On the performance of Shewhart-type synthetic and runs-rules charts combined with an $\bar{X}$ chart

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Abstract

A synthetic $\bar{X}$ chart is a combination of a conforming run-length chart and an $\bar{X}$ chart, or equivalently, a *2-of-*(*H+1*) runs-rules (RR) chart with a head-start feature. However, a synthetic $\bar{X}$ chart combined with an $\bar{X}$ chart is called a Synthetic-$\bar{X}$ chart. In this article, we build a framework for Shewhart Synthetic-$\bar{X}$ and improved RR (i.e. *1-of-1* or *2-of-*(*H+1*) without head-start) charts by conducting an in-depth zero-state and steady-state study to gain insight into the design of different classes of these schemes and their performance using the Markov chain imbedding technique. More importantly, we propose a modified side-sensitive Synthetic-$\bar{X}$ chart, and then using overall performance measures we show that this new chart has a uniformly better performance than its Shewhart competitors. We also provide easy to use tables for each of the chart's design parameters to aid practical implementation. Moreover, a performance comparison with their corresponding counterparts (i.e. synthetic $\bar{X}$ and RR charts) is conducted.

Keywords: Extra quadratic loss, Runs-rules, Steady-state, Synthetic chart, Zero-state

1. Introduction

In statistical process control and monitoring (SPCM) field, control charts are usually used to distinguish between the chance and the assignable causes of variation. When a process has only chance causes of variation present, it is said to be statistically in-control (IC), otherwise, the process is said to be out-of-control (OOC). Assume that $\{X_{ij}: i \geq 1; j = 1, 2,\ldots, n\}$ is a sequence of samples from iid $N(\mu_0, \sigma_0^2)$ distribution where μ_0 and σ_0^2 are the specified IC mean and variance, respectively. Let $\bar{X}_i$ denote the plotting statistic calculated from $\{X_{ij}\}$ at sampling point i. A Shewhart control chart that is usually used to monitor $\bar{X}_i$ is called the $\bar{X}$ chart and it signals when a single plotting statistic falls above the upper control limit (UCL) or below the lower control limit (LCL) which are given by

$$UCL = \mu_0 + k\sigma_0, CL = \mu_0, LCL = \mu_0 - k\sigma_0, \quad (1)$$

where k is the distance of the control limits from the center line (CL).

A Shewhart $\bar{X}$ chart is known to be more effective in detecting (sudden) large process shifts in the process mean, however, it is poor in detecting small and moderate mean shifts. Hence, there has been a variety of alternative control charts proposed in the literature to efficiently monitor the process mean. Amongst the most popular control charts, we point out a few; these are cumulative sum (CUSUM), exponentially weighted moving average (EWMA) and control charts based on the change point model. In order to further increase the sensitivity of Shewhart, CUSUM and EWMA $\bar{X}$

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charts, various adaptations and generalizations of these basic charts have been considered, for example, the variable sampling interval, variable sample size, variable sample size and interval, variable parameter, double sampling, supplementing runs-rules (RR) and integrating these schemes with other monitoring schemes; see for instance Celano et al.[1], Reynolds and Arnold[2], Daudin[3], Koutras et al.[4], Scariano and Calzada[5], Costa[6], Wu and Spedding[7], Khoo et al.[8], Abbas et al.[9,10], Haq et al.[11], etc. In this paper, we concentrate on the two latter adaptations i.e. the supplementary RR schemes and integrating the Shewhart $\bar{X}$ scheme with the conforming run-length (CRL) scheme – called a synthetic chart; we discuss these two next.

Firstly, a "run" is defined as an uninterrupted sequence of the same elements bordered at each end by other types of elements, see Balakrishnan and Koutras[12]. Koutras et al.[4] gave a literature review on Shewhart-type charts with supplementary RR to improve an $\bar{X}$ chart in detecting small shifts. Most notable of these, is the paper by Klein[13], where the author proposed two powerful rules that improved the ability of the $\bar{X}$ chart to detect small shifts, by means of a Markov chain approach, whereby the control limits in Equation (1) can be adjusted to give the desired ARL_0. The two rules suggested by Klein[13] are the *2-of-2* and *2-of-3*. Later, Acosta-Mejia[14] studied the general form of these rules i.e. w*-of-*$(w+v)$ where $w \geq 2$ and $v \geq 0$ are specified positive integers. Note that both Klein[13] and Acosta-Mejia[14] showed that these rules require at least w plotting statistics before an OOC event can be observed, hence, the $\bar{X}$ chart is more effective than these rules in detecting large shifts as it requires 1 plotting statistic to issue a signal. Thus, to increase the performance of the rules in Klein[13], Khoo and Ariffin[15] proposed two improved runs-rules (IRR) $\bar{X}$ schemes which are a combination of the basic $\bar{X}$ chart and the two rules of Klein[13] i.e. *1-of-1* or *2-of-2* and *1-of-1* or *2-of-3*. Then, Acosta-Mejia[14] studied the general form of the Khoo and Ariffin[15] rules i.e. *1-of-1* or w*-of-*$(w+v)$, which signal an OOC event when one plotting statistic falls beyond either the LCL or UCL; or when w out of $w+v$ plotting statistics fall in between the UCL (LCL) and upper (lower) warning limits (i.e. UWL (LWL)), respectively. That is, the limits of the IRR $\bar{X}$ scheme are given by

$$UCL = \mu_0 + k_1\sigma_0,\ UWL = \mu_0 + k_2\sigma_0,\ CL = \mu_0,\ LWL = \mu_0 - k_2\sigma_0,\ LCL = \mu_0 - k_1\sigma_0 \qquad (2)$$

where k_1 and k_2 are the distances of the control and warning limits from the CL, respectively, with $k_1 > k_2$. Khoo and Ariffin[15] showed that the IRR retains the small shift sensitivity of the rules in Klein[13] and has a better performance for large shifts compared to the Klein[13] rules.

Secondly, a synthetic $\bar{X}$ chart to monitor the process mean consists of two sub-charts, one, a basic $\bar{X}$ chart and a second, a CRL chart. For a synthetic chart, an OOC signal is not based on a single plotting statistic falling beyond the control limits in Equation (1), instead, when a sample produces a value beyond the control limits in Equation (1), that sample is marked as nonconforming and the control procedure moves to the second sub-chart and a signal is obtained depending on the outcome of the *CRL* sub-chart. Note that when a sample produces a value falling between *LCL* and *UCL*, then that sample is marked as conforming. Bourke[16] defined a *CRL* as the number of conforming samples between two consecutive nonconforming samples, inclusive of the nonconforming sample at the end. The *CRL* chart signals when an observed *CRL* value is less than or equal to some threshold, say *H* (an integer, greater or equal to 1), which is defined to be the threshold / control limit of the *CRL* chart. To make the run-length analysis of the synthetic chart easier, Davis and Woodall[17] showed that a synthetic chart is a special case of a RR chart i.e. a *2-of-*(*H*+1) with a head-start (HS) feature. The HS feature implies that we assume that (at time 0) the first observation is nonconforming, consequently, we need at least one other nonconforming sample within the next *H* sampling points, for a *2-of-*(*H*+1) runs-type chart to issue a signal. Similar to Khoo and Ariffin[15], Wu et al.[18] improved the sensitivity of the synthetic $\bar{X}$ chart of Wu and Spedding[7] to efficiently monitor both small and large shifts called a Synthetic-$\bar{X}$ chart. A Synthetic-$\bar{X}$ chart consists of two sub-charts, one, a synthetic $\bar{X}$ chart and second, a basic $\bar{X}$ chart. An OOC signal is observed when one plotting statistic falls beyond either the *LCL* or *UCL* in Equation (2); or when $CRL \leq H$.

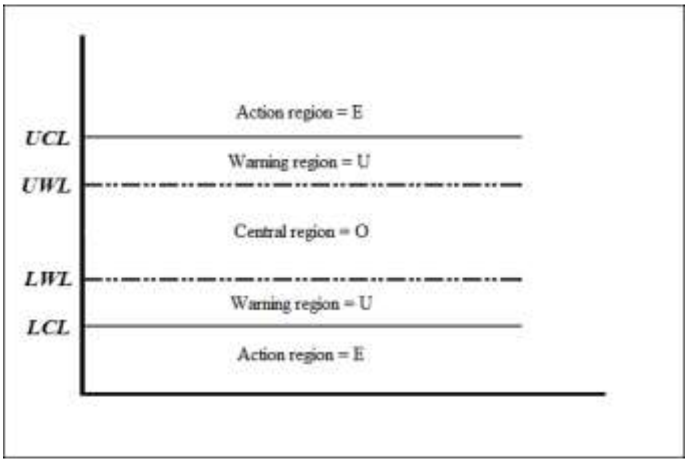
The performance of the RR and synthetic $\bar{X}$ charts based on Equation (1) have been investigated and reported in Shongwe and Graham[19]. The focus of this paper is on the performance of the IRR and Synthetic-$\bar{X}$ charts with the limits in Equation (2) – these are summarized in Table I. For these charts, when a sample produces a value between the *UCL* (*LCL*) and *UWL* (*LWL*), that sample is marked as nonconforming, whereas, a value between the *UWL* and *LWL* is marked as conforming.

Table I: Types of IRR and Synthetic-$\bar{X}$ charts

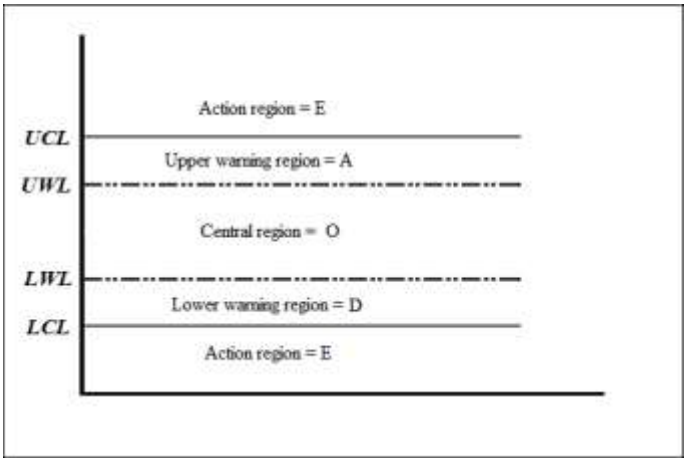
IRR $\bar{X}$ charts	Synthetic-$\bar{X}$ charts
(i) Non-side-sensitive (NSS) *1-of-1* or *w-of-*(*w*+*v*): • Discussed here – (IRR1)	(i) NSS *1-of-1* or *2-of-*(*H*+1): • Wu et al.[18] – (SC1)
(ii) Standard side-sensitive (SSS) *1-of-1* or *w-of-*(*w*+*v*): • Khoo and Ariffin[15] – (IRR2)	(ii) SSS *1-of-1* or *2-of-*(*H*+1): • Discussed here – (SC2)
(iii) Revised side-sensitive (RSS) *1-of-1* or *w-of-*(*w*+*v*): • Discussed here – (IRR3)	(iii) RSS *1-of-1* or *2-of-*(*H*+1): • Machado and Costa[21] – (SC3)
(iv) Modified side-sensitive (MSS) *1-of-1* or *w-of-*(*w*+*v*): • Antzoulakos and Rakitzis[22] – (IRR4)	(iv) MSS *1-of-1* or *2-of-*(*H*+1): • Proposed in this paper – (SC4)

Firstly, there are four types of IRR $\bar{X}$ schemes that are based on the limits in Equation (2) which are as follows:

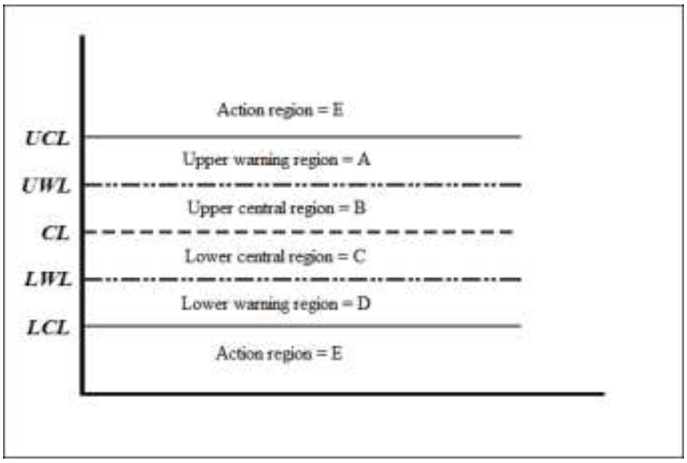
(i) the NSS *1-of-1* or *w-of-*(*w*+*v*) (adopted from Derman and Ross[20] – denoted by IRR1) with the charting regions shown in Figure 1(a). Hence, IRR1 gives an OOC signal when either one plotting statistic falls on the action region (i.e. region E), or when *w* nonconforming

(ii) 

(a) NSS regions



(b) SSS / RSS regions



(c) MSS regions

Figure 1: The control / warning limits and the corresponding regions

samples out of $w+v$ successive samples fall on the warning region (i.e. region U), which are separated by at most v conforming samples that fall on the central region (i.e. region O).

(iii)the SSS *1-of-1* or *w-of*-($w+v$) (by Khoo and Ariffin[15] – denoted by IRR2) with the charting regions shown in Figure 1(b). Hence, IRR2 gives an OOC signal when either one plotting statistic falls on the action region (i.e. region E), or when w nonconforming samples out of $w+v$ successive samples fall on the upper (lower) warning region which are separated by at most v samples that fall on either the central region or the lower (upper) warning region, respectively.

