ON THE $\partial\overline{\partial}$-LEMMA AND BOTT-CHERN COHOMOLOGY

DANIELE ANGELLA AND ADRIANO TOMASSINI

Abstract. On a compact complex manifold X, we prove a Frölicher-type inequality for Bott-Chern cohomology and we show that the equality holds if and only if X satisfies the $\partial\overline{\partial}$-Lemma.

Introduction

An important cohomological invariant for compact complex manifolds is provided by the Dolbeault cohomology. While in the compact Kähler case the Hodge decomposition theorem states that the Dolbeault cohomology groups give a decomposition of the de Rham cohomology, this holds no more true, in general, for non-Kähler manifolds.
Nevertheless, on a compact complex manifold X, the Hodge-Frölicher spectral sequence $E_1^{\bullet,\bullet} \simeq H^{\bullet,\bullet}_{\overline{\partial}}(X) \Rightarrow H^{\bullet}_{dR}(X;\mathbb{C})$, see [4], links Dolbeault cohomology to de Rham cohomology, giving in particular the *Frölicher inequality*:

$$\text{for every } k \in \mathbb{N}\,, \qquad \sum_{p+q=k} \dim_{\mathbb{C}} H^{p,q}_{\overline{\partial}}(X) \;\geq\; \dim_{\mathbb{C}} H^{k}_{dR}(X;\mathbb{C})\,.$$

Other important tools to study the geometry of compact complex (especially, non-Kähler) manifolds are the *Bott-Chern* and *Aeppli* cohomologies, that is,

$$H^{\bullet,\bullet}_{BC}(X) \;:=\; \frac{\ker\partial\cap\ker\overline{\partial}}{\operatorname{im}\partial\overline{\partial}} \qquad \text{and} \qquad H^{\bullet,\bullet}_{A}(X) \;:=\; \frac{\ker\partial\overline{\partial}}{\operatorname{im}\partial+\operatorname{im}\overline{\partial}}\,.$$

While they coincide with the Dolbeault cohomology in the Kähler case, they supply further informations on the complex structure of a non-Kähler manifold. Fixing a Hermitian metric, as a consequence of the Hodge theory, one has that they are finite-dimensional $\mathbb{C}$-vector spaces and that there is an isomorphism between Bott-Chern and Aeppli cohomologies.
These cohomology groups have been recently studied by J.-M. Bismut in the context of Chern characters (see [2]) and by L.-S. Tseng and S.-T. Yau in the framework of generalized geometry and type II string theory (see [8]).
A very special condition in complex geometry from the cohomological point of view is provided by the $\partial\overline{\partial}$*-Lemma*: namely, a compact complex manifold is said to satisfy the $\partial\overline{\partial}$-Lemma if every ∂-closed, $\overline{\partial}$-closed, d-exact complex form is $\partial\overline{\partial}$-exact. For example, compact Kähler manifolds or, more in general, manifolds in *class* $\mathcal{C}$ *of Fujiki* (that is, compact complex manifolds admitting a proper Kähler modification) satisfy the $\partial\overline{\partial}$-Lemma (see the paper by P. Deligne, Ph. Griffiths, J. Morgan and D. Sullivan, [3]).

In this note, we study the relations between Bott-Chern and Aeppli cohomologies and $\partial\overline{\partial}$-Lemma.
More precisely, we prove the following result, stating a Frölicher-type inequality also for Bott-Chern and Aeppli cohomologies and giving a characterization of the validity of the $\partial\overline{\partial}$-Lemma just in terms of the dimensions of $H^{\bullet,\bullet}_{BC}(X;\mathbb{C})$.

