ON THE PERFECT MATCHING INDEX OF BRIDGELESS CUBIC GRAPHS

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Abstract. If G is a bridgeless cubic graph, Fulkerson conjectured that we can find 6 perfect matchings $M_1, \ldots, M_6$ of G with the property that every edge of G is contained in exactly two of them and Berge conjectured that its edge set can be covered by 5 perfect matchings. We define $\tau(G)$ as the least number of perfect matchings allowing to cover the edge set of a bridgeless cubic graph and we study this parameter. The set of graphs with perfect matching index 4 seems interesting and we give some informations on this class.

1. Introduction

The following conjecture is due to Fulkerson, and appears first in [6].

Conjecture 1.1. *If G is a bridgeless cubic graph, then there exist 6 perfect matchings $M_1, \ldots, M_6$ of G with the property that every edge of G is contained in exactly two of $M_1, \ldots, M_6$.*

If G is 3-edge-colourable, then we may choose three perfect matchings M_1, M_2, M_3 so that every edge is in exactly one. Taking each of these twice gives us 6 perfect matchings with the properties described above. Thus, the above conjecture holds trivially for 3-edge-colorable graphs. There do exist bridgeless cubic graphs which are not $3-$edge-colourable (for instance the Petersen graph), but the above conjecture asserts that every such graph is close to being 3-edge-colourable.

If Fulkerson's conjecture were true, then deleting one of the perfect matchings from the double cover would result in a covering of the graph by 5 perfect matchings. This weaker conjecture was proposed by Berge (see Seymour [12]).

Conjecture 1.2. *If G is a bridgeless cubic graph, then there exists a covering of its edges by 5 perfect matchings.*

Since the Petersen graph does not admit a covering by less that 5 perfect matchings (see section 3), 5 in the above conjecture can not be changed into 4 and the following weakening of conjecture 1.2 (suggested by Berge) is still open.

Conjecture 1.3. *There exists a fixed integer k such that the edge set of every bridgeless cubic graph can be written as a union of k perfect matchings.*

Another consequence of the Fulkerson conjecture would be that every bridgeless cubic graph has 3 perfect matchings with empty intersection (take any 3 of the 6 perfect matchings given by the conjecture). The following weakening of this (also suggested by Berge) is still open.

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Conjecture 1.4. *There exists a fixed integer k such that every bridgeless cubic graph has a list of k perfect matchings with empty intersection.*

For $k = 3$ this conjecture is known as the Fan Raspaud Conjecture.

Conjecture 1.5. [3] *Every bridgeless cubic graph contains perfect matching M_1, M_2, M_3 such that*

$$M_1 \cap M_2 \cap M_3 = \emptyset$$

While some partial results exist concerning conjecture 1.5 (see [17]), we have noticed no result in the literature concerning the validity of Conjecture 1.1 or Conjecture 1.4 for te usual classes of graphs which are examined when dealing with the $5-$flow conjecture of Tutte [15] or the cycle double conjecture of Seymour [11] and Szekeres [13]. Hence for bridgeless cubic graphs with *oddness* 2 (a $2-$factor contains exactly tow odd cycles) it is known that the $5-$flow conjecture holds true as well as the cycle double conjecture (see Zhang [18] for a comprehensive study of this subject).

Let G be a bridgeless cubic graph, we shall say that the set $\mathcal{M} = \{M_1, \ldots M_k\}$ ($k \geq 3$) of perfect matchings is a *$k-$covering* when each edge is contained in at least one of theses perfect matchings. A *Fulkerson covering* is a $6-$covering where each edge appears exactly twice. Since every edge of a bridgeless cubic graph is contained in a perfect matching (see [10]) the minimum number $\tau(G)$ of perfect matchings covering its edge set is well defined. We shall say that $\tau(G)$ is the *perfect matching index* of G. We obviously have that $\tau(G) = 3$ if and only if G is $3-$edge-colourable.

2. Preliminaries results

Proposition 2.1. *let G be a cubic graph with a $k-$covering $\mathcal{M} = \{M_1, \ldots, M_k\}$ ($k \geq 3$) then G is bridgeless.*

Proof Assume that $e \in E(G)$ is an isthmus, then the edges incident to e are not covered by any perfect matching of G and $\mathcal{M}$ is not a $K-$covering, a contradiction. □

2.1. $2-$cut connection.

Let G_1, G_2 be two bridgeless cubic graph and $e_1 = u_1v_1 \in E(G_1)$, $e_2 = u_2v_2 \in E(G_1)$ be two edges. Construct a new graph $G = G_1 \bigodot G_2$

$$G = [G_1 \setminus \{e_1\}] \cup [G_2 \setminus \{e_2\}] \cup \{u_1u_2, v_1v_2\}$$

Proposition 2.2. *Let G_1 be a cubic graph such that $\tau(G_1) = k \geq 3$ and let G_2 be any cubic bridgeless graph, then $\tau(G_1 \bigodot G_2) \geq k$*

Proof Let $G = G_1 \bigodot G_2$. Assume that $k' = \tau(G) < k$ and let $\mathcal{M} = \{M_1, \ldots, M_{k'}\}$ be a $k'-$covering of G Any perfect matching of G must intersect the $2-$edge cut $\{u_1v_1, u_2v_2\}$ in two edges or has no edge in common with that set. Thus any perfect matching in $\mathcal{M}$ leads to a perfect matching of G_1. Hence we should have a $k'-$covering of the edge set of G_1, a contradiction. □

2.2. 3−**cut connection.** Let G_1, G_2 be two bridgeless cubic graph and $u \in V(G_1)$, $v \in V(G_2)$ be two vertices with $N(u) = \{u_1, u_2, u_3\}$ and $N(v) = \{v_1, v_2, v_3\}$. Construct a new graph $G = G_1 \otimes G_2$

$$G = [G_1 \setminus \{u\}] \cup [G_2 \setminus \{v\}] \cup \{u_1v_1, u_2v_2, u_3v_3\}$$

It is well known that the resulting graph $G_1 \otimes G_2$ is bridgeless. The 3−edge cut $\{u_1v_1, u_2v_2, u_3v_3\}$ will be called the *principal* 3−*edge cut*.

Proposition 2.3. *Let G_1 be a cubic graph such that $\tau(G_1) = k \geq 3$ and let G_2 be any cubic bridgeless graph. Let $k' = \tau(G_1 \otimes G_2)$ and let $\mathcal{M} = \{M_1, \dots, M_{k'}\}$ be a k'−covering of $G_1 \otimes G_2$. Then one of the followings is true*

(1) $k' \geq k$
(2) *There is a perfect matching $M_i \in \mathcal{M}$ ($1 \leq i \leq k$) containing the principal 3−edge cut*

Proof Assume that $k' < k$. Any perfect matching of $G_1 \otimes G_2$ must intersect the principal 3−edge cut in one or three edges. If none of the perfect matchings in $\mathcal{M}$ contains the principal 3−edge cut, then any perfect matching in $\mathcal{M}$ leads to a perfect matching of G_1 and any edge of G_1 is covered by one of these perfect matchings. Hence we should have a k'−covering of the edge set of G_1, a contradiction. □

3. On graphs with perfect matching index 4

A natural question is to investigate the class of graphs for which the perfect matching index is 4.