(iv) the RSS *1-of-1* or *w-of-*($w+v$) (adopted from Machado and Costa[21] – denoted by IRR3) with the charting regions shown in Figure 1(b). Hence, IRR3 gives an OOC signal when either one plotting statistic falls on the action region (i.e. region E), or when w nonconforming samples out of $w+v$ successive samples fall on the upper (lower) warning region which are separated by at most v conforming samples that fall on the central region, respectively.

(v) the MSS *1-of-1* or *w-of-*($w+v$) (by Antzoulakos and Rakitzis[22] – denoted by IRR4) with the charting regions shown in Figure 1(c). Hence, IRR4 gives an OOC signals when either one plotting statistic falls on the action region (i.e. region E), or when w nonconforming samples out of $w+v$ successive samples fall on the upper (lower) warning region which are separated by at most v conforming samples that fall on the upper (lower) central region, respectively.

Secondly, there are three types of Synthetic-$\bar{X}$ schemes that are based on the limits in Equation (2) which are as follows:

(i) the NSS Synthetic-$\bar{X}$ chart (by Wu et al.[18] – denoted by SC1) with the charting regions shown in Figure 1(a). Hence, SC1 gives an OOC signal when either one plotting statistic falls on the action region (i.e. region E), or when two nonconforming samples out of H+1 successive samples fall on the warning region (i.e. region U), which are separated by at most H-1 conforming samples that fall on the central region (i.e. region O).

(ii) the SSS Synthetic-$\bar{X}$ chart (adopted from Davis and Woodall[17] – denoted by SC2) with the charting regions shown in Figure 1(b). Hence, SC2 gives an OOC signal when either one plotting statistic falls on the action region (i.e. region E), or when two nonconforming samples out of H+1 successive samples fall on the upper (lower) warning regions which are separated by at most H-1 samples that fall on either the central region or the lower (upper) warning region, respectively.

(iii) the RSS Synthetic-$\bar{X}$ chart (by Machado and Costa[21] – denoted by SC3) with the charting regions shown in Figure 1(b). Hence, SC3 gives an OOC signal when either one plotting statistic falls on the action region (i.e. region E), or when two nonconforming samples out of H+1 successive samples fall on the upper (lower) warning region which are separated by at most v conforming samples that fall on the central region, respectively.

For a fair comparison, we only consider IRR charts with w = 2 so that v = H-1 and $w+v$ = H+1. Unlike the *2-of-*(H+1) rules based on Equation (1) with two parameters i.e. H and k; Equation (2) has three design parameters, i.e. H, k_1 and k_2. The three types of Synthetic-$\bar{X}$ charts in Table I (i.e. SC1, SC2 and SC3) have runs-type rules similar to those in IRR1, IRR2 and IRR3, respectively. Therefore, the aim of this paper is to supplement on the work done by the authors listed in Table I, by proposing a Synthetic-$\bar{X}$ chart that has runs-type rules similar to those in Antzoulakos and Rakitzis[22], called the modified side-sensitive Synthetic-$\bar{X}$ chart, denoted by SC4. That is, this paper makes a contribution to both the synthetic-type and runs-type $\bar{X}$ schemes by:

- prosing a new Shewhart-type Synthetic-$\bar{X}$ chart;
- using a Markov chain approach, we employ a design criterion with more emphasis on the overall performance to investigate the effectiveness of the *1-of-1* or *2-of-*(H+1) $\bar{X}$ runs-type charts as H increases i.e. $H \leq 20$.
- we give recommendations on what the optimal value of H should be used, so that the corresponding chart (i.e. IRR1, IRR2, IRR3, IRR4, SC1, SC2, SC3 and SC4) each results in the best overall performance; depending on the upper bound on the range of shifts.
- to supplement on the zero-state (ZS) average run-length (*ARL*) performance of the IRR2 and IRR4 in Khoo and Ariffin[15] and Antzoulakos and Rakitzis[22], respectively, we study the steady-state (SS) performance of these charts. Moreover, we evaluate the ZS performance of the SC3 scheme, then propose its ZS and SS mode IRR version (i.e. IRR3) and we evaluate the ZS and SS performance of the IRR1 and SC2 schemes.
- Finally, we compare the overall performance of the schemes in Table I with their corresponding counterparts that are based on Equation (1) discussed in Shongwe and Graham[19].

The goal of this paper is to compare a variety of Shewhart synthetic-type and runs-type $\bar{X}$ schemes to monitor the process mean for normally distributed data. Thus, the schemes discussed herein will not outperform more advanced schemes like the basic EWMA, CUSUM, or EWMA and CUSUM with IRR, respectively, (with or without a HS feature); an interested reader may see for instance [5, 8-11, 23]. Thus, this paper must be considered as a framework for quality practitioners who utilise Shewhart synthetic and runs-rules charts. The rest of the paper is as follows: In Section 2, we present the operation of the MSS Synthetic-$\bar{X}$ chart as well as the ZS and SS performance measures using the Markov chain imbedding technique discussed in the Appendix. In Section 3, we evaluate the OOC performance of the proposed Synthetic-$\bar{X}$ chart and compare its specific shift and overall performance with the Shewhart-type charts listed in Table I. In Section 4, we compare the overall performance of the schemes in Table I with their counterparts based on Equation (1). Finally, in Section 5 we give concluding remarks.

2. Operation and design considerations

2.1 Operation of the modified side-sensitive Synthetic-$\bar{X}$ chart

Consider Figure 1(c), the MSS Synthetic-$\bar{X}$ (i.e. SC4) chart signals when either one plotting statistic falls on the action region (i.e. region E), or when two nonconforming samples out of H+1 successive samples fall on the upper (lower) warning region which are separated by at most H-1 conforming samples that fall on the upper (lower) central region, respectively. This means that the SC4 chart signals when *all* the H+1 consecutive samples that lead to an OOC event fall on one side of the CL. To clearly describe the operation of the SC4 chart, we need to define two types of CRLs i.e. lower CRL (denoted by CRL_{L}) and upper CRL (denoted by CRL_{U}). A CRL_{L} is the number conforming samples (i.e. falling on region C) that are plotted in between the two consecutive nonconforming samples on region D, inclusive of the nonconforming sample at the end. However, a CRL_{U} is the number conforming samples (i.e. falling on region B) that are plotted in between the two consecutive nonconforming samples on region A, inclusive of the nonconforming sample at the end. Note that the absence of a conforming sample implies that either the CRL_{U} or CRL_{L} equals 1. Thus the SC4 chart operates as follows:

Step (i) On the next sampling point, take a sample of size n and compute $\bar{X}_i$.

Step (ii) If $\bar{X}_i \leq LCL$ or $\bar{X}_i \geq UCL$ go to Step (vii).

Step (iii) If $LWL < \bar{X}_i < UWL$ then return to Step (i).

Step (iv) If $LCL < \bar{X}_i \leq LWL$ go to Step (v), or if $UWL \leq \bar{X}_i < UCL$ go to Step (vi).

Step (v) If $CRL_{\mathrm{L}} \leq H$ go to Step (vii), otherwise return to Step (i).

Step (vi) If $CRL_{\mathrm{U}} \leq H$ go to Step (vii), otherwise return to Step (i).

Step (vii) Issue an OOC signal, and then take necessary corrective action to find and remove the assignable causes. Then return to Step (i).

2.2 Design considerations

The Markov chain imbedding technique, its properties as well as the construction of the TPMs for each of the schemes listed in Table I is discussed in the Appendix. Moreover, illustrative examples with H =1 and 5 are shown in the Appendix.

The performance of a chart at some specific shift is usually measured by ARL given by

$$ARL(\delta) = \boldsymbol{\xi}(\mathbf{I} - \mathbf{Q}(\delta))^{-1}\mathbf{1} \quad (3)$$

where $\boldsymbol{\xi}$ is the initial probability vector (depending on whether a ZS or SS mode is of interest by the user). In SPCM, ZS and SS modes are used to characterize short and long term run-length properties of a control chart, respectively. The ZS run-length is the number of sampling points at which the chart first signals given it begins in some initial state, that is, we assume that the mean shift always takes place at the beginning of the process and thus all sample points within the span are taken under the OOC condition. However, the SS run-length is the number of sampling points at which the chart first signals given that the process begins and stays IC for a very long time, then at some random time, an OOC signal is observed, see Champ[24], Davis and Woodall[17], Zhang and Wu[25] and Machado and Costa[21,26]. This is further discussed in the Appendix. The *ARL* based on the ZS mode is denoted by *ZSARL*, whereas that based on the SS mode is denoted by *SSARL*. While the *ZSARL* and *SSARL* at $\delta = 0$ are the same, the OOC (i.e. $\delta \neq 0$) *ZSARL* and *SSARL* are usually not the same.

Because the *ARL* only measures the performance of a chart at some specific shifts, a number of authors have suggested the use of additional indices to measure the overall performance of the charts, see for example, Wu et al.[27], Abujiya et al.[28] and Machado and Costa[21]. Since it is usually unknown what specific shift value(s) a control chart should be optimized for, Wu et al.[27] stated that it is more efficient to design a chart such that it has a better overall performance than its competitors. Thus, when the aim is to measure the overall performance of the chart over a range of shifts (i.e. $0 < \delta \leq \delta_{max}$, where δ_{max} is the upper bound of the mean shift that is of interest by the user), the objective function must be defined in terms of the extra quadratic loss (*EQL*) i.e.

$$EQL = \frac{1}{\delta_{max}} \int_0^{\delta_{max}} \delta^2 \times ARL(\delta)\, d\delta = minimum. \tag{4}$$

Since it is generally assumed that all mean shifts within the range $0 < \delta \leq \delta_{max}$ occur with equal probability, a uniform distribution of δ is implied, see Wu et al.[27] and Machado and Costa[21]. Hence Equation (4) may equivalently be written as

$$EQL = \frac{1}{\delta_{max}} \sum_0^{\delta_{max}} \delta^2 \times ARL(\delta) = minimum. \tag{5}$$

Here, we use a step shift (δ) of size 0.1. In addition to the *EQL*, Wu et al.[27] suggested the performance comparison index (*PCI*) to measure the relative effectiveness of two different charts. In this paper, we shall use the SC4 scheme as the benchmark; hence the *PCI* is given by

$$PCI = \frac{EQL}{EQL_{SC4}}, \tag{6}$$

where EQL_{SC4} is *EQL* of the 'benchmark' SC4 scheme. Also, the ratio of the *ARL*s is usually used to measure the overall effectiveness of a benchmark chart against other competitors, see Wu et al.[27]. Hence, assuming a uniform distribution in δ, the average ratio of *ARL*s is given by

$$ARARL = \frac{1}{\delta_{max}} \sum_0^{\delta_{max}} \frac{ARL(\delta)}{ARL_{SC4}(\delta)} \tag{7}$$

where $ARL_{SC4}(\delta)$ is the *ARL* produced by the SC4 scheme. If the value of *PCI* or *ARARL* is larger than one, the competing chart will produce larger OOC *ARL* over a larger shift range and / or to a larger degree compared to the SC4 scheme and thus, the competing chart is relatively less effective.

However, if the *PCI* or *ARARL* is smaller than one, the competing chart will have higher overall effectiveness than the SC4 scheme.

In the next section, we conduct an empirical study of the SC4 scheme and compare its performance to the schemes listed in Table I for the ZS and SS mode.

3. Discussion

Firstly, we use Equations (2) and (3) in conjunction with Equations (8) to (15) and Tables IX to XII (in the Appendix) to determine the corresponding values of k_2 for k_1 = 3.1, 3.2,…, 5.0 when H = 1, 2,…, 20 so that ARL_0 = 200, 370.4, 500, 1000. However, due to space limitation, only the results relating to ARL_0 = 370.4 are reported here. Hence, in Table II we show the values of k_2 for k_1 = 3.1, 3.5, 4.0, 5.0 when H = 1, 2,…, 20 such that ARL_0 = 370.4. In Table II, we observe the following: (i) As H increases, the values of k_2 for the SC4 and IRR4 schemes converge to some specific value for each k_1 – this is not the case for the other schemes; (ii) When H = 1, we see that IRR2≡IRR3≡IRR4 and SC2≡SC3≡SC4 in ZS mode, whereas in SS mode, IRR1≡SC1 and IRR2≡IRR3≡IRR4≡SC2≡SC3≡SC4 – this is explained in Tables IX to XII in the Appendix.

Secondly, in Tables III and IV, we compute the *EQL* and *PCI* when H = 1 and 5 with δ_{max} = 5. For each scheme, as k_1 increase from 3.1 to 5.0 (for some corresponding k_2), the smallest *EQL* value is boldfaced. For instance, the S4 scheme in ZS mode with H = 1, k_1 = 3.7, k_2 = 1.8167 results in an *EQL* = 161.65. Furthermore, for each k_1 we compute the *PCI* (in bracket) with the SC4 scheme used as the benchmark in ZS and SS, respectively. We see that when k_1 = 4.0, the (SC2, SC3, SC4) schemes are 19.13%, 34.83% and 55.78% better than the (SC1), (IRR2, IRR3, IRR4) and (IRR1) schemes in ZS mode, respectively; whereas in SS mode, (SC2, SC3, SC4, IRR2, IRR3, IRR4) schemes are 15.33% better than the IRR1 and SC1 schemes. When H = 5, the ZS *EQL* of the SC4 scheme is uniformly better than the other schemes, whereas the SS *EQL* of the SC4 and IRR4 are jointly better than the other schemes.

