x, y - x\}.$$

This gives that either $2j - 2j' + x = y + x$ or $2j - 2j' + x = 2y - x$. In the first case this gives $y = 2j - 2j'$, contradicting $\frac{1}{4}y > 2n$. The second case gives $x = y + j' - j$. Since $x < y$ it follows that $j > j'$ and, thus, $i \leq i' < j' < j$. However, since $\mathrm{dist}(r_{j'}^{i'}, v) = x = y + j' - j$, it follows that

$$\mathrm{dist}(u_n^t, v) = \mathrm{dist}(u_1^b, u_{i',j',2}^t, w_2^{i',j'}, w_1^{i',j'}, v)$$
$$\overset{(9)}{=} (n-j')y + i' + (j' - i' + \frac{1}{2})y + j - j'$$
$$= (n - i' + \frac{1}{2})y + i' - j' + j.$$

Furthermore,

$$\begin{aligned}
\operatorname{dist}(u_n^t, u) =\ & \operatorname{dist}(u_n^b, u_{i,j,2}^t, w_2^{i,j}, w_1^{i,j}) - \operatorname{dist}(w_1^{i,j}, u) \\
\overset{(9)}{=}\ & (n-j)y + i + (j - i + \tfrac{1}{2})y \\
& - ((i' - i - 1)y + j - j' + x) \\
=\ & (n - i + \tfrac{1}{2})y + i \\
& - ((i' - i - 1)y + j - j' + x) \\
=\ & (n - i' + \tfrac{3}{2})y + i - j + j' - x \\
=\ & (n - i' + \tfrac{1}{2})y + i.
\end{aligned}$$

Hence,

$$\begin{aligned}
\operatorname{dist}(u_n^b, u) - \operatorname{dist}(u_n^b, v) &= i - (i' - j' + j) \\
&= i - i' + j' - j.
\end{aligned}$$

Recall that $i \leq i'$ and $j' < j$. Thus, $\operatorname{dist}(u_n^b, u) - \operatorname{dist}(u_n^b, v) < 0$ and, hence, u and v are separated by u_n^b. $\qquad\square$