Proposition 3.1. *Let G be a cubic graph with a 4−covering $\mathcal{M} = \{M_1, M_2, M_3, M_4\}$ then*

(1) *Every edge is contained in exactly one or two perfect matchings of $\mathcal{M}$.*
(2) *The set M of edges contained in exactly two perfect matchings of $\mathcal{M}$ is a perfect matching.*
(3) *If $\tau(G) = 4$ then $\forall i \neq j \in \{1, 2, 3, 4\} \quad M_i \cap M_j \neq \emptyset$.*

Proof Let v be any vertex of G, each edge incident with v must be contained in some perfect matching of $\mathcal{M}$ and each perfect matching must be incident with v. We have thus exactly one edge incident with v which is covered by exactly two perfect matchings of $\mathcal{M}$ while the two other edges are covered by exactly one perfect matching. We get thus immediately Items 1 and 2.

When $\tau(G) = 4$, G is not a 3−edge colourable graph. Assume that we have two perfect matchings with an empty intersection. These two perfect matchings lead to an even 2−factor and hence a a 3−edge colouring of G, a contradiction. □

In the following the edges of the matching M described in item 2 of Proposition 3.1 will be said to be *covered twice*.

Proposition 3.2. *Let G be a cubic graph such that $\tau(G) = 4$ then G has at least 12 vertices*

Proof Let $\mathcal{M} = \{M_1, M_2, M_3, M_4\}$ be a covering of the edge set of G into 4 perfect matchings. From Proposition 3.1 we must have at least 6 edges in the perfect matching formed with the edges covered twice in $\mathcal{M}$. Hence, G must have at least

12 vertices as claimed. □

From Proposition 3.2, we obviously have that the Petersen graph has a perfect matching index equal to 5.

Proposition 3.3. *Let G be a cubic graph such that $\tau(G) = 4$ and let $\mathcal{M} = \{M_1, M_2, M_3, M_4\}$ be a covering of its edge set into 4 perfect matchings then for each j ($j = 1 \ldots 4$) $\mathcal{M} - M_j$ is a set of 3 perfect matchings satisfying the Fan Raspaud conjecture.*

Proof Obvious since, by Item 1 of Proposition 3.1 any edge is contained in exactly one or two perfect matchings of $\mathcal{M}$. □

Let G be a cubic graph with 3 perfect matchings M_1, M_2 and M_3 having an empty intersection. Since such a graph satisfy the Fan Raspaud conjecture, when considering these three perfect matchings, we shall say that (M_1, M_2, M_3) is an *FR-triple*. When a cubic graph has a FR-triple we define T_i ($i = 0, 1, 2$) as the set of edges that belong to precisely i matchings of the FR-triple. Thus (T_0, T_1, T_2) is a partition of the edge set.

Proposition 3.4. *Let G be a cubic graph with 3 perfect matchings M_1, M_2 and M_3 having an empty intersection. Then the set $T_0 \cup T_2$ is a set of disjoint even cycles. Moreover, the edges of T_0 and T_2 alternate along these cycles.*

Proof Let v be a vertex incident to a edge of T_0. Since v must be incident to each perfect matching and since the three perfect matchings have an empty intersection, one of the remaining edges incident to v must be contained into 2 perfect matchings while the other is contained in exactly one perfect matching. The result follows. □

Let G be a bridgeless cubic graph and let C and C' be distinct odd cycles of G. Assume that there are three distinct edges namely xx', yy' and zz' such that x, y and z are vertices of C while x', y', z' are vertices of C' which determine on C and on C' edge-disjoint paths of odd length then we shall say that (xx', yy', zz') is a *good triple* and that the pair of cycles $\{C, C'\}$ is a *good pair*.

Theorem 3.5. *Let G be a cubic graph which has a $2-$factor F whose odd cycles can be arranged into good pairs $\{C_1, D_1\}$,$\{C_2, D_2\}$, ..., $\{C_k, D_k\}$. Then $\tau(G) \leq 4$.*

Proof For each good pair $\{C_i, D_i\}$ let $(c_i^1 d_i^1, c_i^2 d_i^2, c_i^3 d_i^3)$ be a good triple of C_i and D_i, c_i^1, c_i^2, c_i^3 being vertices of C_i while d_i^1, d_i^2 and ${}_i^3$ are on D_i. In order to construct a set $\mathcal{M} = \{M_1, M_2, M_3, M_4\}$ of 4 perfect matchings covering the edge set of G we let M_1 as the perfect matching of G obtained by deleting the edges of the $2-$factor.

Let A_j be the set of edges $\{c_i^j d_i^j | i = 1 \ldots k\}$. We construct a perfect matching M_j ($j = 2, 3, 4$) of G such that $M_1 \cap M_j = A_j$. For each good pair $\{C_i, D_i\}$ ($i = 1 \ldots k$), we add to A_j the unique perfect matching contained in $E(C_i) \cup E(D_i)$ when the two vertices c_i^j and d_i^j are deleted. We get hence 3 matchings B_j ($j = 2, 3, 4$) where each vertex contained in a good pair is saturated. If the $2-$factor contains some even cycles, we add first a perfect matching contained in the edge set of these even cycles to B_2. We obtain thus a perfect matching M_2 whose intersection with M_1 is reduced to A_2. The remaining edges of these even cycles are added to B_3 and to

B_4, leading to the perfect matchings M_3 and M_4. Let us remark that each edge of these even cycles are contained in $M_2 \cup M_3$.

We claim that each edge of G is contained in at least one of $\mathcal{M} = \{M_1, M_2, M_3, M_4\}$. Since M_1 is the perfect matching which complements in G the $2-$factor F, the above remark says that we have just to prove that each edge of each good pair is covered by some perfect matching of $\mathcal{M}$. By construction, no edge is contained in $M_1 \cup M_2 \cup M_3$ which means that (M_1, M_2, M_3) is an FR-triple. In the same way, (M_1, M_3, M_4) and (M_1, M_2, M_4) are FR-triples. The edges of $T_0 \cup T_2$ induced by the FR-triple (M_1, M_2, M_3) on each good pair $\{C_i, D_i\}$ is the even cycle Γ_i using $c_i^1 d_i^1$ and $c_i^2 d_i^2$, the odd path of C_i joining c_i^1 to c_i^2 and the odd path of D_i joining d_i^1 to d_i^2. In the same way, edges of $T_0 \cup T_2$ induced by the FR-triple (M_1, M_3, M_4) on each good pair $\{C_i, D_i\}$ is the even cycle Λ_i using $c_i^2 d_i^2$ and $c_i^3 d_i^3$, the odd path of C_i joining c_i^2 to c_i^3 and the odd path of D_i joining d_i^2 to d_i^3. It is an easy task to see that these two cycles Γ_i and Λ_i have the only edge $c_i^2 d_i^2$ in common. Hence each edge of $\Gamma_i \cap T_0$ is contained into M_4 while each edge of $\Lambda_i \cap T_0$ is contained into M_2. The result follows. □

3.1. **On balanced matchings.** A set $A \subseteq E(G)$ is a *balanced matching* when we can find 2 perfect matchings M_1 and M_2 such that $A = M_1 \cap M_2$. Let $B(G)$ be the set of balanced matchings of G, we define $b(G)$ as the minimum size of a any set $A \in B(G)$, we have:

Proposition 3.6. *Let G be a cubic graph such that $\tau(G) = 4$ then $b(G) \leq \frac{n}{12}$.*

Proof Let $\mathcal{M} = \{M_1, M_2, M_3, M_4\}$ be a covering of the edge set of G into 4 perfect matchings and let M be the perfect matching of edges contained in exactly two perfect matchings of $\mathcal{M}$ (Iem 2 of Proposition **??**). Since $M_i \cap M_j \neq \emptyset \ \forall i \neq j \in \{1, 2, 3, 4\}$ by Proposition **??**, these 6 balanced matchings partition M. Hence, one of them must have at most $\frac{|M|}{6} = \frac{n}{12}$ edges.