Thirdly, The OOC *ZSARL* and *SSARL* performance for each of the schemes with minimum *EQL* highlighted in bold in Tables III and IV are shown in Tables V and VI, respectively. We see that the SC4 scheme has a uniformly better OOC *ZSARL* performance; however, in SS, the SC4 scheme is outperformed by the other schemes in some cases once δ > 3. Based on Tables III to VI, the overall performance measure i.e. *EQL*, *PCI* and *ARARL* of the SC4 scheme are either less or equal to those of the other schemes (with equality occurring only when H = 1). The results in these tables are further illustrated in Figure 2.

Table II: The values of k_2 for the zero-state and steady-state *1-of-1* or *2-of-*(H+1) IRR and Synthetic-$\bar{X}$ charts when k_1 = 3.1, 3.5, 4.0, 5.0, H = 1,…, 20 and ARL_0 = 370.4

	k_1 = 3.1											
	Zero-state								Steady-state			
H	IRR1	IRR2	IRR3	IRR4	SC1	SC2	SC3	SC4	IRR1 & SC1	IRR2 & SC2	IRR3 & SC3	IRR4 & SC4
1	2.1705	2.0393	2.0393	2.0393	2.1884	2.0664	2.0664	2.0664	2.1710	2.0398	2.0398	2.0398
2	2.2907	2.1690	2.1677	2.1150	2.3144	2.2046	2.2035	2.1469	2.2914	2.1698	2.1685	2.1157
3	2.3568	2.2406	2.2386	2.1425	2.3846	2.2822	2.2807	2.1765	2.3576	2.2416	2.2395	2.1433
4	2.4017	2.2894	2.2869	2.1544	2.4327	2.3359	2.3340	2.1894	2.4027	2.2906	2.2881	2.1553
5	2.4354	2.3262	2.3232	2.1600	2.4693	2.3767	2.3746	2.1954	2.4367	2.3276	2.3246	2.1609
6	2.4622	2.3555	2.3522	2.1626	2.4984	2.4096	2.4073	2.1982	2.4636	2.3571	2.3538	2.1635
7	2.4843	2.3809	2.3762	2.1638	2.5227	2.4378	2.4346	2.1996	2.4859	2.3826	2.3779	2.1648
8	2.5030	2.4004	2.3965	2.1644	2.5433	2.4605	2.4580	2.2002	2.5048	2.4023	2.3984	2.1654
9	2.5192	2.4182	2.4141	2.1647	2.5613	2.4809	2.4784	2.2006	2.5211	2.4203	2.4162	2.1657
10	2.5335	2.4339	2.4296	2.1649	2.5772	2.4991	2.4964	2.2007	2.5355	2.4362	2.4319	2.1659
11	2.5461	2.4480	2.4435	2.1649	2.5914	2.5153	2.5126	2.2008	2.5484	2.4504	2.4459	2.1659
12	2.5575	2.4606	2.4559	2.1650	2.6041	2.5300	2.5273	2.2008	2.5599	2.4632	2.4585	2.1660
13	2.5678	2.4721	2.4672	2.1650	2.6158	2.5435	2.5407	2.2009	2.5704	2.4749	2.4700	2.1660
14	2.5773	2.4826	2.4775	2.1650	2.6265	2.5558	2.5530	2.2009	2.5799	2.4855	2.4805	2.1660
15	2.5859	2.4922	2.4870	2.1650	2.6363	2.5672	2.5644	2.2009	2.5887	2.4953	2.4902	2.1660
16	2.5939	2.5012	2.4958	2.1650	2.6454	2.5778	2.5749	2.2009	2.5969	2.5044	2.4991	2.1660
17	2.6014	2.5095	2.5040	2.1650	2.6539	2.5877	2.5848	2.2009	2.6044	2.5129	2.5074	2.1660
18	2.6083	2.5172	2.5116	2.1650	2.6619	2.5969	2.5940	2.2009	2.6115	2.5208	2.5152	2.1660
19	2.6147	2.5245	2.5187	2.1650	2.6693	2.6056	2.6027	2.2009	2.6181	2.5282	2.5225	2.1660
20	2.6208	2.5313	2.5254	2.1650	2.6763	2.6138	2.6109	2.2009	2.6243	2.5352	2.5294	2.1660

	k_1 = 3.5											
	Zero-state								Steady-state			
H	IRR1	IRR2	IRR3	IRR4	SC1	SC2	SC3	SC4	IRR1 & SC1	IRR2 & SC2	IRR3 & SC3	IRR4 & SC4
1	1.9698	1.8221	1.8221	1.8221	1.9818	1.8401	1.8401	1.8401	1.9703	1.8227	1.8227	1.8227
2	2.1046	1.9671	1.9646	1.9056	2.1207	1.9909	1.9886	1.9269	2.1054	1.9680	1.9654	1.9064
3	2.1794	2.0475	2.0435	1.9357	2.1985	2.0754	2.0720	1.9585	2.1804	2.0486	2.0446	1.9366
4	2.2306	2.1026	2.0976	1.9486	2.2523	2.1340	2.1298	1.9721	2.2319	2.1039	2.0989	1.9496
5	2.2693	2.1442	2.1384	1.9545	2.2931	2.1786	2.1738	1.9782	2.2708	2.1458	2.1399	1.9555
6	2.3002	2.1775	2.1710	1.9572	2.3260	2.2145	2.2092	1.9811	2.3019	2.1793	2.1727	1.9583
7	2.3259	2.2072	2.1980	1.9585	2.3534	2.2463	2.2389	1.9825	2.3277	2.2092	2.2000	1.9596
8	2.3477	2.2287	2.2210	1.9591	2.3768	2.2703	2.2643	1.9832	2.3498	2.2309	2.2232	1.9602
9	2.3666	2.2492	2.2410	1.9594	2.3973	2.2929	2.2865	1.9835	2.3689	2.2516	2.2435	1.9605
10	2.3834	2.2673	2.2586	1.9595	2.4154	2.3129	2.3062	1.9836	2.3858	2.2699	2.2613	1.9606
11	2.3983	2.2835	2.2744	1.9596	2.4317	2.3309	2.3240	1.9837	2.4010	2.2863	2.2773	1.9607
12	2.4118	2.2981	2.2886	1.9597	2.4464	2.3472	2.3401	1.9837	2.4146	2.3011	2.2917	1.9608
13	2.4240	2.3114	2.3015	1.9597	2.4599	2.3621	2.3548	1.9837	2.4271	2.3147	2.3048	1.9608
14	2.4352	2.3236	2.3133	1.9597	2.4723	2.3758	2.3683	1.9837	2.4385	2.3271	2.3169	1.9608
15	2.4456	2.3348	2.3242	1.9597	2.4837	2.3886	2.3809	1.9837	2.4490	2.3385	2.3280	1.9608
16	2.4552	2.3452	2.3343	1.9597	2.4944	2.4004	2.3926	1.9837	2.4588	2.3491	2.3383	1.9608
17	2.4641	2.3549	2.3437	1.9597	2.5043	2.4115	2.4035	1.9837	2.4679	2.3590	2.3479	1.9608
18	2.4724	2.3640	2.3525	1.9597	2.5136	2.4219	2.4138	1.9837	2.4764	2.3683	2.3569	1.9608
19	2.4802	2.3725	2.3607	1.9597	2.5224	2.4317	2.4235	1.9837	2.4844	2.3771	2.3653	1.9608
20	2.4875	2.3806	2.3685	1.9597	2.5307	2.4410	2.4326	1.9837	2.4919	2.3853	2.3733	1.9608

	k_1 = 4											
	Zero-state								Steady-state			
H	IRR1	IRR2	IRR3	IRR4	SC1	SC2	SC3	SC4	IRR1 & SC1	IRR2 & SC2	IRR3 & SC3	IRR4 & SC4
1	1.9370	1.7866	1.7866	1.7866	1.9483	1.8035	1.8035	1.8035	1.9376	1.7872	1.7872	1.7872
2	2.0742	1.9341	1.9313	1.8713	2.0893	1.9564	1.9539	1.8914	2.0750	1.9350	1.9322	1.8721
3	2.1503	2.0159	2.0115	1.9018	2.1684	2.0422	2.0384	1.9232	2.1514	2.0170	2.0126	1.9027
4	2.2025	2.0720	2.0664	1.9148	2.2230	2.1015	2.0968	1.9369	2.2038	2.0734	2.0678	1.9158
5	2.2420	2.1144	2.1079	1.9207	2.2645	2.1467	2.1413	1.9431	2.2435	2.1160	2.1095	1.9218
6	2.2735	2.1483	2.1410	1.9235	2.2979	2.1832	2.1772	1.9460	2.2752	2.1501	2.1429	1.9246
7	2.2997	2.1787	2.1685	1.9248	2.3258	2.2155	2.2072	1.9473	2.3016	2.1808	2.1706	1.9259
8	2.3220	2.2005	2.1919	1.9254	2.3497	2.2398	2.2329	1.9480	2.3241	2.2028	2.1942	1.9265
9	2.3414	2.2214	2.2123	1.9257	2.3705	2.2626	2.2554	1.9483	2.3437	2.2239	2.2148	1.9268
10	2.3585	2.2398	2.2302	1.9258	2.3890	2.2829	2.2753	1.9484	2.3610	2.2426	2.2330	1.9269
11	2.3738	2.2564	2.2462	1.9259	2.4056	2.3012	2.2933	1.9485	2.3765	2.2593	2.2492	1.9270
12	2.3876	2.2713	2.2607	1.9259	2.4206	2.3177	2.3096	1.9485	2.3905	2.2744	2.2639	1.9270
13	2.4001	2.2848	2.2738	1.9259	2.4344	2.3329	2.3245	1.9485	2.4033	2.2882	2.2773	1.9270
14	2.4116	2.2973	2.2859	1.9259	2.4470	2.3468	2.3382	1.9486	2.4150	2.3009	2.2895	1.9270
15	2.4222	2.3088	2.2970	1.9259	2.4587	2.3597	2.3509	1.9486	2.4258	2.3126	2.3008	1.9270
16	2.4320	2.3194	2.3073	1.9259	2.4696	2.3718	2.3627	1.9486	2.4358	2.3235	2.3113	1.9270
17	2.4412	2.3294	2.3169	1.9259	2.4797	2.3830	2.3738	1.9486	2.4451	2.3336	2.3211	1.9270
18	2.4497	2.3387	2.3258	1.9259	2.4893	2.3936	2.3842	1.9486	2.4538	2.3431	2.3303	1.9270
19	2.4577	2.3474	2.3342	1.9259	2.4982	2.4036	2.3940	1.9486	2.4621	2.3520	2.3389	1.9270
20	2.4653	2.3556	2.3421	1.9259	2.5067	2.4130	2.4033	1.9486	2.4698	2.3604	2.3470	1.9270

	k_1 = 5											
	Zero-state								Steady-state			
H	IRR1	IRR2	IRR3	IRR4	SC1	SC2	SC3	SC4	IRR1 & SC1	IRR2 & SC2	IRR3 & SC3	IRR4 & SC4
1	1.9323	1.7815	1.7815	1.7815	1.9435	1.7983	1.7983	1.7983	1.9329	1.7821	1.7821	1.7821
2	2.0698	1.9294	1.9265	1.8664	2.0849	1.9515	1.9490	1.8863	2.0706	1.9303	1.9274	1.8672
3	2.1462	2.0114	2.0069	1.8969	2.1641	2.0374	2.0335	1.9182	2.1472	2.0125	2.0080	1.8979
4	2.1985	2.0676	2.0620	1.9100	2.2188	2.0969	2.0921	1.9318	2.1998	2.0690	2.0633	1.9110
5	2.2381	2.1101	2.1035	1.9159	2.2604	2.1422	2.1367	1.9380	2.2396	2.1117	2.1051	1.9169
6	2.2697	2.1441	2.1367	1.9186	2.2939	2.1787	2.1726	1.9409	2.2714	2.1459	2.1386	1.9197
7	2.2960	2.1747	2.1643	1.9199	2.3219	2.2111	2.2026	1.9423	2.2979	2.1767	2.1664	1.9210
8	2.3183	2.1964	2.1878	1.9205	2.3458	2.2354	2.2284	1.9429	2.3205	2.1987	2.1900	1.9216
9	2.3378	2.2174	2.2081	1.9208	2.3667	2.2583	2.2509	1.9432	2.3401	2.2199	2.2107	1.9219
10	2.3549	2.2359	2.2261	1.9209	2.3852	2.2786	2.2709	1.9434	2.3575	2.2386	2.2289	1.9221
11	2.3703	2.2525	2.2422	1.9210	2.4019	2.2969	2.2889	1.9435	2.3730	2.2554	2.2452	1.9221
12	2.3841	2.2674	2.2567	1.9210	2.4169	2.3135	2.3052	1.9435	2.3871	2.2706	2.2599	1.9222
13	2.3967	2.2810	2.2699	1.9210	2.4307	2.3287	2.3201	1.9435	2.3998	2.2844	2.2733	1.9222
14	2.4082	2.2935	2.2819	1.9211	2.4434	2.3426	2.3339	1.9435	2.4116	2.2971	2.2856	1.9222
15	2.4188	2.3051	2.2931	1.9211	2.4551	2.3556	2.3466	1.9435	2.4224	2.3089	2.2969	1.9222
16	2.4287	2.3157	2.3034	1.9211	2.4660	2.3677	2.3585	1.9435	2.4325	2.3198	2.3075	1.9222
17	2.4379	2.3257	2.3130	1.9211	2.4762	2.3789	2.3696	1.9435	2.4418	2.3299	2.3173	1.9222
18	2.4464	2.3350	2.3220	1.9211	2.4858	2.3896	2.3800	1.9435	2.4506	2.3395	2.3265	1.9222
19	2.4545	2.3438	2.3304	1.9211	2.4948	2.3995	2.3898	1.9435	2.4588	2.3484	2.3351	1.9222
20	2.4621	2.3520	2.3383	1.9211	2.5033	2.4090	2.3991	1.9435	2.4666	2.3569	2.3432	1.9222