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Theorem (see Theorem A and Theorem B)**.** *Let X be a compact complex manifold. Then, for every $k \in \mathbb{N}$, the following inequality holds:*

$$\sum_{p+q=k} \left(\dim_{\mathbb{C}} H^{p,q}_{BC}(X) + \dim_{\mathbb{C}} H^{p,q}_{A}(X)\right) \;\geq\; 2\, \dim_{\mathbb{C}} H^{k}_{dR}(X;\mathbb{C}) \;. \tag{5}$$

Moreover, the equality in (5) *holds for every $k \in \mathbb{N}$ if and only if X satisfies the $\partial\overline{\partial}$-Lemma.*

As a consequence of the previous theorem, we obtain another proof of the stability of the $\partial\overline{\partial}$-Lemma under small deformations of the complex structure (see [10, 11]), see Corollary 2.7.

1. Preliminaries and notations

In this section, we recall some notions and results we need in the sequel.
Let X be a compact complex manifold of complex dimension n.

The *Bott-Chern cohomology* of X is the bi-graded algebra

$$H^{\bullet,\bullet}_{BC}(X) \;:=\; \frac{\ker\partial \cap \ker\overline{\partial}}{\operatorname{im}\partial\overline{\partial}}$$

and the *Aeppli cohomology* of X is the bi-graded $H^{\bullet,\bullet}_{BC}(X)$-module

$$H^{\bullet,\bullet}_{A}(X) \;:=\; \frac{\ker\partial\overline{\partial}}{\operatorname{im}\partial + \operatorname{im}\overline{\partial}} \;.$$

The identity induces the natural maps of (bi-)graded $\mathbb{C}$-vector spaces

$$\begin{tikzcd}
 & H^{\bullet,\bullet}_{BC}(X) \arrow[dl] \arrow[d] \arrow[dr] & \\
H^{\bullet,\bullet}_{\partial}(X) \arrow[dr] & H^{\bullet}_{dR}(X;\mathbb{C}) \arrow[d] & H^{\bullet,\bullet}_{\overline{\partial}}(X) \arrow[dl] \\
 & H^{\bullet,\bullet}_{A}(X) &
\end{tikzcd}$$

In general, the maps above are neither injective nor surjective. The compact complex manifold X is said to *satisfy the $\partial\overline{\partial}$-Lemma* if and only if

$$\ker\partial \cap \ker\overline{\partial} \cap \operatorname{im}\mathrm{d} \;=\; \operatorname{im}\partial\overline{\partial} \;,$$

that is, if and only if the map $H^{\bullet,\bullet}_{BC}(X) \to H^{\bullet}_{dR}(X;\mathbb{C})$ is injective. This turns out to be equivalent to say that all the maps above are isomorphisms, see [3, Remark 5.16]. As already reminded, compact Kähler manifolds and, more in general, compact complex manifolds in class $\mathcal{C}$ of Fujiki, [5], satisfy the $\partial\overline{\partial}$-Lemma, see [3, Corollary 5.23].

There is a Hodge theory also for Bott-Chern and Aeppli cohomologies, see [7]. More precisely, fixed a Hermitian metric on X, one has that

$$H^{\bullet,\bullet}_{BC}(X) \;\simeq\; \ker\tilde{\Delta}_{BC} \qquad \text{and} \qquad H^{\bullet,\bullet}_{A}(X) \;\simeq\; \ker\tilde{\Delta}_{A} \;,$$

where

$$\tilde{\Delta}_{BC} \;:=\; \left(\partial\overline{\partial}\right)\left(\partial\overline{\partial}\right)^{*} + \left(\partial\overline{\partial}\right)^{*}\left(\partial\overline{\partial}\right) + \left(\overline{\partial}^{*}\partial\right)\left(\overline{\partial}^{*}\partial\right)^{*} + \left(\overline{\partial}^{*}\partial\right)^{*}\left(\overline{\partial}^{*}\partial\right) + \overline{\partial}^{*}\overline{\partial} + \partial^{*}\partial$$

and

$$\tilde{\Delta}_{A} \;:=\; \partial\partial^{*} + \overline{\partial}\overline{\partial}^{*} + \left(\partial\overline{\partial}\right)^{*}\left(\partial\overline{\partial}\right) + \left(\partial\overline{\partial}\right)\left(\partial\overline{\partial}\right)^{*} + \left(\overline{\partial}\partial^{*}\right)^{*}\left(\overline{\partial}\partial^{*}\right) + \left(\overline{\partial}\partial^{*}\right)\left(\overline{\partial}\partial^{*}\right)^{*}$$

are 4-th order elliptic self-adjoint differential operators. In particular, one gets that

$$\dim_{\mathbb{C}} H^{\bullet,\bullet}_{\sharp}(X) \;<\; +\infty \qquad \text{for } \sharp \in \left\{\overline{\partial},\, \partial,\, BC,\, A\right\} .$$