□

In [14] Kaiser, Král and Norine proved

Theorem 3.7. *Any bridgeless cubic graph contains 2 perfect matchings whose union cover at least $\frac{9n}{10}$ edges of G.*

From Theorem 3.7, we can find two perfect matchings with an intersection having at most $\frac{n}{10}$ edges in any cubic bridgeless graph. It can be proved (see [4]) that for any cyclically 4-edge connected cubic graph G, either $b(G) \leq \frac{n}{14}$ or any perfect matching contains an odd cut of size 5.

3.2. **On classical snarks.** As usual a *snark* is a non $3-$edge colourable bridgeless cubic graph. In Figure 1 is depicted one of the two the Blanuša snarks on 18 vertices [1]. In bold we have drawn a $2-$factor (each cycle has length 9) and the dashed edges connect the triple (x, y, z) of one cycle to the triple (x', y', z') of the second cycle. It is a routine matter to check that (xx', yy', zz') is a good triple and Theorem 3.5 allows us to say that this graph has perfect matching index 4. In the same way the second Blanuša snark on 18 vertices depicted in Figure 2 can be covered by 4 perfect matchings by using Theorem 3.5.

For an odd $k \geq 3$ the Flower Snark F_k intoduced by Isaac (see [8]) is the cubic graph on $4k$ vertices $x_0, x_1, \ldots x_{k-1},\ y_0, y_1, \ldots y_{k-1},\ z_0, z_1, \ldots z_{k-1},\ t_0, t_1, \ldots t_{k-1}$

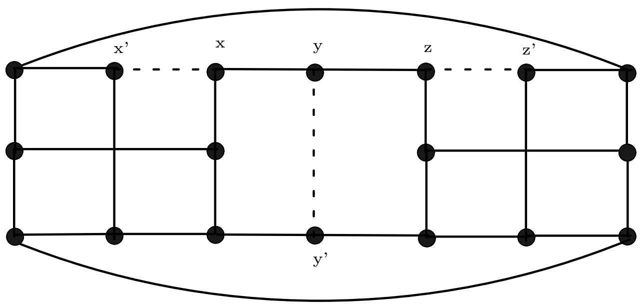

FIGURE 1. Blanuša snark #1

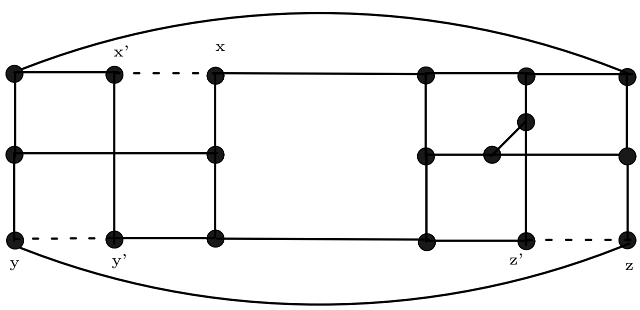

FIGURE 2. Blanuša snark #2

such that $x_0x_1 \dots x_{k-1}$ is an induced cycle of length k, $y_0y_1 \dots y_{k-1}$ $z_0z_1 \dots z_{k-1}$ is an induced cycle of length $2k$ and for $i = 0 \dots k-1$ the vertex t_i is adjacent to x_i, y_i and z_i. The set $\{t_i, x_i, y_i, z_i\}$ induces the claw C_i. In Figure 3 we have a representation of F_5, the half edges (to the left and to the right in the figure) with same labels are identified.

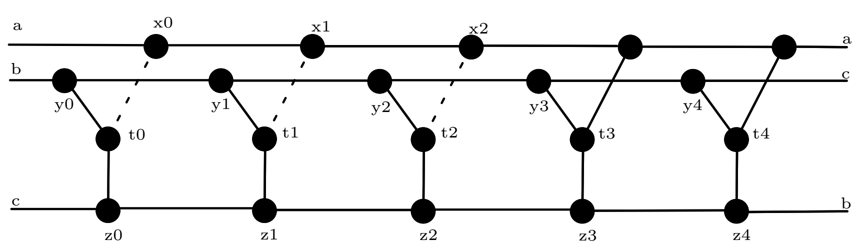

FIGURE 3. J_5

Theorem 3.8. $\tau(F_k) = 4$.

Proof Let $k = 2p+1 \geq 3$ and let $C = x_0x_1 \dots x_{2p}$, $D = y_0t0z_0z_1t_1y_1 \dots y_{2i}t_{2i}z_{2i}z_{2i+1}t_{2i+1}y_{2i+1} \dots y_{2p}t_{2p}z_{2p}$ $(0 \leq i \leq)$ be the odd cycles of lengths $2k+1$ and $3 \times (2k+1)$ respectively which partition F_k (in bold in Figure 3. It is a routine matter to check that the edges x_0t_0, x_1t_1 and x_2t_2 form a good triple (dashed edges in Figure 3). Hence (C, D) is a $2-$factor of G and it is a good pair. The result follows from Theorem 3.5. □

Let H be the graph depicted in Figure 4

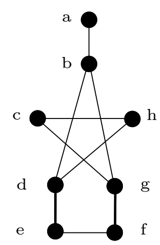

FIGURE 4. H

Let G_k (k odd) be a cubic graph obtained from k copies of H ($H_0 \dots H_{k-1}$ where the name of vertices are indexed by i) in adding edges a_ia_{i+1}, c_ic_{i+1}, e_ie_{i+1}, f_if_{i+1} and h_ih_{i+1} (subscripts are taken modulo k).

If $k = 5$, then G_k is known as the Goldberg snark. Accordingly, we refer to all graphs G_k as Goldberg graphs. The graph G_5 is shown in Figure 5. The half edges (to the left and to the right in the figure) with same labels are identified.