Table III: The EQL and PCI (in brackets) of the *1-of-1* or *2-of-*(H+1) IRR and Synthetic-$\bar{X}$ charts when H = 1, δ_{max} = 5 and ARL_0 = 370.4

	1	3.1	3.2	3.3	3.4	3.5	3.6	3.7	3.8	3.9	4.0	4.1	4.2	4.3	4.4	4.5	4.6	4.7	4.8	4.9	5.0
ZS	IRR1	225.82 (1.2497)	**223.95** (1.3165)	225.69 (1.3634)	228.81 (1.4010)	232.60 (1.4338)	236.62 (1.4626)	240.72 (1.4891)	244.78 (1.5134)	248.75 (1.5363)	252.57 (1.5578)	256.21 (1.5781)	259.70 (1.5974)	263.02 (1.6158)	266.19 (1.6336)	269.17 (1.6503)	271.99 (1.6663)	274.66 (1.6817)	277.15 (1.6961)	279.48 (1.7095)	281.67 (1.7223)
	IRR2,IRR3,IRR4	207.93 (1.1507)	201.94 (1.1871)	**200.94** (1.2138)	201.95 (1.2365)	203.95 (1.2572)	206.52 (1.2765)	209.37 (1.2952)	212.39 (1.3132)	215.50 (1.3310)	218.61 (1.3483)	221.68 (1.3654)	224.69 (1.3821)	227.62 (1.3984)	230.46 (1.4143)	233.20 (1.4297)	235.83 (1.4447)	238.31 (1.4592)	240.68 (1.4729)	242.94 (1.4860)	245.05 (1.4984)
	SC1	198.56 (1.0989)	191.59 (1.1263)	189.33 (1.1437)	**188.83** (1.1562)	189.16 (1.1660)	189.85 (1.1735)	190.71 (1.1798)	191.59 (1.1846)	192.39 (1.1882)	193.15 (1.1913)	193.80 (1.1937)	194.35 (1.1955)	194.79 (1.1967)	195.19 (1.1978)	195.49 (1.1986)	195.73 (1.1991)	195.90 (1.1995)	196.08 (1.2000)	196.19 (1.2000)	196.29 (1.2002)
	SC2, SC3, SC4	180.69 (1.0000)	170.11 (1.0000)	165.54 (1.0000)	163.32 (1.0000)	162.23 (1.0000)	161.78 (1.0000)	**161.65** (1.0000)	161.74 (1.0000)	161.92 (1.0000)	162.13 (1.0000)	162.35 (1.0000)	162.58 (1.0000)	162.77 (1.0000)	162.95 (1.0000)	163.11 (1.0000)	163.23 (1.0000)	163.32 (1.0000)	163.40 (1.0000)	163.48 (1.0000)	163.55 (1.0000)
SS	IRR1, SC1	225.07 (1.0857)	**222.79** (1.1082)	224.21 (1.1220)	227.07 (1.1318)	230.59 (1.1389)	234.39 (1.1440)	238.31 (1.1480)	242.18 (1.1505)	245.98 (1.1523)	249.62 (1.1533)	253.10 (1.1536)	256.43 (1.1535)	259.60 (1.1532)	262.60 (1.1525)	265.44 (1.1516)	268.13 (1.1507)	270.67 (1.1497)	273.04 (1.1486)	275.29 (1.1477)	277.39 (1.1467)
	IRR2,IRR3,IRR4, SC2, SC3, SC4	207.31 (1.0000)	201.04 (1.0000)	**199.82** (1.0000)	200.63 (1.0000)	202.47 (1.0000)	204.88 (1.0000)	207.59 (1.0000)	210.50 (1.0000)	213.46 (1.0000)	216.45 (1.0000)	219.40 (1.0000)	222.30 (1.0000)	225.12 (1.0000)	227.85 (1.0000)	230.49 (1.0000)	233.02 (1.0000)	235.44 (1.0000)	237.72 (1.0000)	239.86 (1.0000)	241.90 (1.0000)

Table IV: The EQL and PCI (in brackets) of the *1-of-1* or *2-of-*(H+1) IRR and Synthetic-$\bar{X}$ charts when H = 5, δ_{max} = 5 and ARL_0 = 370.4

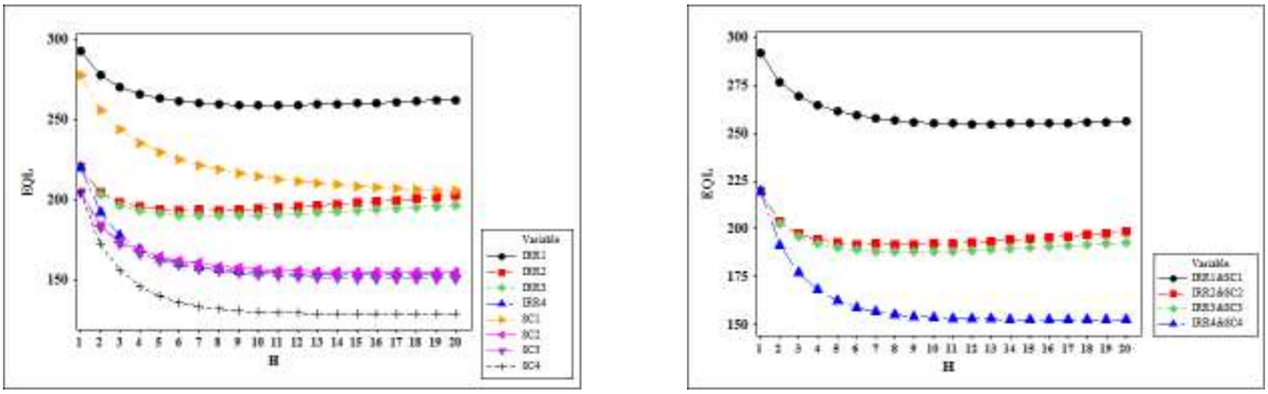
	1	3.1	3.2	3.3	3.4	3.5	3.6	3.7	3.8	3.9	4.0	4.1	4.2	4.3	4.4	4.5	4.6	4.7	4.8	4.9	5.0
ZS	IRR1	223.15 (1.4232)	219.07 (1.5074)	**219.02** (1.5618)	220.62 (1.6049)	223.04 (1.6422)	225.90 (1.6754)	229.02 (1.7065)	232.22 (1.7353)	235.47 (1.7627)	238.68 (1.7888)	241.85 (1.8137)	244.94 (1.8377)	247.95 (1.8605)	250.85 (1.8824)	253.66 (1.9037)	256.32 (1.9237)	258.87 (1.9428)	261.27 (1.9608)	263.57 (1.9779)	265.71 (1.9942)
	IRR2	205.80 (1.3126)	198.89 (1.3685)	**197.24** (1.4065)	197.70 (1.4382)	199.21 (1.4667)	201.35 (1.4933)	203.81 (1.5187)	206.53 (1.5434)	209.33 (1.5670)	212.21 (1.5904)	215.09 (1.6131)	217.95 (1.6351)	220.75 (1.6564)	223.49 (1.6771)	226.15 (1.6973)	228.71 (1.7165)	231.17 (1.7349)	233.53 (1.7526)	235.74 (1.7691)	237.83 (1.7849)
	IRR3	205.37 (1.3098)	198.28 (1.3644)	**196.50** (1.4012)	196.86 (1.4321)	198.32 (1.4602)	200.40 (1.4863)	202.84 (1.5114)	205.51 (1.5357)	208.30 (1.5593)	211.15 (1.5825)	214.03 (1.6051)	216.86 (1.6269)	219.67 (1.6483)	222.39 (1.6689)	225.06 (1.6891)	227.62 (1.7083)	230.08 (1.7267)	232.41 (1.7442)	234.64 (1.7608)	236.71 (1.7765)
	IRR4	191.16 (1.2192)	183.72 (1.2641)	**181.79** (1.2963)	181.98 (1.3238)	183.25 (1.3492)	185.14 (1.3731)	187.38 (1.3962)	189.87 (1.4188)	192.47 (1.4409)	195.16 (1.4626)	197.87 (1.4839)	200.59 (1.5049)	203.28 (1.5253)	205.90 (1.5451)	208.47 (1.5646)	210.95 (1.5832)	213.32 (1.6010)	215.62 (1.6182)	217.78 (1.6343)	219.82 (1.6498)
	SC1	187.97 (1.1988)	177.29 (1.2199)	172.45 (1.2297)	169.88 (1.2358)	168.44 (1.2402)	167.67 (1.2435)	167.23 (1.2461)	167.01 (1.2481)	166.91 (1.2495)	**166.89** (1.2507)	166.92 (1.2518)	166.94 (1.2524)	166.98 (1.2529)	167.02 (1.2533)	167.04 (1.2536)	167.08 (1.2540)	167.10 (1.2541)	167.11 (1.2541)	167.13 (1.2542)	167.12 (1.2542)
	SC2	172.16 (1.0980)	159.35 (1.0965)	153.50 (1.0946)	150.28 (1.0932)	148.38 (1.0925)	147.22 (1.0919)	146.49 (1.0915)	146.04 (1.0914)	145.78 (1.0913)	145.60 (1.0912)	145.50 (1.0912)	145.46 (1.0913)	145.43 (1.0912)	145.41 (1.0912)	**145.41** (1.0913)	145.42 (1.0914)	145.41 (1.0913)	145.42 (1.0914)	145.42 (1.0913)	145.42 (1.0914)
	SC3	171.84 (1.0959)	158.88 (1.0932)	152.92 (1.0905)	149.64 (1.0886)	147.70 (1.0875)	146.50 (1.0865)	145.75 (1.0860)	145.30 (1.0858)	145.01 (1.0856)	144.83 (1.0855)	144.74 (1.0855)	144.68 (1.0855)	144.65 (1.0854)	144.63 (1.0853)	144.62 (1.0854)	**144.62** (1.0854)	144.62 (1.0854)	144.62 (1.0854)	144.63 (1.0854)	144.64 (1.0855)
	SC4	156.79 (1.0000)	145.33 (1.0000)	140.24 (1.0000)	137.47 (1.0000)	135.82 (1.0000)	134.83 (1.0000)	134.20 (1.0000)	133.82 (1.0000)	133.58 (1.0000)	133.43 (1.0000)	133.34 (1.0000)	133.29 (1.0000)	133.27 (1.0000)	133.26 (1.0000)	133.24 (1.0000)	**133.24** (1.0000)	133.25 (1.0000)	133.25 (1.0000)	133.25 (1.0000)	133.25 (1.0000)
SS	IRR1, SC1	221.46 (1.1641)	216.44 (1.1867)	**215.70** (1.1973)	216.68 (1.2032)	218.60 (1.2069)	220.99 (1.2088)	223.64 (1.2096)	226.47 (1.2098)	229.33 (1.2093)	232.19 (1.2082)	235.01 (1.2068)	237.77 (1.2053)	240.46 (1.2034)	243.06 (1.2014)	245.54 (1.1992)	247.95 (1.1973)	250.24 (1.1952)	252.39 (1.1932)	254.44 (1.1912)	256.37 (1.1895)
	IRR2, SC2	204.57 (1.0753)	197.01 (1.0802)	**194.88** (1.0818)	194.91 (1.0823)	196.08 (1.0825)	197.86 (1.0823)	200.05 (1.0820)	202.48 (1.0817)	205.01 (1.0810)	207.64 (1.0805)	210.28 (1.0798)	212.90 (1.0792)	215.47 (1.0783)	218.00 (1.0775)	220.44 (1.0766)	222.80 (1.0758)	225.06 (1.0750)	227.24 (1.0743)	229.27 (1.0734)	231.20 (1.0728)
	IRR3, SC3	204.14 (1.0731)	196.41 (1.0769)	194.16 (1.0777)	**194.11** (1.0779)	195.21 (1.0777)	196.98 (1.0775)	199.13 (1.0770)	201.52 (1.0766)	204.05 (1.0760)	206.66 (1.0754)	209.27 (1.0747)	211.89 (1.0740)	214.48 (1.0733)	216.98 (1.0725)	219.42 (1.0717)	221.80 (1.0710)	224.07 (1.0702)	226.22 (1.0695)	228.26 (1.0686)	230.19 (1.0681)
	IRR4, SC4	190.24 (1.0000)	182.39 (1.0000)	180.15 (1.0000)	**180.08** (1.0000)	181.13 (1.0000)	182.82 (1.0000)	184.89 (1.0000)	187.19 (1.0000)	189.64 (1.0000)	192.17 (1.0000)	194.73 (1.0000)	197.28 (1.0000)	199.83 (1.0000)	202.32 (1.0000)	204.75 (1.0000)	207.10 (1.0000)	209.37 (1.0000)	211.53 (1.0000)	213.59 (1.0000)	215.52 (1.0000)