By the definition of the Laplacians, it follows that

$$\begin{aligned} u \;\in\; \ker \tilde{\Delta}_{BC} \qquad &\Leftrightarrow \qquad \partial u \;=\; \overline{\partial} u \;=\; \left(\partial\overline{\partial}\right)^* u \;=\; 0 \\ &\Leftrightarrow \qquad \partial^* \left(*u\right) \;=\; \overline{\partial}^* \left(*u\right) \;=\; \partial\overline{\partial} \left(*u\right) \;=\; 0 \qquad \Leftrightarrow \qquad *u \;\in\; \ker \tilde{\Delta}_A \end{aligned}$$

and hence the duality

$$*\colon H^{p,q}_{BC}(X) \stackrel{\simeq}{\to} H^{n-q,n-p}_{A}(X) ,$$

for every $p, q \in \mathbb{N}$.

As a matter of notation, for every $p, q \in \mathbb{N}$, for every $k \in \mathbb{N}$ and for $\sharp \in \left\{\overline{\partial},\, \partial,\, BC,\, A\right\}$, we will denote

$$h^{p,q}_{\sharp} \;:=\; \dim_{\mathbb{C}} H^{p,q}_{\sharp}(X) \qquad \text{and} \qquad h^{k}_{\sharp} \;:=\; \sum_{p+q=k} h^{p,q}_{\sharp} ,$$

and we will denote the Betti numbers by

$$b_k \;:=\; \dim_{\mathbb{C}} H^{k}_{dR}(X;\mathbb{C}) .$$

Recall that, by conjugation and by the duality induced by the Hodge-$*$-operator associated to a given Hermitian metric, for every $p, q \in \mathbb{N}$ and for every $k \in \mathbb{N}$, one has the following equalities:

$$h^{p,q}_{BC} \;=\; h^{q,p}_{BC} \;=\; h^{n-p,n-q}_{A} \;=\; h^{n-q,n-p}_{A} \qquad \text{and} \qquad h^{p,q}_{\overline{\partial}} \;=\; h^{q,p}_{\partial} \;=\; h^{n-p,n-q}_{\overline{\partial}} \;=\; h^{n-q,n-p}_{\partial} ,$$

and therefore

$$h^{k}_{BC} \;=\; h^{2n-k}_{A} \qquad \text{and} \qquad h^{k}_{\overline{\partial}} \;=\; h^{k}_{\partial} \;=\; h^{2n-k}_{\overline{\partial}} \;=\; h^{2n-k}_{\partial} ;$$

lastly, recall that the Hodge-$*$-operator (of any given Riemannian metric on X) yields, for every $k \in \mathbb{N}$, the equality

$$b_k \;=\; b_{2n-k} .$$

2. Proof of Theorems A and B

In this section, we prove the main results, stating a Frölicher-type inequality for Bott-Chern and Aeppli cohomologies and giving therefore a characterization of compact complex manifolds satisfying the $\partial\overline{\partial}$-Lemma in terms of the dimensions of their Bott-Chern cohomology groups.