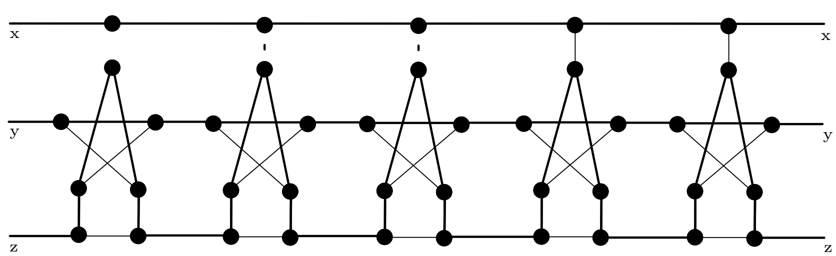

FIGURE 5. Goldberg snark G_5

Theorem 3.9. *$\tau(G_k) = 4$.*

Proof Let $k = 2p + 1 \geq 3$ and let $C = a_0a_1 \dots a_{2p}$, $D = e_0d_0b_0g_0f_0\ e_1d_1b_1g_1f_1 \dots e_id_ib_ig_if_i \dots e_{2p}d_{2p}b_{2p}g_{2p}f_{2p}$ $(0 \leq i \leq)$ be the odd cycles of lengths $2k + 1$ and $5 \times (2k + 1)$ respectively and $E = c_0h_0c_1h_1 \dots c_ih_i \dots c_{2p}h_{2p}$ the cycle of length $4k$ of G_k. This set of 3 cycles is a $2-$factor of G_k (in bold in Figure 5). At last, a_0b_0, a_1b_1 and a_2b_2 are edges of G (dashed edges in Figure 5). Then (a_0b_0, a_1b_1, a_2b_2) is a good triple. Hence (C, D, E) is a $2-$factor of G where (C, D) is a good pair . The result follows from Theorem 3.5. □

3.3. **On permutation graphs.** A cubic graph G is called a *permutation graph* if G has a $2-$factor F such that F is the union of two chordless cycles C and C'. Let M be the perfect matching $G - F$. A subgraph homeomorphic to the Petersen graph with no edge of M subdivided is called a $M - P_{10}$. Ellingham [2] showed that a permutation graph without any $M - P_{10}$ is $3-$edge colourable.

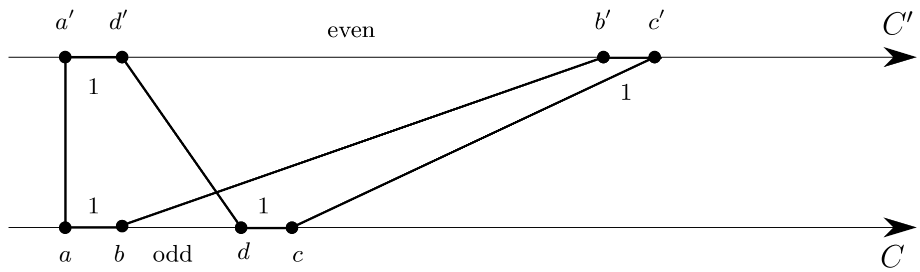

FIGURE 6. The cycle on 8 vertices described in Lemma 3.11

In general, we do not know whether a permutation graph distinct from the Petersen graph is $3-$edge colourable or not. It is an easy task to construct a cyclically $4-$edge connected permutation graph which is a snark (consider the two Blanusa snarks on 18 vertices for exemple) and Zhang [18] conjectured:

Conjecture 3.10. *Let G be a $3-$connected cyclically $5-$edge connected permutation graph. If G is a snark, then G must be the Petersen graph.*

Let us consider a permutation graph G with a 2-factor F having two cycles C and C'. Two distinct vertices of C say x and y determine on C two paths with x and y as end-points. In order to be unambiguous when considering those paths from their end-points we give an orientation to C. Thus $C(x, y)$ will denote in the following the path of C that starts with the vertex x and ends with the vertex x according to the orientation of C.The notation $C'(x', y')$ is defined similarly when x' and y' are vertices of C'.

In order to determine which permutation graphs have a perfect matching index less than 4 we state the following tool (see Figure 6) :

Lemma 3.11. *Let G be a permutation graph with a 2-factor containing precisely two odd cycles C and C'. Assume that $\chi'(G) = 4$ and that (C, C') is not a good pair. Let ab be an edge of C such that the odd path determined on C' with the neighbors of a and b, say a' and b' respectively, has minimum length. Assume that C and C' have an orientation such that $C(a, b)$ is an edge and $C'(a', b')$ has odd length.*
Then there must exist 4 additional vertices c and d on C and their neighbors on C', say c' and d' respectively, verifying :

- *the paths $C'(a', d')$, $C'(b', c')$ and $C(d, c)$ are edges.*
- *the path $C(b, d)$ is odd and the path $C'(d', b')$ is even.*

Proof Observe first that a' and b' are not adjacent otherwise the cycle obtained with the paths $C(b, a)$ and $C'(b', a')$ together with the edges aa' and bb' would be hamiltonian, a contradiction since it is assumed that $\chi'(G) = 4$.

Since the path $C'(a', b')$ is odd there must be a neighbor of b' on $C'(b', a')$, say c'. Let c be the neighbor of c' on C. The path $C(b, c)$ has even length, otherwise (aa', bb', cc') would be a good triple and (C, C') a good pair, a contradiction.

It follows that the vertex c has a neighbor, say d on $C(b, c)$ and $C(b, d)$ has odd length.

Let d' be the neighbor of d on C'. It must be pointed out that d' is a vertex of $C'(a', b')$. As a matter of fact if on the contrary d' belongs to $C'(c', a')$ we would have a good triple with (dd', cc', bb') when $C'(c', d')$ has odd length and with (aa', bb', dd') when $C'(c', d')$ is an even path; a contradiction in both cases.

But now by the choice of the edge ab the length of $C'(a', b')$ cannot be greater than $C'(d', c')$, thus d' is adjacent to a' and the path $C'(d', b')$ has even length. □

We have :

Theorem 3.12. *Let G be a permutation graph then $\tau(G) \leq 4$ or G is the Petersen graph.*

Proof Let C and C' the $2-$factor of chordless cycles which partition $V(G)$ and We can assume that G is not $3-$edge colourable otherwise $\tau(G) = 3$ and there is nothing to prove. Hence, C and C' have both odd lengths. In addition we assume that (C, C') is not a good pair, otherwise we are done by Theorem 3.5.

Let x_1x_2 be an edge of C such that the odd path determined on C' with the neighbors of x_1 and x_2, say y_1 and y_2 respectively, has minimum length.

We choose to orient C from x_1 to x_2 and to orient C' from y_1 to y_2. Thus $C(x_1, x_2)$ is an edge and $C'(y_1, y_2)$ is an odd path.

By Lemma 3.11 we must have two vertices x_3 and x_4 on C and their neighbors y_3 and y_4 on C' such that $C(x_4, x_3)$, $C'(y_1, y_4)$, $C'(y_2, y_3')$ are edges, $C(x_2, x_4)$ being an odd path while $C'(y_4, y_2)$ has even length.

Claim 1. *The vertices y_1 and y_3 are adjacent.*

Proof Assume not.

The odd path $C'(y_4, y_3)$ having the same length than $C'(y_1, y_2)$ we may apply Lemma 3.11 on the edge x_4x_3 ($x_4 = a$, $x_3 = b$). Thus there is edges; say x_5y_5 and x_6y_6, x_5 and x_6 being vertices of C, y_5 and y_6 vertices of C', the paths $C(x_6, x_5)$, $C'(y_4, y_6)$ and $C'(y_3, y_5)$ having length 1. Moreover the paths $C(x_3, x_6)$ and $C'(y_6, y_2)$ are odd. Since it is assumed that y_1 and y_3 are independent we have $y_5 \neq y_1$ and $x_5 \neq x_1$.

Observe that the paths $C'(y_1, y_2)$ and $C'(y_6, y_5)$ have the same length, thus we apply Lemma 3.11 again with $a = x_6$ and $b = x_5$.