Table V: The zero-state and steady-state OOC *ARL* values of the *1-of-1* or *2-of-*(H+1) IRR and Synthetic-$\bar{X}$ charts when H = 1, δ_{max} = 5 and ARL_0 = 370.4

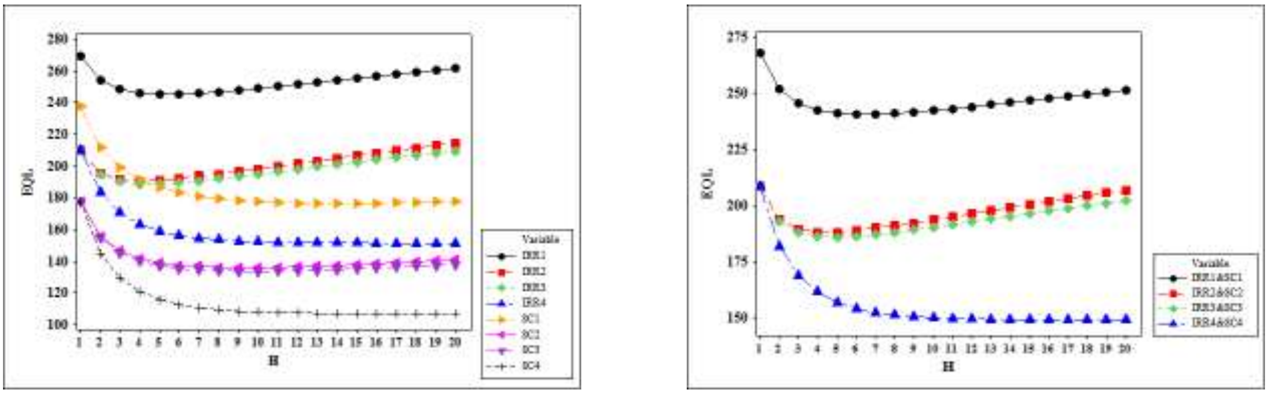
	Zero-state				Steady-state	
	IRR1	IRR2, IRR3, IRR4	SC1	SC2, SC3, SC4	IRR1, SC1	IRR2, IRR3, IRR4, SC2, SC3, SC4
k_1	3.2	3.3	3.4	3.7	3.2	3.3
k_2	2.0700	1.8756	2.0014	1.8167	2.0705	1.8762
δ = 0	370.4	370.4	370.4	370.4	370.4	370.4
0.1	352.10	343.56	352.32	**341.14**	352.06	**343.53**
0.2	305.31	280.70	305.65	**274.03**	305.24	**280.64**
0.3	246.93	212.25	246.66	**203.01**	246.83	**212.16**
0.4	190.96	155.04	189.46	**145.31**	190.83	**154.93**
0.5	144.09	112.30	141.32	**103.22**	143.95	**112.18**
0.6	107.61	81.74	103.91	**73.72**	107.47	**81.62**
0.7	80.28	60.17	76.10	**53.24**	80.14	**60.05**
0.8	60.19	44.91	55.88	**38.99**	60.05	**44.80**
0.9	45.50	34.04	41.33	**28.97**	45.38	**33.93**
1	34.78	26.20	30.88	**21.85**	34.66	**26.10**
1.1	26.90	20.49	23.34	**16.73**	26.80	**20.40**
1.2	21.08	16.26	17.87	**13.01**	20.99	**16.18**
1.3	16.74	13.11	13.87	**10.26**	16.66	**13.03**
1.4	13.47	10.72	10.91	**8.21**	13.40	**10.65**
1.5	10.99	8.89	8.70	**6.67**	10.92	**8.83**
1.6	9.08	7.47	7.04	**5.49**	9.02	**7.41**
1.7	7.60	6.36	5.77	**4.58**	7.54	**6.31**
1.8	6.44	5.48	4.80	**3.88**	6.39	**5.43**
1.9	5.52	4.77	4.04	**3.32**	5.47	**4.73**
2	4.78	4.20	3.45	**2.88**	4.74	**4.16**
2.1	4.19	3.74	2.98	**2.53**	4.15	**3.70**
2.2	3.71	3.36	2.61	**2.25**	3.67	**3.33**
2.3	3.31	3.04	2.31	**2.02**	3.28	**3.01**
2.4	2.99	2.78	2.07	**1.83**	2.96	**2.75**
2.5	2.71	2.56	1.87	**1.68**	2.69	**2.53**
2.6	2.49	2.37	1.71	**1.56**	2.46	**2.34**
2.7	2.29	2.21	1.58	**1.45**	2.27	**2.19**
2.8	2.13	2.07	1.47	**1.37**	2.10	**2.05**
2.9	1.99	1.95	1.38	**1.30**	1.97	**1.93**
3	1.86	1.85	1.31	**1.24**	1.84	**1.83**
3.1	1.76	1.75	1.25	**1.19**	**1.74**	**1.74**
3.2	1.67	1.67	1.20	**1.15**	**1.65**	1.66
3.3	1.58	1.60	1.16	**1.12**	**1.57**	1.58
3.4	1.51	1.53	1.13	**1.10**	**1.50**	1.52
3.5	1.45	1.47	1.10	**1.08**	**1.44**	1.46
3.6	1.39	1.42	1.08	**1.06**	**1.38**	1.41
3.7	1.34	1.37	1.06	**1.05**	**1.33**	1.36
3.8	1.30	1.33	1.05	**1.04**	**1.29**	1.32
3.9	1.26	1.29	1.04	**1.03**	**1.25**	1.28
4	1.22	1.25	1.03	**1.02**	**1.22**	1.24
4.1	1.19	1.22	**1.02**	**1.02**	**1.19**	1.21
4.2	1.16	1.19	1.02	**1.01**	**1.16**	1.18
4.3	1.14	1.16	**1.01**	**1.01**	**1.13**	1.16
4.4	1.12	1.14	**1.01**	**1.01**	**1.11**	1.13
4.5	1.10	1.12	1.01	**1.00**	**1.10**	1.11
4.6	1.08	1.10	1.01	**1.00**	**1.08**	1.09
4.7	1.07	1.08	**1.00**	**1.00**	**1.07**	1.08
4.8	1.06	1.07	**1.00**	**1.00**	**1.05**	1.07
4.9	1.04	1.05	**1.00**	**1.00**	**1.04**	1.05
5	1.04	1.04	**1.00**	**1.00**	**1.03**	1.04
EQL	223.95	200.94	188.83	**161.65**	222.79	**199.82**
ARARL	1.4084	1.2827	1.1499	**1.0000**	1.1004	**1.0000**
PCI	1.3854	1.2430	1.1681	**1.0000**	1.1149	**1.0000**

Table VI: The zero-state and steady-state OOC *ARL* values of the *1-of-1* or *2-of-*(H+1) IRR and Synthetic-$\bar{X}$ charts when H = 5, δ_{max} = 5 and ARL_0 = 370.4

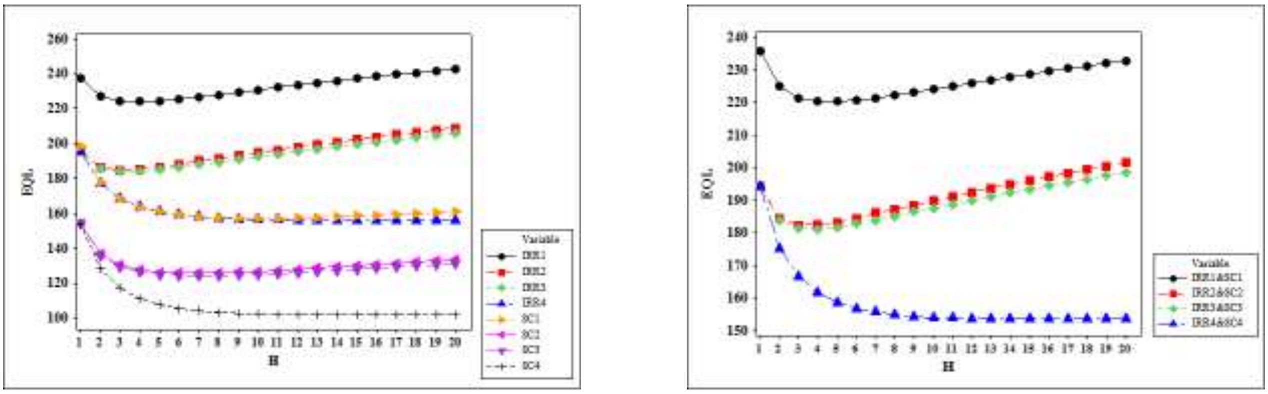
	Zero-state								Steady-state			
	IRR1	IRR2	IRR3	IRR4	SC1	SC2	SC3	SC4	IRR1,SC1	IRR2,SC2	IRR3,SC3	IRR4,SC4
k_1	3.3	3.3	3.3	3.3	4.0	4.5	4.6	4.6	3.3	3.3	3.4	3.4
k_2	2.3105	2.1891	2.1842	2.0053	2.2645	2.1426	2.1369	1.9383	2.3119	2.1907	2.1577	1.9752
δ = 0	370.4	370.3	370.4	370.4	370.3	370.3	370.3	**370.3**	370.5	370.4	370.4	**370.3**
0.1	350.52	340.37	340.05	336.56	349.68	334.93	334.28	**330.02**	350.47	340.42	338.59	**334.69**
0.2	300.04	272.16	271.21	261.74	296.98	257.71	256.12	**245.45**	299.85	272.10	267.50	**257.28**
0.3	238.21	200.92	199.70	186.97	231.95	181.82	179.96	**166.77**	237.88	200.76	195.06	**181.66**
0.4	180.36	143.78	142.61	129.59	171.15	124.39	122.75	**110.21**	179.93	143.55	138.23	**124.81**
0.5	133.26	102.53	101.57	89.80	122.24	84.94	83.67	**73.02**	132.79	102.27	97.91	**85.98**
0.6	97.68	73.84	73.09	63.09	86.10	58.64	57.71	**49.13**	97.20	73.57	70.21	**60.19**
0.7	71.80	54.02	53.46	45.23	60.59	41.15	40.48	**33.75**	71.34	53.76	51.24	**43.08**
0.8	53.26	40.25	39.83	33.18	42.96	29.42	28.95	**23.72**	52.84	40.00	38.14	**31.58**
0.9	40.04	30.56	30.25	24.91	30.85	21.46	21.12	**17.08**	39.65	30.32	28.96	**23.72**
1	30.57	23.64	23.41	19.15	22.52	15.97	15.73	**12.61**	30.22	23.43	22.42	**18.25**
1.1	23.72	18.63	18.46	15.05	16.74	12.13	11.96	**9.54**	23.41	18.43	17.69	**14.37**
1.2	18.72	14.94	14.81	12.08	12.69	9.40	9.28	**7.39**	18.44	14.76	14.21	**11.56**
1.3	15.01	12.18	12.08	9.89	9.81	7.43	7.34	**5.86**	14.76	12.02	11.61	**9.48**
1.4	12.23	10.08	10.01	8.25	7.73	5.99	5.93	**4.76**	12.01	9.94	9.63	**7.93**
1.5	10.12	8.47	8.41	6.99	6.22	4.93	4.88	**3.94**	9.92	8.34	8.11	**6.74**
1.6	8.49	7.22	7.17	6.01	5.09	4.12	4.09	**3.34**	8.31	7.10	6.93	**5.81**
1.7	7.22	6.22	6.19	5.24	4.25	3.51	3.48	**2.88**	7.05	6.11	5.99	**5.08**
1.8	6.21	5.43	5.40	4.63	3.61	3.04	3.02	**2.52**	6.06	5.33	5.24	**4.50**
1.9	5.41	4.79	4.77	4.13	3.12	2.67	2.65	**2.24**	5.27	4.70	4.64	**4.03**
2	4.76	4.26	4.24	3.72	2.73	2.38	2.37	**2.03**	4.64	4.18	4.14	**3.64**
2.1	4.23	3.83	3.81	3.38	2.43	2.15	2.14	**1.85**	4.11	3.75	3.73	**3.31**
2.2	3.79	3.46	3.45	3.09	2.19	1.96	1.95	**1.71**	3.68	3.39	3.38	**3.04**
2.3	3.42	3.16	3.15	2.85	1.99	1.81	1.80	**1.59**	3.32	3.09	3.09	**2.81**
2.4	3.11	2.90	2.89	2.64	1.83	1.68	1.67	**1.50**	3.02	2.83	2.85	**2.62**
2.5	2.85	2.67	2.67	2.46	1.70	1.57	1.57	**1.42**	2.77	2.61	2.63	**2.44**
2.6	2.62	2.48	2.47	2.31	1.59	1.48	1.48	**1.35**	2.55	2.43	2.45	**2.29**
2.7	2.43	2.31	2.31	2.17	1.50	1.41	1.40	**1.29**	2.36	2.26	2.29	**2.16**
2.8	2.26	2.16	2.16	2.05	1.42	1.34	1.34	**1.25**	2.19	2.12	2.15	**2.05**
2.9	2.11	2.03	2.03	1.94	1.36	1.29	1.29	**1.20**	2.05	1.99	2.02	**1.94**
3	1.98	1.92	1.92	1.84	1.30	1.24	1.24	**1.17**	1.93	1.88	1.91	**1.85**
3.1	1.87	1.82	1.81	1.75	1.25	1.20	1.20	**1.14**	1.82	1.78	1.82	**1.77**
3.2	1.76	1.72	1.72	1.67	1.21	1.17	1.17	**1.12**	1.72	**1.69**	1.73	**1.69**
3.3	1.67	1.64	1.64	1.60	1.18	1.14	1.14	**1.10**	1.63	**1.61**	1.65	1.62
3.4	1.59	1.57	1.57	1.54	1.15	1.12	1.12	**1.08**	1.56	**1.54**	1.58	1.56
3.5	1.52	1.50	1.50	1.48	1.12	1.10	1.09	**1.06**	1.49	**1.48**	1.52	1.50
3.6	1.46	1.44	1.44	1.42	1.10	1.08	1.08	**1.05**	1.43	**1.42**	1.46	1.45
3.7	1.40	1.39	1.39	1.38	1.08	1.06	1.06	**1.04**	**1.37**	**1.37**	1.40	1.40
3.8	1.35	1.34	1.34	1.33	1.07	1.05	1.05	**1.03**	1.33	**1.32**	1.36	1.35
3.9	1.30	1.30	1.30	1.29	1.05	1.04	1.04	**1.03**	**1.28**	**1.28**	1.31	1.31
4	1.26	1.26	1.26	1.25	1.04	1.03	1.03	**1.02**	**1.24**	**1.24**	1.27	1.27
4.1	1.23	1.22	1.22	1.22	1.03	1.03	1.03	**1.02**	**1.21**	**1.21**	1.24	1.24
4.2	1.19	1.19	1.19	1.19	1.03	1.02	1.02	**1.01**	**1.18**	**1.18**	1.21	1.21
4.3	1.17	1.16	1.16	1.16	1.02	1.02	1.02	**1.01**	**1.15**	**1.15**	1.18	1.18
4.4	1.14	1.14	1.14	1.14	1.02	**1.01**	**1.01**	**1.01**	**1.13**	**1.13**	1.15	1.15
4.5	1.12	1.12	1.12	1.12	**1.01**	**1.01**	**1.01**	**1.01**	**1.11**	**1.11**	1.13	1.13
4.6	1.10	1.10	1.10	1.10	1.01	1.01	1.01	**1.00**	**1.09**	**1.09**	1.11	1.11
4.7	1.08	1.08	1.08	1.08	1.01	1.01	1.01	**1.00**	**1.08**	**1.08**	1.09	1.09
4.8	1.07	1.07	1.07	1.07	1.01	**1.00**	**1.00**	**1.00**	**1.06**	**1.06**	1.08	1.08
4.9	1.06	1.06	1.06	1.06	**1.00**	**1.00**	**1.00**	**1.00**	**1.05**	**1.05**	1.06	1.06
5	1.04	1.04	1.04	1.04	**1.00**	**1.00**	**1.00**	**1.00**	**1.04**	**1.04**	1.05	1.05
EQL	219.02	197.24	196.50	181.79	166.89	145.41	144.62	**133.24**	215.70	194.88	194.11	**180.08**
ARARL	1.7686	1.5852	1.5789	1.4432	1.2797	1.1078	1.1014	**1.0000**	1.2072	1.0886	1.0834	**1.0000**
PCI	1.6438	1.4803	1.4748	1.3643	1.2525	1.0913	1.0854	**1.0000**	1.1978	1.0822	1.0779	**1.0000**