First of all, we need to recall two exact sequences from [9]. Let X be a compact complex manifold of complex dimension n. Following J. Varouchas, one defines the finite-dimensional bi-graded vector spaces

$$A^{\bullet,\bullet} \;:=\; \frac{\operatorname{im}\overline{\partial} \cap \operatorname{im}\partial}{\operatorname{im}\partial\overline{\partial}} , \quad B^{\bullet,\bullet} \;:=\; \frac{\ker\overline{\partial} \cap \operatorname{im}\partial}{\operatorname{im}\partial\overline{\partial}} , \quad C^{\bullet,\bullet} \;:=\; \frac{\ker\partial\overline{\partial}}{\ker\overline{\partial} + \operatorname{im}\partial}$$

and

$$D^{\bullet,\bullet} \;:=\; \frac{\operatorname{im}\overline{\partial} \cap \ker\partial}{\operatorname{im}\partial\overline{\partial}} , \quad E^{\bullet,\bullet} \;:=\; \frac{\ker\partial\overline{\partial}}{\ker\partial + \operatorname{im}\overline{\partial}} , \quad F^{\bullet,\bullet} \;:=\; \frac{\ker\partial\overline{\partial}}{\ker\overline{\partial} + \ker\partial} .$$

For every $p, q \in \mathbb{N}$ and $k \in \mathbb{N}$, we will denote

$$a^{p,q} \;:=\; \dim_{\mathbb{C}} A^{p,q} , \quad \ldots , \quad f^{p,q} \;:=\; \dim_{\mathbb{C}} F^{p,q}$$

and

$$a^{k} \;:=\; \sum_{p+q=k} a^{p,q} , \quad \ldots , \quad f^{k} \;:=\; \sum_{p+q=k} f^{p,q} .$$

One has the following exact sequences, see [9, §3.1]:

$$0 \to A^{\bullet,\bullet} \to B^{\bullet,\bullet} \to H^{\bullet,\bullet}_{\overline{\partial}}(X) \to H^{\bullet,\bullet}_{A}(X) \to C^{\bullet,\bullet} \to 0 \tag{1}$$

and

$$0 \to D^{\bullet,\bullet} \to H^{\bullet,\bullet}_{BC}(X) \to H^{\bullet,\bullet}_{\overline{\partial}}(X) \to E^{\bullet,\bullet} \to F^{\bullet,\bullet} \to 0 \,. \tag{2}$$

Note also (see [9, §3.1]) that the conjugation and the maps $\overline{\partial}\colon C^{\bullet,\bullet} \stackrel{\simeq}{\to} D^{\bullet,\bullet+1}$ and $\partial\colon E^{\bullet,\bullet} \stackrel{\simeq}{\to} B^{\bullet+1,\bullet}$ induce, for every $p, q \in \mathbb{N}$, the equalities

$$a^{p,q} \;=\; a^{q,p} \,, \qquad f^{p,q} \;=\; f^{q,p} \,, \qquad d^{p,q} \;=\; b^{q,p} \,, \qquad e^{p,q} \;=\; c^{q,p} \tag{3}$$

and

$$c^{p,q} \;=\; d^{p,q+1} \,, \qquad e^{p,q} \;=\; b^{p+1,q} \,,$$

from which one gets, for every $k \in \mathbb{N}$, the equalities

$$d^k \;=\; b^k \,, \qquad e^k \;=\; c^k \qquad \text{and} \qquad c^k \;=\; d^{k+1} \,, \qquad e^k \;=\; b^{k+1} \,.$$

Remark 2.1. Note that the argument used to prove the duality between Bott-Chern and Aeppli cohomology groups, see [7], can be applied to show also the dualities between $A^{\bullet,\bullet}$ and $F^{\bullet,\bullet}$ and between $C^{\bullet,\bullet}$ and $D^{\bullet,\bullet}$.

We can now state a Frölicher-type inequality for Bott-Chern and Aeppli cohomologies. While, on every compact complex manifold, one has the Frölicher inequality $h^k_{\overline{\partial}} \geq b_k$ for every $k \in \mathbb{N}$, see [4], this holds no more true for h^k_{BC}, as the following example shows.