Let y_7 be the neighbor of y_5 on $C'y_5, y_1)$ and x_7 be the neighbor of y_7 on C. We know that x_7 is a vertex of $C(x_5, x_1)$ at even distance of x_5. The vertex x_8 being the neighbor of x_7 on $C(x_5, x_7)$ and y_8 the neighbor of x_8 on C', we have that y_8 is the neighbor of y_6 on $C'(y_6, y_2)$.

The path $C'(y_8, y_2)$ has even length, hence there must be on this path a neighbor of y_8 distinct from y_2, say y_9. Let x_9 be the neighbor of y_9 on C.

The vertex x_9 belongs to $C(x_7, x_1)$. Otherwise when x_9 is on $C(x_2, x_4)$; if the path $C(x_2, x_9)$ is odd we can find a good triple, namely (x_8y_8, x_9y_9, x_2y_2) on the other case we have the good triple (x_9y_9, x_4y_4, x_1y_1). A contradiction in both cases.

We get a similar contradiction if x_9 belongs to $C(x_3, x_6)$ by considering the triples (x_5y_5, x_9y_9, x_8y_8) or (x_9y_9, x_4y_4, x_2y_2).

Finally, when x_9 is a vertex of $C(x_5, x_8)$ a contradiction occurs with the triple $(x_5y_5, x_9y_9, x_7, y_7)$ if $C(x_5, x_9)$ is odd and with the tripe (x_8y_8, x_9y_9, x_6y_6) otherwise.

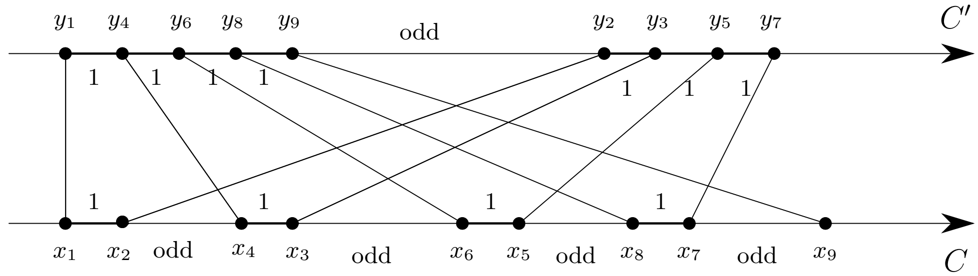

FIGURE 7. Situation at the end of Claim 2

Observe that the path $C(x_7, x_9)$ must be odd or (x_9y_9, x_7y_7, x_8y_8) would be a good triple, a contradiction.

But now (x_9y_9, x_5y_5, x_4y_4) is a good triple, a contradiction which proves the Claim (see Figure 7). □

From now on we assume that y_3y_1 is an edge.

The path $C(x_1, x_3)$ being odd there must be a neighbor of x_3 on $C(x_3, x_1)$ distinct from x_1, let x_5 be this vertex. It's neighbor on C', say y_5, must be on $C'(y_4, y_2)$. Moreover the length of $C'(y_4, y_5)$ is odd otherwise the edges x_5y_5, x_3y_3 and x_1y_1 would form a good triple, a contradiction.

Claim 2. *The paths $C'(y_4, y_5)$ and $C'(y_5, y_2)$ are reduced to edges.*

Proof Assume in a first stage that the neighbor of y_4 on $C'(y_4, y_5)$ is distinct from y_5, let y_6 be this vertex and x_6 be its neighbor on C.

The vertex x_6 cannot belong to $C(x_5, x_1)$, otherwise we would have a good triple (x_3y_3, x_6y_6, x_4y_4) when $C(x_5, x_6)$ is an even path and the good triple (x_4y_4, x_6y_6, x_2y_2) if it's an odd path, contradictions.

Similarly the vertex x_6 cannot belong to $C(x_2, x_4)$. On the contrary we would have a good triple with the edges x_2y_2, x_6y_6 and x_1y_1 when the path $C(x_2, x_6)$ is odd and another good triple with the edges x_4y_4, x_6y_6 and x_1y_1.

On the same manner we can prove that the path $C'(y_5, y_2)$ has length 1. □

It comes from Claim 2 that C' has only 5 vertices. Since both cycles C and C' have the same length C has 5 vertices too and G is the Petersen graph. □

In [16] Watkins proposed two families of generalized Blanuša snarks using the blocks B, A_1 and A_2 described in Figure 8. The generalized Blanuša snarks of type 1 (resp. of type 2) are obtained by considering a number of blocks B and one block A_1 (resp. A_2), these blocks are arranged cyclically, the semi-edges a and b of one block being connected to the semi-edges a, b of the next one. Recently generalized Blanuša snarks were studied in terms of circular chromatic index (see [9, 7]).

The generalized Blanuša snarks are permutation graphs, hence :

Corollary 3.13. *Let G be a generalized Blanuša snarks then $\tau(G) = 4$.*

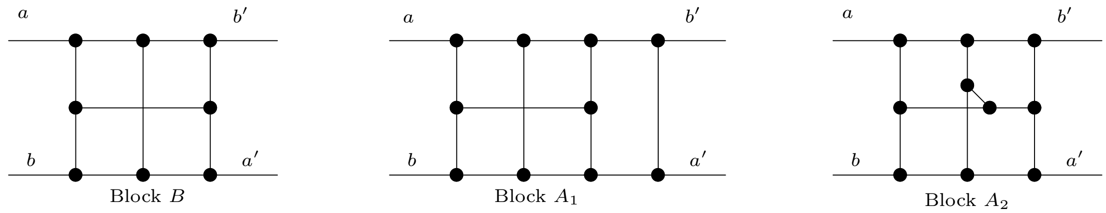

FIGURE 8. Blocks for the construction of generalized Blanuša snarks.

4. ON GRAPHS WITH $\tau \geq 5$

It is an easy task to construct cubic graphs with perfect matching index at least 5 with the help of Proposition 2.2. Take indeed the Petersen graph P and any bridgeless cubic graph G and apply the construction $P \bigodot G$.

Proposition 4.1. *Let G be bridgeless cubic graph with perfect matching index at least 5 and let H be a connected bipartite cubic graph. Then $G \otimes H$ is bridgeless cubic graph with perfect matching index at least 5.*

Proof Assume that $\tau(G \otimes H) = 4$ and let $\mathcal{M} = \{M_1, M_2, M_3, M_4\}$ be a covering of its edge set into 4 perfect matchings. Let $\{aa', bb', cc'\}$ (with a, b and c in G and a', b' and c' in H) be the principal 3−edge cut of $G \otimes H$. From Item 2 of Proposition 2.3 there is perfect matching $M_i \in \mathcal{M}$ such that $\{aa', bb', cc'\} \subseteq M_i$. This is clearly impossible since the set of vertices of H which must be saturated by M_i is partitioned into 2 independent sets whose size differs by one unit. □

Let us consider the following construction. Given four cubic graphs $G_1^{x_1}$, $G_2^{x_2}$, $G_3^{x_3}$, $G_4^{x_4}$ together with a distinguished vertex x_i $(i = 1, 2, 3, 4)$ whose neighbors in $G_i^{x_i}$ are a_i, b_i and c_i, we get a 3-connected cubic graphs in deleting the vertices x_i $(i = 1, 2, 3, 4)$ and connecting the remaining subgraphs as described in Figure 9. In other words we define the cubic graphs denoted $K_4[G_1^{x_1}, G_2x_2, G_3^{x_3}, G_4^{x_4}]$ whose vertex set is

$$\bigcup_{i \in \{1,2,3,4\}} V(G_i^{x_i}) - \bigcup_{i \in \{1,2,3,4\}} \{x_i\}$$

while the edge set is

$$\bigcup_{i \in \{1,2,3,4\}} E(G_i^{x_i}) - \bigcup_{i \in \{1,2,3,4\}} \{a_i x_i, b_i x_i, c_i x_i\} \bigcup \{a_1c_3, b_1a_4, c_1c_2, b_2c_4, a_2c_3, b_3b_4\}.$$

For convenience G_i $(i \in \{1, 2, 3, 4\})$ will denote the induced subgraph of $G_i^{x_i}$ where the vertex x_i has been deleted.