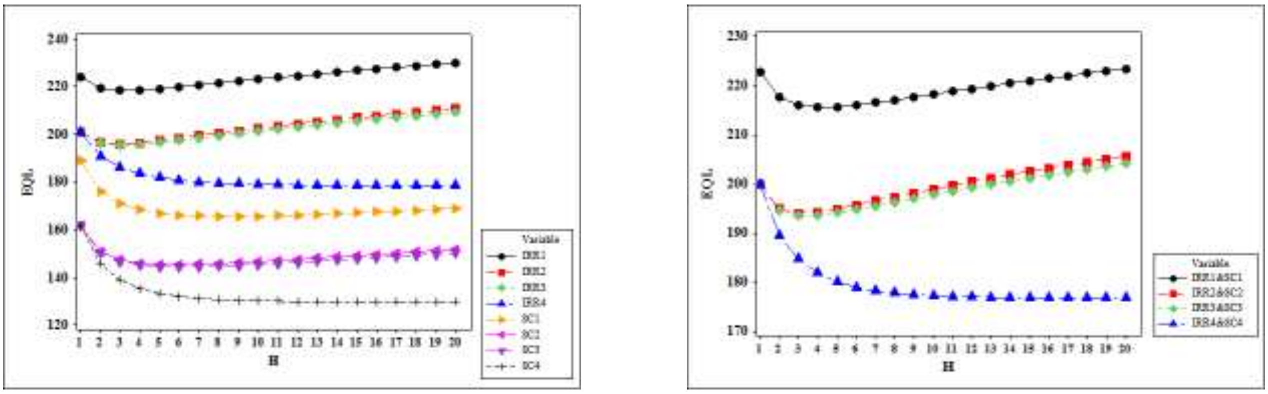
Zero-state (a) $\delta_{max} = 1$ Steady-state



Zero-state (b) $\delta_{max} = 2$ Steady-state



Zero-state (c) $\delta_{max} = 3$ Steady-state



Zero-state (d) $\delta_{max} = 5$ Steady-state

Figure 2: The zero-state and steady-state *EQL* values of the *1-of-1* or *2-of-*(H+1) IRR and Synthetic-$\bar{X}$ charts when H = 1, 2, … , 20 for an ARL_0 = 370.4

Next, we investigate whether the SC4 scheme has a uniformly better overall OOC performance as H varies from 1 to 20 by plotting the values of the minimum ZS and SS *EQL* for each scheme listed in Table I (similar to those in Tables V and VI). It is clear from Figure 2 that the SC4 scheme outperform all the other schemes once $H > 1$, both in ZS and SS modes for a variety of δ_{max} values. Moreover, the convergence of k_2 values as H increases for the SC4 and IRR4 schemes result in the convergence of their *EQL* values. The *EQL* values of the IRR2 and IRR3 as well as SC2 and SC3 schemes are almost equal, however, with those of the IRR3 and SC3 just slightly better. The IRR1 scheme is the worst performing scheme and the SC1 scheme become less effective as δ_{max} decrease. For each of the schemes, we recommend a value of H that corresponds to the minimum of the *EQL* curve depending on the desired δ_{max}. The results in Figure 2 are further illustrated by the results in Tables VII and VIII when $\delta_{max} = 5$. That is, we see that the *PCI*s and *ARARL*s of the IRR2 and IRR3 as well as SC2 and SC3 schemes are relatively close to each other. Also, IRR1 has the worst performance, for instance, when $H = 5$, it is 64.38% and 76.86% worse than the SC4 scheme in ZS with respect to the *PCI* and *ARARL*, respectively. The *PCI*s and *ARARL*s of the SC4 scheme have a uniformly better performance with equality in performance occurring only when $H = 1$ and with the IRR4 in SS.

Table VII: The zero-state and steady-state *PCI* values of the *1-of-1* or *2-of-*(H+1) IRR and Synthetic-$\bar{X}$ charts for $\delta_{max} = 5$ when $H = 1, 2, \ldots, 20$ and $ARL_0 = 370.4$

	Zero-state								Steady-state			
H	IRR1	IRR2	IRR3	IRR4	SC1	SC2	SC3	SC4	IRR1&SC1	IRR2&SC2	IRR3&SC3	IRR4&SC4
1	1.3854	1.2430	1.2430	1.2430	1.1681	1.0000	1.0000	1.0000	1.1149	1.0000	1.0000	1.0000
2	1.5057	1.3485	1.3463	1.3107	1.2088	1.0358	1.0332	1.0000	1.1485	1.0283	1.0267	1.0000
3	1.5719	1.4099	1.4063	1.3405	1.2300	1.0606	1.0565	1.0000	1.1694	1.0507	1.0481	1.0000
4	1.6144	1.4510	1.4463	1.3559	1.2435	1.0783	1.0732	1.0000	1.1853	1.0683	1.0648	1.0000
5	1.6438	1.4803	1.4748	1.3643	1.2525	1.0913	1.0854	1.0000	1.1978	1.0822	1.0779	1.0000
6	1.6653	1.5023	1.4959	1.3692	1.2589	1.1012	1.0946	1.0000	1.2072	1.0931	1.0883	1.0000
7	1.6817	1.5215	1.5123	1.3721	1.2636	1.1112	1.1020	1.0000	1.2149	1.1034	1.0969	1.0000
8	1.6944	1.5334	1.5256	1.3737	1.2673	1.1157	1.1081	1.0000	1.2213	1.1097	1.1041	1.0000
9	1.7050	1.5451	1.5369	1.3748	1.2705	1.1214	1.1133	1.0000	1.2267	1.1164	1.1104	1.0000
10	1.7138	1.5554	1.5464	1.3753	1.2734	1.1266	1.1181	1.0000	1.2314	1.1223	1.1159	1.0000
11	1.7215	1.5645	1.5550	1.3757	1.2762	1.1314	1.1226	1.0000	1.2356	1.1275	1.1209	1.0000
12	1.7286	1.5726	1.5627	1.3758	1.2789	1.1360	1.1268	1.0000	1.2395	1.1324	1.1256	1.0000
13	1.7350	1.5804	1.5700	1.3760	1.2817	1.1404	1.1309	1.0000	1.2430	1.1369	1.1299	1.0000
14	1.7409	1.5874	1.5767	1.3760	1.2844	1.1446	1.1348	1.0000	1.2464	1.1412	1.1339	1.0000
15	1.7468	1.5943	1.5832	1.3761	1.2873	1.1490	1.1388	1.0000	1.2496	1.1453	1.1377	1.0000
16	1.7521	1.6009	1.5893	1.3761	1.2902	1.1531	1.1427	1.0000	1.2527	1.1491	1.1413	1.0000
17	1.7573	1.6071	1.5951	1.3761	1.2931	1.1572	1.1465	1.0000	1.2557	1.1528	1.1448	1.0000
18	1.7622	1.6130	1.6007	1.3761	1.2961	1.1613	1.1503	1.0000	1.2585	1.1564	1.1481	1.0000
19	1.7670	1.6189	1.6061	1.3761	1.2991	1.1652	1.1540	1.0000	1.2612	1.1600	1.1513	1.0000
20	1.7716	1.6244	1.6113	1.3761	1.3021	1.1692	1.1578	1.0000	1.2638	1.1631	1.1544	1.0000

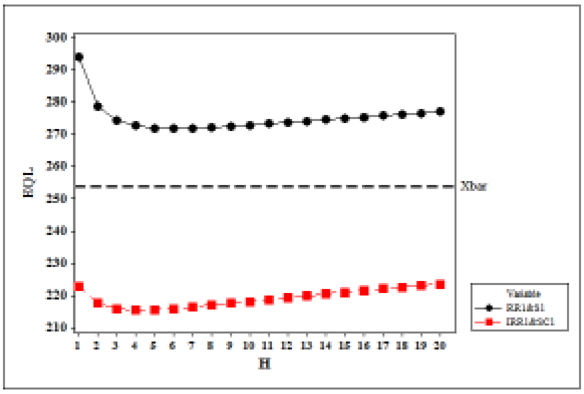
Table VIII: The zero-state and steady-state *ARARL* values of the *1-of-1* or *2-of-*(H+1) IRR and Synthetic-$\bar{X}$ charts for δ_{max} = 5 when H = 1, 2, … , 20 and ARL_0 = 370.4