Example 2.2. Let $\mathbb{H}(3;\mathbb{C})$ be the 3-dimensional complex *Heisenberg group* defined by

$$\mathbb{H}(3;\mathbb{C}) := \left\{ \begin{pmatrix} 1 & z^1 & z^3 \\ 0 & 1 & z^2 \\ 0 & 0 & 1 \end{pmatrix} \in \mathrm{GL}(3;\mathbb{C}) \;:\; z^1,\, z^2,\, z^3 \in \mathbb{C} \right\} \,.$$

Define the *Iwasawa manifold* as the 3-dimensional compact complex manifold given by the quotient

$$\mathbb{I}_3 := \mathbb{H}\left(3;\mathbb{Z}\left[\mathrm{i}\right]\right) \backslash \mathbb{H}(3;\mathbb{C}) \,,$$

where $\mathbb{H}\left(3;\mathbb{Z}\left[\mathrm{i}\right]\right) := \mathbb{H}(3;\mathbb{C}) \cap \mathrm{GL}\left(3;\mathbb{Z}\left[\mathrm{i}\right]\right)$.
The Kuranishi space of $\mathbb{I}_3$ is smooth and depends on 6 effective parameters, see [6]. According to I. Nakamura's classification, the small deformations of $\mathbb{I}_3$ are divided into three classes, *(i)*, *(ii)* and *(iii)*, in terms of their Hodge numbers: such classes are explicitly described by means of polynomial (in)equalities in the parameters, see [6, §3].
The dimensions of the Bott-Chern and Aeppli cohomology groups for $\mathbb{I}_3$ are computed in [7, Proposition 1.2], for the small deformations of $\mathbb{I}_3$ are computed in [1, §5.3] (we refer to it for more details). It turns out that the Bott-Chern cohomology yields a finer classification of the Kuranishi space of $\mathbb{I}_3$. More precisely, $h^{2,2}_{BC}$ assumes different values within class *(ii)*, respectively class *(iii)*, according to the rank of a certain matrix whose entries are related to the complex structure equations with respect to a suitable co-frame (see [1, §4.2]), whereas the numbers corresponding to class *(i)* coincide with those for $\mathbb{I}_3$: this allows a further subdivision of classes *(ii)* and *(iii)* into subclasses *(ii.a)*, *(ii.b)*, and *(iii.a)*, *(iii.b)*. For the sake of completeness, we list here these numbers.

classes	$\mathbf{h^1_{\overline{\partial}}}$	$\mathbf{h^1_{BC}}$	$\mathbf{h^1_{A}}$	$\mathbf{h^2_{\overline{\partial}}}$	$\mathbf{h^2_{BC}}$	$\mathbf{h^2_{A}}$	$\mathbf{h^3_{\overline{\partial}}}$	$\mathbf{h^3_{BC}}$	$\mathbf{h^3_{A}}$	$\mathbf{h^4_{\overline{\partial}}}$	$\mathbf{h^4_{BC}}$	$\mathbf{h^4_{A}}$	$\mathbf{h^5_{\overline{\partial}}}$	$\mathbf{h^5_{BC}}$	$\mathbf{h^5_{A}}$
(i)	5	4	6	11	10	12	14	14	14	11	12	10	5	6	4
(ii.a)	4	4	6	9	8	11	12	14	14	9	11	8	4	6	4
(ii.b)	4	4	6	9	8	10	12	14	14	9	10	8	4	6	4
(iii.a)	4	4	6	8	6	11	10	14	14	8	11	6	4	6	4
(iii.b)	4	4	6	8	6	10	10	14	14	8	10	6	4	6	4
	$\mathbf{b_1 = 4}$			$\mathbf{b_2 = 8}$			$\mathbf{b_3 = 10}$			$\mathbf{b_4 = 8}$			$\mathbf{b_5 = 4}$		

Nevertheless, we can prove the following.