Proposition 4.2. *Let $G_1^{x_1}$, $G_2^{x_2}$, $G_3^{x_3}$ and $G_4^{x_4}$ be 3-connected cubic graphs such that $\tau(G_1^{x_1}) \geq 5$, $\tau(G_2^{x_2}) \geq 5$, G_4 is reduced to a single vertex, say x. Then $\tau(K_4[G_1^{x_1}, G_2^{x_2}, G_3^{x_3}, G_4^{y}]) \geq 5$.*

Proof Let us denote $G = K_4[G_1^{x_1}, G_2^{x_2}, G_3^{x_3}, G_4^{x_4}]$. Observe that $a_4 = b_4 = c_4 = x$.

If $\tau(G) = 3$ the graph G would be 3-edge colourable, but in considering the 3-edge cut $\{a_1a_3, b_1a_4, c_1c_2\}$ we would have $\chi'(G_1^{x_1}) = 3$, a contradiction. Hence

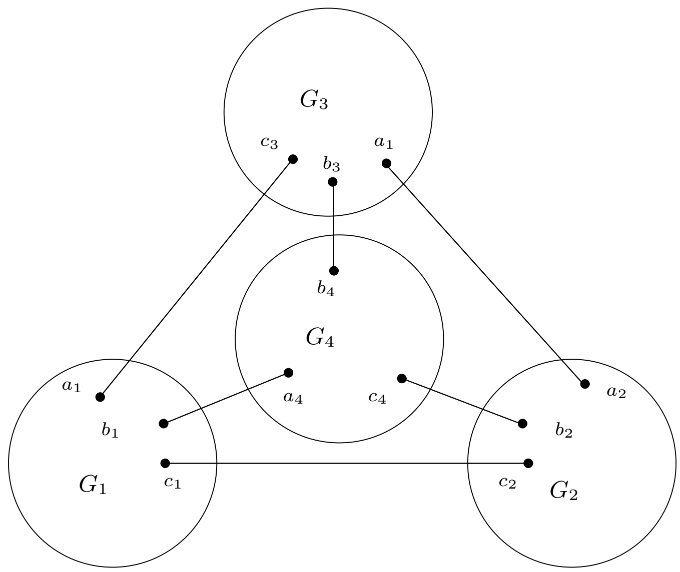

Figure 9. $K_4[G_1^{x_1}, G_2^{x_2}, G_3^{x_3}, G_4^{x_4}]$

$\tau(G) \geq 4$. Assume that $\tau(G) = 4$ and let $\mathcal{M} = \{M_1, M_2, M_3, M_4\}$ be a covering of its edge set into 4 perfect matchings.

From Item 2 of Proposition 2.3 there is perfect matching $M_i \in \mathcal{M}$ such that $\{a_1a_3, b_1a_4, c_1c_2\} \subseteq M_i$. For the same reason, there is perfect matching $M_j \in \mathcal{M}$ such that $\{c_1c_2, xb_2c_3a_2\} \subseteq M_j$. We certainly have $i \neq j$, otherwise the vertex x is incident twice to the same perfect matching M_i. Without loss of generality, we suppose that $i = 1$ and $j = 2$. Hence $c_1c_2 \in M_1 \cap M_2$. If we consider the $3-$edge cut $\{a_1a_3, b_1a_4, c_1c_2\}$, since each perfect matching must intersect this cut in an odd number of edges we must have one of the edges a_1a_3 or b_1x in M_3 while the other must be in M_4. The same holds with the $3-$edge cut $\{c_1c_2, xb_2c_3a_2\}$ and the edges b_2x and a_2c_3. Hence, we can suppose that $a_1a_3 \in M_1 \cap M_3$ and $b_1x \in M_1 \cap M_4$ as well that $b_2x \in M_2 \cap M_3$ and $a_2c_3 \in M_2 \cap M_4$, a contradiction since the set of edges contained into 2 perfect matchings of $\mathcal{M}$ is a perfect matching by Item 2 of Proposition 3.1 and x is incident to two such edges. □

We do not know any cyclically $4-$edge connected cubic graph, distinct from the Petersen graph, having a perfect matching index at least 5 and we propose as an open problem:

Problem 4.3. *Is there any cyclically $4-$edge connected cubic graph distinct from the Petersen graph with a perfect matching index at least 5?*

5. Technical tools.

In fact Theorem 3.5 can be generalized. Let M be a perfect matching, a set $A \subseteq E(G)$ is a $M-$*balanced matching* when we can find a perfect matchings M' such that $A = M \cap M'$. Assume that $\mathcal{M} = \{A, B, C\}$ are 3 pairwise disjoint

$M-$balanced matchings, we shall say that $\mathcal{M}$ is a *good family* whenever the two following conditions are fulfilled:

- i Every odd cycle C of $G\backslash M$ has exactly one vertex incident with one edge of each subset of $\mathcal{M}$ and the three paths determined by these vertices on C are odd.
- ii For every even cycle of $G\backslash M$ there are at least two matchings of $\mathcal{M}$ with no edge incident to the cycle.

Theorem 5.1. *Let G be a bridgeless cubic graph together with a good family $\mathcal{M}$. Then $\tau(G) \leq 4$.*

Sketch of the proof Let us denote M_A (resp. M_B, M_C) a perfect matching such that $M_A \cap M = A$ (resp. $M_B \cap M = B$, $M_C \cap M = C$).

Let $\mathcal{C}$ be a cycle of the 2-factor $G - M$.

When $\mathcal{C}$ is an even cycle, there are precisely two matchings on $\mathcal{C}$, namely $M_{\mathcal{C}}$ and $M'_{\mathcal{C}}$ such that $M_{\mathcal{C}} \cup M'_{\mathcal{C}}$ covers all the edge-set of $\mathcal{C}$. Since there are at least two matchings in $\{M_A, M_B, M_C\}$ that are not incident to $\mathcal{C}$, say M_A and M_B, up to a redistribution of the edges in $M_A \cap \mathcal{C}$ and $M_B \cap \mathcal{C}$ we may assume that $M_{\mathcal{C}} \subset M_A$ and $M'_{\mathcal{C}} \subset M_B$.