	Zero-state								Steady-state			
H	IRR1	IRR2	IRR3	IRR4	SC1	SC2	SC3	SC4	IRR1&SC1	IRR2&SC2	IRR3&SC3	IRR4&SC4
1	1.4084	1.2827	1.2827	1.2827	1.1499	1.0000	1.0000	1.0000	1.1004	1.0000	1.0000	1.0000
2	1.5651	1.4095	1.4073	1.3686	1.2040	1.0377	1.0352	1.0000	1.1393	1.0277	1.0262	1.0000
3	1.6603	1.4887	1.4849	1.4090	1.2381	1.0674	1.0632	1.0000	1.1694	1.0523	1.0497	1.0000
4	1.7230	1.5443	1.5391	1.4308	1.2624	1.0902	1.0848	1.0000	1.1906	1.0724	1.0683	1.0000
5	1.7686	1.5852	1.5789	1.4432	1.2797	1.1078	1.1014	1.0000	1.2072	1.0886	1.0834	1.0000
6	1.8024	1.6166	1.6091	1.4505	1.2924	1.1218	1.1146	1.0000	1.2199	1.1015	1.0955	1.0000
7	1.8315	1.6436	1.6328	1.4549	1.3021	1.1355	1.1253	1.0000	1.2301	1.1136	1.1055	1.0000
8	1.8517	1.6613	1.6520	1.4575	1.3096	1.1428	1.1343	1.0000	1.2312	1.1173	1.1109	1.0000
9	1.8682	1.6780	1.6682	1.4591	1.3159	1.1512	1.1421	1.0000	1.2380	1.1249	1.1181	1.0000
10	1.8819	1.6927	1.6821	1.4600	1.3216	1.1588	1.1493	1.0000	1.2437	1.1317	1.1244	1.0000
11	1.8936	1.7057	1.6943	1.4607	1.3267	1.1659	1.1559	1.0000	1.2488	1.1376	1.1301	1.0000
12	1.9041	1.7172	1.7054	1.4609	1.3316	1.1726	1.1621	1.0000	1.2536	1.1433	1.1362	1.0000
13	1.9137	1.7281	1.7157	1.4611	1.3364	1.1790	1.1682	1.0000	1.2578	1.1484	1.1411	1.0000
14	1.9223	1.7380	1.7252	1.4612	1.3411	1.1852	1.1740	1.0000	1.2618	1.1532	1.1456	1.0000
15	1.9308	1.7476	1.7343	1.4614	1.3458	1.1913	1.1798	1.0000	1.2655	1.1578	1.1499	1.0000
16	1.9385	1.7567	1.7428	1.4614	1.3505	1.1973	1.1854	1.0000	1.2691	1.1621	1.1540	1.0000
17	1.9459	1.7654	1.7509	1.4614	1.3552	1.2031	1.1910	1.0000	1.2726	1.1663	1.1579	1.0000
18	1.9529	1.7735	1.7587	1.4614	1.3599	1.2089	1.1964	1.0000	1.2757	1.1703	1.1616	1.0000
19	1.9598	1.7816	1.7662	1.4615	1.3645	1.2145	1.2017	1.0000	1.2789	1.1743	1.1652	1.0000
20	1.9662	1.7892	1.7734	1.4615	1.3691	1.2201	1.2070	1.0000	1.2818	1.1778	1.1687	1.0000

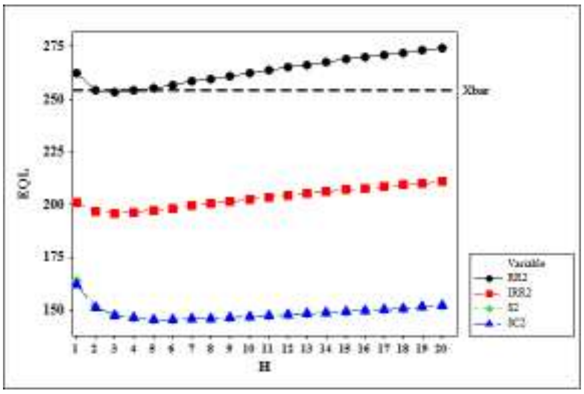
4. Comparison with the *2-of-*(H+1) runs-rules and synthetic $\bar{X}$ charts

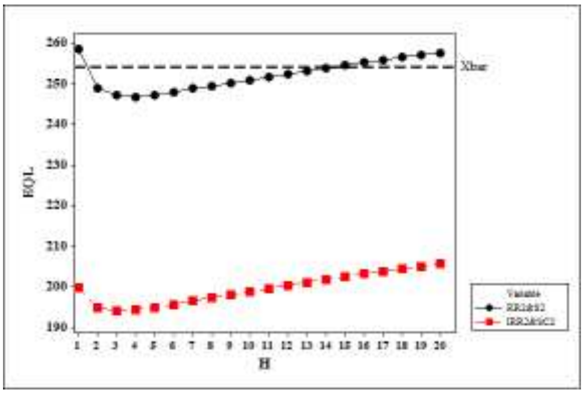
In this section, we compare the overall performance of the schemes in Table I with their corresponding counterparts that are based on Equation (1) which are discussed in Shongwe and Graham[19]. As explained in Figure 1, the synthetic-type and runs-type charts may be classed into four categories i.e. NSS, SSS, RSS and MSS. The NSS, SSS, RSS, MSS synthetic $\bar{X}$ charts were proposed by Wu and Spedding[7], Davis and Woodall[17], Machado and Costa[26], Shongwe and Graham[19] – these are denoted by S1, S2, S3, S4, respectively. The NSS, SSS, RSS, MSS *2-of-*(H+1) RR $\bar{X}$ charts were proposed by Derman and Ross[20], Klein[13], adopted from Machado and Costa[26], Antzoulakos and Rakitzis[29] – these are denoted by RR1, RR2, RR3, RR4, respectively.

In Figure 3, the ZS and SS *EQL*s of these schemes are compared with those listed in Table I when δ_{max} = 5 and we overlaid the *EQL* value of 253.99 of the $\bar{X}$ chart to measure its performance against these schemes. We see that in ZS, the Synthetic-$\bar{X}$ and synthetic $\bar{X}$ charts have an almost equal *EQL* values, with the Synthetic-$\bar{X}$ having just slightly lower *EQL*s. Hence, in ZS it follow that it is better to use the more effective one-sided Synthetic-$\bar{X}$ discussed in Wu et al.[18] rather than the two-sided Synthetic-$\bar{X}$ chart discussed here. However, in SS, there is a significant difference between the Synthetic-$\bar{X}$ and synthetic $\bar{X}$ charts with SC1, SC2, SC3 and SC4 having a better performance than the S1, S2, S3 and S3 schemes, respectively. For RR and IRR schemes, there is a significant performance difference especially when δ_{max} is large, with the IRR1, IRR2, IRR3 and IRR4 better than the RR1, RR2, RR3 and RR4, respectively. Note that as δ_{max} decrease, the *EQL* values of the IRR and RR schemes become closer to each other, respectively. Finally, the $\bar{X}$ chart outperform some of the RR schemes, however, as δ_{max} decrease, it becomes less effective.

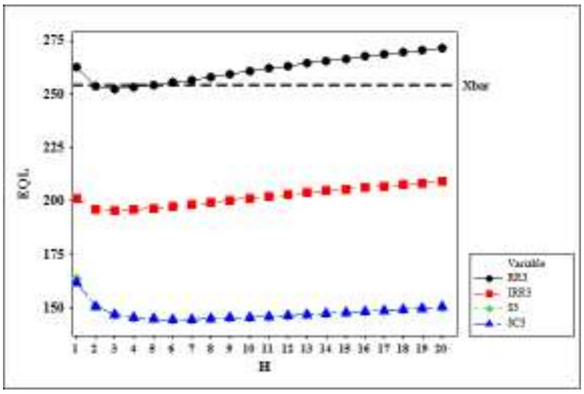


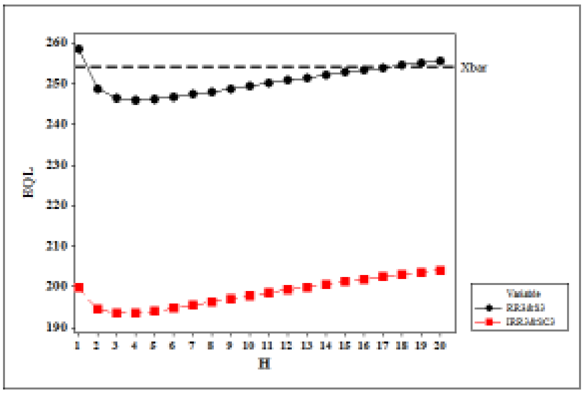
Zero-state (a) NSS Steady-state



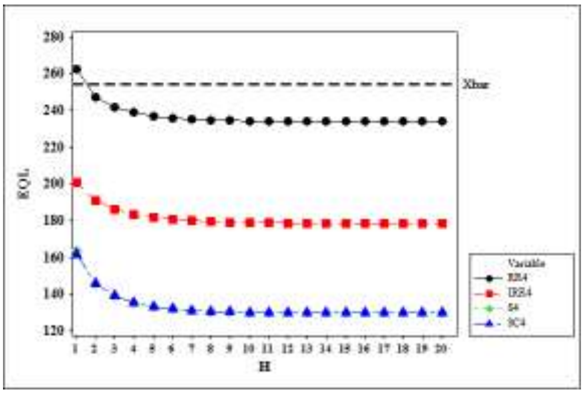


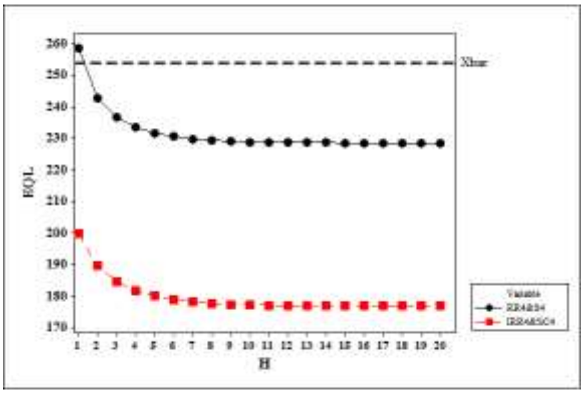
Zero-state (b) SSS Steady-state





Zero-state (c) RSS Steady-state





Zero-state (d) MSS Steady-state

Figure 3: Comparison of the *EQL* values of the IRR and Synthetic- $\bar{X}$ charts with the RR and synthetic $\bar{X}$ charts when H = 1, 2,…, 20 for ARL_0 = 370.4 and δ_{max} = 5

5. Conclusion

In this paper, we proposed a MSS Synthetic-$\bar{X}$ (i.e. SC4) chart using a Markov chain imbedding technique and compared its ZS and SS *ARL*, *EQL*, *PCI* and *ARARL* performance with its seven Shewhart-type competitors. The convergence in the design parameter k_2 as H increases for some given k_1 have a direct effect on the specific shift and overall performance of the SC4 chart. Hence, for all values of H considered here, the new SC4 scheme yields a uniformly better ZS OOC *ARL*, whereas, in SS, the OOC *ARL* is only better for small and moderate shifts. However, the overall performance measures for the SC4 scheme both in ZS and SS are either the same or better than those of the other schemes discussed here; with equality occurring only when H = 1 and with the IRR4 in SS mode. Furthermore, we compared the *EQL* performance of the *1-of-1* or *2-of-*(H+1) IRR and Synthetic-$\bar{X}$ charts with the *2-of-*(H+1) RR and synthetic $\bar{X}$ charts in terms of NSS, SSS, RSS, MSS categories. We observed that there is no significant benefit in adopting the more complicated ZS Synthetic-$\bar{X}$ schemes over ZS synthetic $\bar{X}$ schemes, however, the respective SS Synthetic-$\bar{X}$ schemes have a significant benefit over the SS synthetic $\bar{X}$ schemes.

Note that the results presented here only hold when the observations are from a normal distribution, hence, for other distributions, these will need to be re-evaluated or nonparametric counterparts need to be considered.

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Appendix

The transition probability matrix (TPM) of the Markov chain for any general (integer) value of M > 0 is given by

$$\boldsymbol{P}_{(M+1)\times(M+1)} = \begin{pmatrix} \mathbf{Q}_{(M\times M)} & | & \boldsymbol{r}_{(M\times 1)} \\ - & - & - \\ \mathbf{0}'_{(1\times M)} & | & 1_{(1\times 1)} \end{pmatrix} \quad (8)$$

where $\mathbf{Q}_{(M\times M)}$ is the essential TPM, the vector $\mathbf{r}_{(M\times 1)}$ satisfies $\mathbf{r} = \mathbf{1} - \mathbf{Q1}$ with $\mathbf{1}_{(M\times 1)} = (1\ 1\ \ldots\ 1)^T$ and $\mathbf{0}_{(M\times 1)} = (0\ 0\ \ldots\ 0)^T$. In order to construct the TPM, we follow the Markov chain imbedding technique discussed briefly by Antzoulakos and Rakitzis[22,29], Low et al.[30] and in detail by Fu and Lou[31]. This entails dividing the chart into separate distinct regions (see Figure 1) i.e. let $\{\bar{X}_i; i \geq 1\}$ be a sequence of iid trials taking values in the set ζ_1 = {O, U, E}, ζ_2 = {A, O, D, E} and

ζ_3 = {A, B, C, D, E} for the (IRR1, SC1), (IRR2, SC2, IRR3, SC3) and (IRR4, SC4), respectively. Then, define the probability that a plotting statistic falls in each region:

(i) θ_A denotes the probability that a point falls between the UWL and UCL i.e. $P(\bar{X}_i \in A)$;

(ii) θ_B denotes the probability that a point falls between the CL and UWL i.e. $P(\bar{X}_i \in B)$;

(iii) θ_C denotes the probability that a point falls between the LWL and CL i.e. $P(\bar{X}_i \in C)$;

(iv) θ_D denotes the probability that a point falls between the LCL and LWL i.e. $P(\bar{X}_i \in D)$;

(v) θ_O=θ_B+θ_C denotes the probability that a point falls between the LWL and UWL i.e. $P(\bar{X}_i \in O)$;

(vi) θ_U=θ_A+θ_D denotes the probability that a point falls either between the UWL and UCL, or LCL and LWL i.e. $P(\bar{X}_i \in U)$;

(vii) θ_E denotes the probability that a point falls either above the UCL, or below the LCL i.e. $P(\bar{X}_i \in E)$.