Theorem A. *Let X be a compact complex manifold of complex dimension n. Then, for every $p,q \in \mathbb{N}$, the following inequality holds:*

$$h^{p,q}_{BC} + h^{p,q}_{A} \;\geq\; h^{p,q}_{\overline{\partial}} + h^{p,q}_{\partial} \;. \tag{4}$$

In particular, for every $k \in \mathbb{N}$, the following inequality holds:

$$h^{k}_{BC} + h^{k}_{A} \;\geq\; 2\, b_k \;, \tag{5}$$

where $h^k_{BC} := \sum_{p+q=k} \dim_{\mathbb{C}} H^{p,q}_{BC}(X)$ and $h^k_{A} := \sum_{p+q=k} \dim_{\mathbb{C}} H^{p,q}_{A}(X)$.

Proof. Fix $p,q \in \mathbb{N}$; using the symmetries $h^{p,q}_A = h^{q,p}_A$ and $h^{p,q}_{\overline{\partial}} = h^{q,p}_{\partial}$, the exact sequences (1) and (2) and the equalities (3), we have

$$\begin{aligned} h^{p,q}_{BC} + h^{p,q}_{A} &= h^{p,q}_{BC} + h^{q,p}_{A} \\ &= h^{p,q}_{\overline{\partial}} + h^{q,p}_{\overline{\partial}} + f^{p,q} + a^{q,p} + d^{p,q} - b^{q,p} - e^{p,q} + c^{q,p} \\ &= h^{p,q}_{\overline{\partial}} + h^{p,q}_{\partial} + f^{p,q} + a^{p,q} \\ &\geq h^{p,q}_{\overline{\partial}} + h^{p,q}_{\partial} \;, \end{aligned}$$

which proves (4).
Now, fix $k \in \mathbb{N}$; summing over $(p,q) \in \mathbb{N} \times \mathbb{N}$ such that $p+q=k$, we get

$$\begin{aligned} h^{k}_{BC} + h^{k}_{A} &= \sum_{p+q=k} \left(h^{p,q}_{BC} + h^{p,q}_{A}\right) \\ &\geq \sum_{p+q=k} \left(h^{p,q}_{\overline{\partial}} + h^{p,q}_{\partial}\right) \;=\; h^{k}_{\overline{\partial}} + h^{k}_{\partial} \\ &\geq 2\, b_k \;, \end{aligned}$$

from which we get (5). □

Remark 2.3. Note that small deformations of the Iwasawa manifold in Example 2.2 show that both the inequalities (4) and (5) can be strict.

Remark 2.4. Note that we have actually proved that, for every $k \in \mathbb{N}$,

$$h^{k}_{BC} + h^{k}_{A} \;=\; 2\, h^{k}_{\overline{\partial}} + a^k + f^k \;.$$

We prove now that equality in (5) holds for every $k \in \mathbb{N}$ if and only if the $\partial\overline{\partial}$-Lemma holds; in particular, this gives a characterization of the validity of the $\partial\overline{\partial}$-Lemma just in terms of $\left\{h^k_{BC}\right\}_{k\in\mathbb{N}}$.

Theorem B. *Let X be a compact complex manifold. The equality*

$$h^{k}_{BC} + h^{k}_{A} \;=\; 2\, b_k$$

in (5) holds for every $k \in \mathbb{N}$ if and only if X satisfies the $\partial\overline{\partial}$-Lemma.

Proof. Obviously, if X satisfies the $\partial\overline{\partial}$-Lemma, then, for every $k \in \mathbb{N}$, one has

$$h^{k}_{BC} \;=\; h^{k}_{A} \;=\; h^{k}_{\overline{\partial}} \;=\; b_k$$

and hence, in particular,

$$h^{k}_{BC} + h^{k}_{A} \;=\; 2\, b_k \;.$$

We split the proof of the converse into the following claims.

`Claim 1.` *If $h^k_{BC}+h^k_A = 2\,b_k$ holds for every $k\in\mathbb{N}$, then $E_1\simeq E_\infty$ and $a^k=0=f^k$ for every $k\in\mathbb{N}$.*
Since, for every $k\in\mathbb{N}$, we have

$$2\,b_k \;=\; h^k_{BC}+h^k_A \;=\; 2\,h^k_{\overline{\partial}}+a^k+f^k \;\geq\; 2\,b_k\;,$$

then $h^k_{\overline{\partial}}=b_k$ and $a^k=0=f^k$ for every $k\in\mathbb{N}$.