If $\mathcal{C}$ is an odd cycle we know that $\mathcal{C}$ has precisely one vertex which is incident to A say a, one vertex which is incident to B say b, one vertex which is incident to C say c. Without loss of generality we may assume that there is an orientation of $\mathcal{C}$ such that the path $\mathcal{C}(a, b)$ has odd length and the vertex c in $\mathcal{C}(b, a)$. We know that the path $\mathcal{C}(b, c)$ is odd thus the edge-set of $\mathcal{C}$ is covered with $M_A \cup M_B \cup M_C$. □

In the same manner we can obtain a theorem insuring the existence of a 5-covering.

Assume that $\mathcal{M} = \{A, B, C, D\}$ are 4 pairwise disjoint $M-$balanced matchings, we shall say that $\mathcal{M}$ is a *ice family* whenever the two following conditions are fulfilled:

- i Every odd cycle C of $G\backslash M$ has exactly one vertex incident with one edge of each subset of $\mathcal{M}$ and at least two disjoint paths determined by these vertices on C are odd.
- ii For every even cycle of $G\backslash M$ there are at least two matchings of $\mathcal{M}$with no edge incident to the cycle.

Theorem 5.2. *Let G be a bridgeless cubic graph together with a nice family $\mathcal{M}$. Then $\tau(G) \leq 5$.*

Proof Let us denote M_A (resp. M_B, M_C, M_D) a perfect matching such that $M_A \cap M = A$ (resp. $M_B \cap M = B$,$M_C \cap M = C$,$M_D \cap M = D$).

Let $\mathcal{C}$ be a cycle of the 2-factor $G - M$.

When $\mathcal{C}$ is an even cycle, there is at least two matchings in $\{M_A, M_B, M_C, M_D\}$ that are not incident to $\mathcal{C}$, say M_1 and M_2. As in Theorem 5.1 we may assume that the edge-set of $\mathcal{C}$ is a subset of $M_1 \cup M_2$.

If $\mathcal{C}$ is an odd cycle we know that $\mathcal{C}$ has precisely one vertex which is incident to A say a, one vertex which is incident to B say b, one vertex which is incident to C say c, one vertex which is incident to D say d. Without loss of generality we may assume that there is an orientation of $\mathcal{C}$ such that the path $\mathcal{C}(a, b)$ has odd length and the vertices c and d are in this order in (b, a). We can suppose that the path

(b, c) is even otherwise the edge-set of $\mathcal{C}$ would be covered with $M_A \cup M_B \cup M_C$. But now, since $\mathcal{C}$ is an odd cycle the path $\mathcal{C}(d, a)$ has odd length and the edge-set of $\mathcal{C}$ is a subset of $M_A \cup M_B \cup M_D$ and (M, M_A, M_B, M_C, M_D) is a 5-covering. □

In a forthcoming paper [5] we shall give an analogous theorem insuring the existence of a Fulkerson covering and some applications.

6. Odd or even coverings.

A covering of a bridgeless cubic graph being a set of perfect matchings such that every edge is contained in at least one perfect matching, we define an *odd covering* as a covering such that each edge is contained in an odd number of the members of the covering. In the same way, an *even covering* is a covering such that each edge is contained in an even number (at least 2) members of the covering. The *size* of an odd (or even) covering is its number of members. As soon as a covering is given an even covering is obtained by taking each perfect matching twice.

Proposition 6.1. *Let G be bridgeless cubic graph such that $\tau(G) = 4$. Then G has an odd covering of size 5.*

Proof Let G be a cubic graph such that $\tau(G) = 4$ and let $\mathcal{M} = \{M_1, M_2, M_3, M_4\}$ be a covering of its edge set into 4 perfect matchings. Let M be the perfect matching formed with the edges contained in exactly two perfect matchings of $\mathcal{M}$. Then we can check that $\{M, M_1, M_2, M_3, M_4\}$ cover every edge of G either one time or three times. □

Proposition 6.2. *Let G be a bridgeless cubic graph together with an odd covering $\mathcal{M}$ of size k. Then either G has an odd covering of size $k - 2$ or $\forall M, M' \in \mathcal{M}$ we have $M \neq M'$.*

Proof Assume that there are two identical perfect matchings M and M' in $\mathcal{M}$. Each edge e covered by M (and thus M') must be covered by at least another perfect matching M_e and the set $\mathcal{M} - \{M, M'\}$ is still an odd covering. The result follows. □

Proposition 6.3. *The Petersen graph has no odd covering.*

Proof Let $\mathcal{M}$ be an odd covering of the Petersen graph with minimum size. Then, by Proposition 6.2 $\mathcal{M}$ must be a set of distinct perfect matchings. The Petersen graph has exactly 6 distinct perfect matchings (inducing a Fulkerson covering, that is an even covering) and it is an easy task to check that any subset of 5 perfect matchings is not an odd covering. Since $\tau(Petersen) = 5$, the result follows. □

Seymour ([12]) remarked that the edge set of the Petersen graph is not expressible as a symmetric difference (mod 2) of its perfects matchings.

Problem 6.4. *Which bridgeless cubic graph can be provided with an odd covering ?*

We remark that $3-$edge-colorable cubic graphs as well as bridgeless cubic graph with perfect matching index 4 have an odd covering (with size 3 and 5 respectively).

Proposition 6.5. *Let G be bridgeless cubic graph without any odd covering and let H be a connected bipartite cubic graph. Then $G \otimes H$ has no odd covering.*

Proof Assume that $G \otimes H$ can be provided with an odd covering $\mathcal{M}$. Let $\{aa', bb', cc'\}$ (with a, b and c in G and a', b' and c' in H) be the principal $3-$edge cut of $G \otimes H$. None of the perfect matchings of $\mathcal{M}$ can contain the principal $3-$edge cut since the set of vertices of H which must be saturated by such a perfect matching is partitioned into 2 independent sets whose size differs by one unit. Hence every perfect matching of $M \in \mathcal{M}$ contains exactly one edge in $\{aa', bb', cc'\}$ and leads to a perfect matching M' of G. The set $\mathcal{M}'$ of perfect matchings so obtained is an odd covering of G, a contradiction. □

Proposition 6.6. *Let $G_1^{x_1}$ and $G_2^{x_2}$ be cubic graphs with distinguished vertices x_1 and x_2 such that $\tau(G_i^{x_i}) \geq 5$ $(i = 1, 2)$ and $\tau_{odd}(G_i^{x_i}) \neq 5$ $(i = 1, 2)$. Let $G_4^{x'}$ and $G_3^{y'}$ be two copies of the cubic graph on two vertices and $G = K_4[G_1^{x_1}, G_2^{x_2}, G_3^{y'}, G_4^{x'}]$, then $\tau(G) \geq 5$ and if $\tau_{odd}G$ is defined then $\tau_{odd}(G) \neq 5$.*

Proof Let x and y be respectively the unique vertex of G_4, G_3 (see Figure 9 where G_4 is reduced to a single vertex x and G_3 is reduced to y). We know by Proposition 4.2 that$\tau(G) \geq 5$. Assume that $\tau_{odd}(G) = 5$ and let $\mathcal{M} = \{M_1, M_2, M_3, M_4, M_5\}$ be an odd 5-covering. The perfect matchings of $\mathcal{M}$ are pairwise distinct otherwise by Proposition 6.2 either $G_1^{x_1}$ or $G_2^{x_2}$ would be 3-edge colorable, a contradiction. Observe that each vertex is incident to one edge that belongs to precisely three matchings of $\mathcal{M}$, the two other edges being covered only once. Moreover, the set of edges that belong to 3 matchings of $\mathcal{M}$ is a perfect matching itself.

The 3-edge cut $\{a_1y, b_1x, c_1c_2\}$ must be entirely contained in some matching of $\mathcal{M}$, say M_i otherwise we would have a 5- odd covering of $G_1^{x_1}$, a contradiction. Similarly there is a perfect matching in $\mathcal{M}$, say M_j that contains the edges c_1c_2, b_2x, a_2y. Thus the edge c_1c_2 must belong to 3 matchings of $\mathcal{M}$. Without loss of generality we assume that $i = 1$, $j = 2$ and $c_1c_2 \in M_1 \cap M_2 \cap M_3$.

If $ya_1 \in M_3$, since a perfect matching intersects any odd cut in an odd number of edges we have $xb_1 \in M_3$, it follows that the edge ya_1 must be a member of a third matching of $\mathcal{M}$ as well as the edge xb_1. If for some k we have $ya_1 \in M_k$ and $xb_1 \in M_k$, $k \in \{2, 4, 5\}$, k being obviously distinct from 2 M_k intersects the 3-edge cut in an even number of edges, a contradiction. Hence we may assume that $ya_1 \in M_4$ and $xb_1 \in M_5$. But now the edge xy is covered by none of the matchings of $\mathcal{M}$, a contradiction. Consequently $ya_1 \notin M_3$, similarly $xb_1 \notin M_3$.

If $ya_1 \in M_4$ this edge must belong to a third matching of $\mathcal{M}$ which is M_5. Since the set of edges that are covered 3 times is a perfect matching $xb_1 \in M_4 \cap M_5$. But in this case the edge c_1c_2 would belong to M_4 and M_5, a contradiction.

It follows that ya_1 as well as xb_2 are covered only once and the edge xy belongs to 3 matchings of $\mathcal{M}$, that is $xy \in M_3 \cap M_4 \cap M_5$. But now, neither M_4 nor M_5 intersect the edge-cut $\{ya_1, xb_1, c_1c_2\}$ a contradiction since a perfect matching must intersect every odd edge-cut in an odd number of edges. □

The graph G depicted in Figure 10 is an example of cubic graphs with a 7-odd covering and a perfect matching index equals to 5. We know by Proposition 6.6

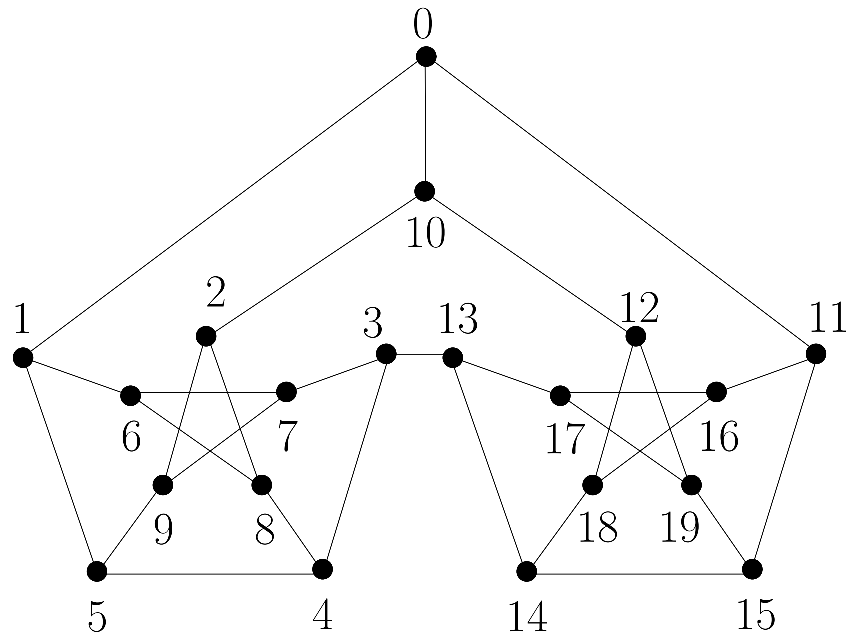

FIGURE 10. A graph G such that $\tau(G) = 5$ and $\tau_{odd}(G) = 7$.

that $\tau_{odd}(G) \geq 7$. As a matter of fact, this graph has 20 distinct perfect matchings and among all the 7-tuples of perfect matchings (77520) 64 form an odd-covering. Let us give below such a 7-tuple.

$\{0-10,\ 1-5,\ 2-9,\ 3-13,\ 4-8,\ 6-7,\ 11-15,\ 12-19,\ 14-18,\ 16-17\}$
$\{0-1,\ 2-8,\ 3-4,\ 5-9,\ 6-7,\ 10-12,\ 11-15,\ 13-14,\ 16-18,\ 17-19\}$
$\{0-1,\ 2-10,\ 3-13,\ 4-5,\ 6-8,\ 7-9,\ 11-15,\ 12-19,\ 14-18,\ 16-17\}$
$\{0-1,\ 2-10,\ 3-13,\ 4-8,\ 5-9,\ 6-7,\ 11-16,\ 12-18,\ 14-15,\ 17-19\}$
$\{0-11,\ 1-5,\ 2-9,\ 3-13,\ 4-8,\ 6-7,\ 10-12,\ 14-15,\ 16-18,\ 17-19\}$
$\{0-11,\ 1-5,\ 2-9,\ 3-13,\ 4-8,\ 6-7,\ 10-12,\ 14-18,\ 15-19,\ 16-17\}$
$\{0-11,\ 1-6,\ 2-10,\ 3-7,\ 4-8,\ 5-9,\ 12-19,\ 13-17,\ 14-15,\ 16-18\}$

Moreover the following perfect matchings form a 5-covering.

$\{0-1,\ 2-10,\ 3-13,\ 6-8,\ 7-9,\ 4-5,\ 12-19,\ 16-17,\ 14-18,\ 11-15\}$
$\{2-9,\ 1-6,\ 7-9,\ 4-5,\ 3-13,\ 0-11,\ 10-12,\ 14-15,\ 16-18,\ 17-19\}$
$\{1-6,\ 7-9,\ 2-8,\ 5-4,\ 0-10,\ 12-18,\ 17-19,\ 14-15,\ 11-15,\ 3-13\}$
$\{0-1,\ 2-8,\ 6-7,\ 5-9,\ 3-4,\ 10-12,\ 13-17,\ 14-18,\ 15-19,\ 11-16\}$
$\{1-6,\ 5-9,\ 4-8,\ 3-7,\ 2-10,\ 0-11,\ 12-18,\ 13-14,\ 15-19,\ 16-17\}$

We do not know any example of graph G for which τ_{odd} is defined and with $\tau(G) = \tau_{odd}(G) = 5$. We just observe that in such a graph every vertex would be incident to an edge belonging to 3 perfect matchings and to precisely two edges covered only once. The set of edges covered by 3 perfect matchings being a perfect matching itself.

Problem 6.7. *Is it true that every bridgeless cubic graph has an even covering where each edge appears twice or* 4 *times ?*

The answer is yes for $3-$edge-colorable cubic graphs and for bridgeless cubic graphs with perfect matching index 4 since such graphs have an even covering of size 8.

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