`Claim 2.` *Fix $k\in\mathbb{N}$. If $a^{k+1}:=\sum_{p+q=k+1}\dim_{\mathbb{C}} A^{p,q}=0$, then the natural map*

$$\bigoplus_{p+q=k} H^{p,q}_{BC}(X)\to H^k_{dR}(X;\mathbb{C})$$

is surjective.
Let $\mathfrak{a}=[\alpha]\in H^k_{dR}(X;\mathbb{C})$. We have to prove that $\mathfrak{a}$ admits a representative whose pure-type components are d-closed. Consider the pure-type decomposition of α:

$$\alpha \;=:\; \sum_{j=0}^{k}(-1)^j\;\alpha^{k-j,j}\;,$$

where $\alpha\in\wedge^{k-j,j}X$. Since $\mathrm{d}\,\alpha=0$, we get that

$$\partial\alpha^{k,0}=0\;,\qquad \overline{\partial}\alpha^{k-j,j}-\partial\alpha^{k-j-1,j+1}=0 \text{ for } j\in\{0,\ldots,k-1\}\;,\qquad \overline{\partial}\alpha^{0.k}=0\;;$$

by the hypothesis $a^{k+1}=0$, for every $j\in\{0,\ldots,k-1\}$, we get that,

$$\overline{\partial}\alpha^{k-j,j} \;=\; \partial\alpha^{k-j-1,j+1} \;\in\; \left(\mathrm{im}\,\overline{\partial}\cap\mathrm{im}\,\partial\right)\cap\wedge^{k-j,j+1}X \;=\; \mathrm{im}\,\partial\overline{\partial}\cap\wedge^{k-j,j+1}X$$

and hence there exists $\eta^{k-j-1,j}\in\wedge^{k-j-1,j}X$ such that

$$\overline{\partial}\alpha^{k-j,j} \;=\; \partial\overline{\partial}\eta^{k-j-1,j} \;=\; \partial\alpha^{k-j-1,j+1}\;.$$

Define

$$\eta \;:=\; \sum_{j=0}^{k-1}(-1)^j\;\eta^{k-j-1,j}\;\in\;\wedge^{k-1}(X;\mathbb{C})\;.$$

The claim follows noting that

$$\begin{aligned}
\mathfrak{a} \;&=\; [\alpha] \;=\; [\alpha+\mathrm{d}\,\eta]\\
&=\; \left[\left(\alpha^{k,0}+\partial\eta^{k-1,0}\right)+\sum_{j=1}^{k-1}(-1)^j\;\left(\alpha^{k-j,j}+\partial\eta^{k-j-1,j}-\overline{\partial}\eta^{k-j,j-1}\right)\right.\\
&\qquad \left.+\,(-1)^k\;\left(\alpha^{0,k}-\overline{\partial}\eta^{0,k-1}\right)\right]\\
&=\; \left[\alpha^{k,0}+\partial\eta^{k-1,0}\right]+\sum_{j=1}^{k-1}(-1)^j\;\left[\alpha^{k-j,j}+\partial\eta^{k-j-1,j}-\overline{\partial}\eta^{k-j,j-1}\right]\\
&\qquad +\,(-1)^k\;\left[\alpha^{0,k}-\overline{\partial}\eta^{0,k-1}\right]\;,
\end{aligned}$$

that is, each of the pure-type components of $\alpha+\mathrm{d}\,\eta$ is both ∂-closed and $\overline{\partial}$-closed.

`Claim 3.` *If* $h^k_{BC} \geq b_k$ *and* $h^k_{BC} + h^k_A = 2\, b_k$ *for every* $k \in \mathbb{N}$*, then* $h^k_{BC} = b_k$ *for every* $k \in \mathbb{N}$*.*
If n is the complex dimension of X, then, for every $k \in \mathbb{N}$, we have

$$b_k \;\leq\; h^k_{BC} \;=\; h^{2n-k}_A \;=\; 2\, b_{2n-k} - h^{2n-k}_{BC} \;\leq\; b_{2n-k} \;=\; b_k$$

and hence $h^k_{BC} = b_k$ for every $k \in \mathbb{N}$.

Now, by `Claim 1`, we get that $a^k = 0$ for each $k \in \mathbb{N}$; hence, by `Claim 2`, for every $k \in \mathbb{N}$ the map

$$\bigoplus_{p+q=k} H^{p,q}_{BC}(X) \to H^k_{dR}(X;\mathbb{C})$$

is surjective and hence, in particular, $h^k_{BC} \geq b_k$. By `Claim 3` we get therefore that $h^k_{BC} = b_k$ for every $k \in \mathbb{N}$. Hence, the natural map $H^{\bullet,\bullet}_{BC}(X) \to H^\bullet_{dR}(X;\mathbb{C})$ is actually an isomorphism, which is equivalent to say that X satisfies the $\partial\overline{\partial}$-Lemma. □

Remark 2.5. We note that, using the exact sequences (2) and (1), one can prove that, on a compact complex manifold X and for every $k \in \mathbb{N}$,

$$\begin{aligned} e^k \;&=\; \left(h^k_{\overline{\partial}} - h^k_{BC}\right) + f^k + c^{k-1} \\ &=\; \left(h^k_{\overline{\partial}} - h^k_{BC}\right) - \left(h^{k-1}_{\overline{\partial}} - h^{k-1}_{A}\right) + f^k - a^{k-1} + e^{k-2} \,. \end{aligned}$$

Remark 2.6. Note that $E_1 \simeq E_\infty$ is not sufficient to have the equality $h^k_{BC} + h^k_A = 2\, b_k$ for every $k \in \mathbb{N}$: a counter-example is provided by small deformations of the Iwasawa manifold, see Example 2.2.

Using Theorem B, we get another proof of the following result (see, e.g., [10, 11]).

Corollary 2.7. *Satisfying the* $\partial\overline{\partial}$*-Lemma is a stable property under small deformations of the complex structure.*

Proof. Let $\{X_t\}_t$ be a complex-analytic family of compact complex manifolds. Since, for every $k \in \mathbb{N}$, the dimensions $h^k_{BC}(X_t)$ are upper-semi-continuous functions at t (see, e.g., [7]), while the dimensions $b_k(X_t)$ are constant in t, one gets that, if X_{t_0} satisfies the equality $h^k_{BC}\left(X_{t_0}\right) + h^k_A\left(X_{t_0}\right) = 2\, b_k\left(X_{t_0}\right)$ for every $k \in \mathbb{N}$, the same holds true for X_t with t near t_0. □

It could be interesting to construct a compact complex manifold (of any complex dimension greater or equal to 3) such that $E_1 \simeq E_\infty$ and $h^{p,q}_{\overline{\partial}} = h^{p,q}_{\partial}$ for every $p, q \in \mathbb{N}$ but for which the $\partial\overline{\partial}$-Lemma does not hold.
A compact complex manifold X whose double complex $\left(\wedge^{\bullet,\bullet}X,\, \partial,\, \overline{\partial}\right)$ has the form in Figure 1 provides such an example.

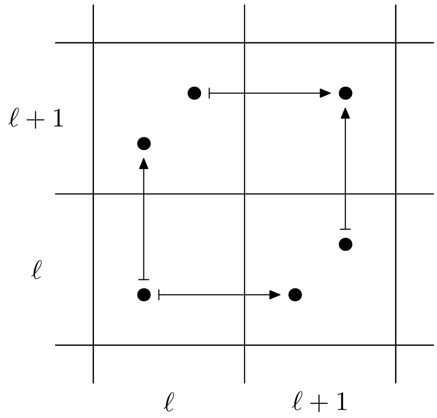

FIGURE 1. An abstract example

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