For some sample size, n, suppose that the values of μ_0 and σ_0^2 are known. Thus the probabilities of a plotting statistic falling in a specific region are given by

$$\begin{aligned}
\theta_A(\delta) &= P(UWL \le \bar{X} < UCL) = \Phi(k_1 - \delta\sqrt{n}) - \Phi(k_2 - \delta\sqrt{n}) \\
\theta_B(\delta) &= P(CL \le \bar{X} < UWL) = \Phi(k_2 - \delta\sqrt{n}) - \Phi(-\delta\sqrt{n}) \\
\theta_C(\delta) &= P(LWL < \bar{X} \le CL) = \Phi(-\delta\sqrt{n}) - \Phi(-k_2 - \delta\sqrt{n}) \\
\theta_D(\delta) &= P(LCL < \bar{X} \le LWL) = \Phi(-k_2 - \delta\sqrt{n}) - \Phi(-k_1 - \delta\sqrt{n}) \\
\theta_O(\delta) &= P(LWL < \bar{X} < UWL) = \theta_B(\delta) + \theta_C(\delta) \\
\theta_U(\delta) &= P(UWL \le \bar{X} < UCL) + P(LCL < \bar{X} \le LWL) = \theta_A(\delta) + \theta_D(\delta) \\
\theta_E(\delta) &= P(\bar{X} \ge UCL) + P(\bar{X} \le LCL) = 1 - \Phi(k_1 - \delta\sqrt{n}) + \Phi(-k_1 - \delta\sqrt{n});
\end{aligned} \tag{9}$$

respectively, where $\Phi(\cdot)$ denotes the cumulative distribution function (cdf) of the standard normal distribution and δ is the shift parameter expressed in terms of the standard deviation units and we let CL = 0.

Moreover, we need to define the compound patterns that result in an OOC event (which is also known as the waiting time until the first occurrence of an OOC signal). For example, the sequence of plotting statistics 'AA' indicates two consecutive plotting statistics falling in region A, whereas 'ABA' indicates the first plotting statistic falling in region A, the second in region B and the third in region A, etc. The symbol '±' is used to denote the assumption that (at time 0) the first observation falls either on the upper or lower warning region (i.e. HS feature), so that '±A' indicate the first plotting statistic falls either on region A or D and the second on region A. Following Fu and Lou[31], we let the sequences of conforming and nonconforming samples, say $\Lambda_l =$ ABBA, to be the l^{th} simple pattern within a sequence of n four-state trials from, say, set ζ_3. Furthermore, let Ψ_r to be r^{th} simple pattern with the sequence of states starting with a HS state, say Ψ_r = {±OOA}. Then, define Λ as a compound pattern if it is the union of ω distinct simple patterns i.e. $\Lambda = \Lambda_1 \cup \Lambda_2 \cup$

$\ldots \cup \Lambda_\omega$. Similarly, define Ψ as a compound pattern if it is the union of υ distinct simple patterns i.e. $\Psi = \Psi_1 \cup \Psi_2 \cup \ldots \cup \Psi_\upsilon$. Let W denote the waiting time for the first occurrence of either Λ or Ψ. Then the run-length distribution of a control chart coincides with the waiting time distribution of W. That is, the run-length distribution of the chart becomes the waiting time until the first occurrence of one of the patterns, $\Lambda_1,\ldots,\Lambda_\omega$, $\Psi_1,\ldots,\Psi_\upsilon$ and these are the absorbing states of the Markov chain, where $\omega+\upsilon$ denotes the number of patterns (or sequences) of the $\bar{X}_i$ that cause the chart to signal. Then we define the Markov chain with the state space, Ω, operating on $\{\bar{X}_i; i \geq 1\}$ as follows:

- the absorbing state – corresponding to the union of $\Lambda_1,\ldots,\Lambda_\omega$, $\Psi_1,\ldots,\Psi_\upsilon$; in order to reduce the dimension of the TPM, the $\omega+\upsilon$ absorbing states which signal the entrance of the Markov chain to each of the $\omega+\upsilon$ distinct simple patterns may be substituted by a single absorbing state, denoted by OOC;
- the sub-patterns – corresponding to the distinct first element(s) of the simple pattern $\Lambda_1,\ldots,\Lambda_\omega$ and $\Psi_1,\ldots,\Psi_\upsilon$ without the last element. These sub-patterns are non-absorbing and are denoted by $\eta_1,\ldots,\eta_\tau$ and $\psi_1,\ldots,\psi_\kappa$ where $\tau \leq \omega$ and $\kappa < \upsilon$, respectively, with one of these η_i equal to the transient state, denoted by ϕ, corresponding to the IC central region in Figure 1.

For any $H > 0$, the dimension (i.e. M+1) of the TPM in Equation (8) for the schemes listed in Table I are given by

$$\text{IRR1: } (H+1)+1, \quad \text{IRR2: } (H^2+H+1)+1, \quad \text{IRR3: } (2H+1)+1, \quad \text{IRR4: } (2H+1)+1 \tag{10a}$$

and

$$\text{SC1:}(H+1)+1, \text{SC2:}[(H^2+H+1)+H]+1, \text{SC3:}[(2H+1)+H]+1, \text{SC4:}[(2H+1)+(2H-1)]+1, \tag{10b}$$

respectively. That is, M, the dimension of the essential TPM, is given by

$$M = \begin{cases} \tau+\kappa & \text{for SC2, SC3, SC4} \\ \tau & \text{for SC1, IRR1, IRR2, IRR3, IRR4} \end{cases} \tag{11}$$

where

$$\kappa = \begin{cases} H & \text{for SC2, SC3} \\ 2H-1 & \text{for SC4.} \end{cases} \tag{12}$$

Based on this, for any $H > 0$, we define the state spaces as follows

$$\begin{aligned} &\text{IRR1 \& SC1: } \Omega = \{\eta_1 = \phi, \eta_2, \ldots, \eta_\tau; \text{OOC}\} \\ &\text{IRR2, IRR3, IRR4: } \Omega = \{\eta_1,\ldots,\eta_{\frac{(\tau+1)}{2}-1}, \eta_{\frac{(\tau+1)}{2}} = \phi, \eta_{\frac{(\tau+1)}{2}+1}, \ldots, \eta_\tau; \text{OOC}\} \\ &\text{SC2, SC3, SC4: } \Omega = \{\eta_1,\ldots,\eta_{\frac{(\tau+1)}{2}-1}, \eta_{\frac{(\tau+1)}{2}} = \phi, \eta_{\frac{(\tau+1)}{2}+1}, \ldots, \eta_\tau; \psi_1, \ldots, \psi_\kappa; \text{OOC}\} \end{aligned} \tag{13}$$

In this paper, we considered H = 1, 2,…, 20, hence, in Table IX we show the dimension of the TPMs for each scheme listed in Table I.

Table IX: The dimension (M+1)×(M+1) of the TPM of the embedded Markov chain approach for the *1-of-1* or *2-of-*(H+1) IRR and Synthetic-$\bar{X}$ charts when H = 1,…, 20

H	IRR1	IRR2	IRR3	IRR4	SC1	SC2	SC3	SC4
1	3×3	4×4	4×4	4×4	3×3	5×5	5×5	5×5
2	4×4	8×8	6×6	6×6	4×4	10×10	8×8	9×9
3	5×5	14×14	8×8	8×8	5×5	17×17	11×11	13×13
4	6×6	22×22	10×10	10×10	6×6	26×26	14×14	17×17
5	7×7	32×32	12×12	12×12	7×7	37×37	17×17	21×21
6	8×8	44×44	14×14	14×14	8×8	50×50	20×20	25×25
7	9×9	58×58	16×16	16×16	9×9	65×65	23×23	29×29
8	10×10	74×74	18×18	18×18	10×10	82×82	26×26	33×33
9	11×11	92×92	20×20	20×20	11×11	101×101	29×29	37×37
10	12×12	112×112	22×22	22×22	12×12	122×122	32×32	41×41
11	13×13	134×134	24×24	24×24	13×13	145×145	35×35	45×45
12	14×14	158×158	26×26	26×26	14×14	170×170	38×38	49×49
13	15×15	184×184	28×28	28×28	15×15	197×197	41×41	53×53
14	16×16	212×212	30×30	30×30	16×16	226×226	44×44	57×57
15	17×17	242×242	32×32	32×32	17×17	257×257	47×47	61×61
16	18×18	274×274	34×34	34×34	18×18	290×290	50×50	65×65
17	19×19	308×308	36×36	36×36	19×19	325×325	53×53	69×69
18	20×20	344×344	38×38	38×38	20×20	362×362	56×56	73×73
19	21×21	382×382	40×40	40×40	21×21	401×401	59×59	77×77
20	22×22	422×422	42×42	42×42	22×22	442×442	62×62	81×81

Note that $\mathbf{q}_{(1\times M)}$ is the row vector of initial probabilities associated with the ZS mode so that the initial state on the TPM corresponds to the value of 1. From Equation (13), it follows that the initial state is given by

$$\text{IRR1: } \mathbf{q}_{(1\times M)} = (1\ 0\ 0\ \ldots\ 0) \text{ i.e. the } 1^{\text{st}} \text{ one}$$
$$\text{SC1: } \mathbf{q}_{(1\times M)} = (0\ 1\ 0\ \ldots\ 0) \text{ i.e. the } 2^{\text{nd}} \text{ one}$$
$$\text{IRR2, IRR3, IRR4: } \mathbf{q}_{(1\times M)} = (0\ 0\ \ldots\ 1_{\frac{(\tau+1)}{2}}\ \ldots 0\ 0) \text{ i.e. the } \left(\frac{\tau+1}{2}\right)^{\text{th}} \text{ one} \quad (14)$$
$$\text{SC2, SC3, SC4: } \mathbf{q}_{(1\times M)} = (0\ 0\ \ldots 0\ 1_{(\tau+1)}\ \ldots 0) \text{ i.e. the } (\tau+1)^{\text{th}} \text{ one}$$

In SS, the vector $\mathbf{q}$ is replaced by a vector $\mathbf{s}$, i.e. the SS initial probability vector given by

$$\boldsymbol{\xi} = \mathbf{s}_{(1\times M)} = (s_1\ s_2\ \ldots\ s_M) \quad (15)$$

where the sum of the elements in Equation (15) sum to unity. Champ[24] showed that when $\mathbf{Q_0}$ is obtained from $\mathbf{Q}(0)$ (i.e. δ = 0) after diving each element by its corresponding row sum, then $\mathbf{s}_{(1\times M)}$ is a vector such that $\mathbf{s} = \mathbf{Q_0}'\mathbf{s}$ subject to $\sum_{i=1}^{M} s_i = 1$.

For illustration purpose, in Table X we give the compound patterns of the eight schemes in Table I when $H \le 5$ with charting regions in Figure 1.

Table X: Compound patterns[1] of the Markov chain imbedding technique for the *1-of-1* or *2-of-*(*H*+1) IRR and Synthetic-$\bar{X}$ charts when $H \leq 5$

H	IRR1 / SC1	IRR2 / SC2	IRR3 / SC3	IRR4 / SC4
1	E, UU	E, AA, **±A** DD, **±D**	E, AA, **±A** DD, **±D**	E, AA, **±A** DD, **±D**
2	UOU	AOA, ADA, **±OA** DOD, DAD, **±OD**	AOA, **±OA** DOD, **±OD**	ABA, **±BA** DCD, **±CD**
3	UOOU	AOOA, AODA, ADOA, **±OOA** DOOD, DOAD, DAOD, **±OOD**	AOOA, **±OOA** DOOD, **±OOD**	ABBA, **±BBA** DCCD, **±CCD**
4	UOOOU	AOOOA, AOODA, AODOA, ADOOA, **±OOOA** DOOOD, DOOAD, DOAOD, DAOOD, **±OOOD**	AOOOA, **±OOOA** DOOOD, **±OOOD**	ABBBA, **±BBBA** DCCCD, **±CCCD**
5	UOOOOU	AOOOOA, AOOODA, AOODOA, AODOOA, ADOOOA, **±OOOOA** DOOOOD, DOOOAD, DOOAOD, DOAOOD, DAOOOD, **±OOOOD**	AOOOOA, **±OOOOA** DOOOOD, **±OOOOD**	ABBBBA, **±BBBBA** DCCCCD, **±CCCCD**

The breakdown of these compound patterns is done in Table XI by first defining the absorbing states and then obtaining the corresponding non-absorbing states from these. Then Table IX and Equations (10) to (13) are used to construct the TPMs in Table XII for *H* = 1 and 5. The construction of the TPMs of the other values of *H* follows in a similar manner.

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[1] The boldfaced simple patterns with a HS feature (i.e. starting with '±') correspond to Ψ, see Table XI.

Table XI: Components of the TPMs of the IRR and Synthetic-$\bar{X}$ charts when H = 1 and 5

H	Type		Λ	Ψ	η	ψ	Ω
1	IRR1	η_1= {O}	Λ_1={E}, Λ_2={UU}	None	η_2={U}	None	{ϕ,η_2;OOC}
	SC1						
	IRR2	η_2= {O}	Λ_1={E}, Λ_2={AA}, Λ_3={DD}	None	η_1={A}, η_3={D}	None	{η_1,ϕ,η_3;OOC}
	IRR3						
	IRR4	η_2= {B,C}					
	SC2	η_2= {O}		Ψ_1={±A}, Ψ_2={±D}		ψ_1={±}	{η_1,ϕ,η_3;ψ_1;OOC}
	SC3						
	SC4	η_2= {B,C}					

Table XI: (continued)

H	Type		Λ	Ψ	η	ψ	Ω
5	IRR1 SC1	η_1= {O}	Λ_1={E}, Λ_2={UU}, Λ_3={UOU}, Λ_4={UOOU}, Λ_5={UOOOU}, Λ_6={UOOOOU}	None	η_2={U}, η_3={UO}, η_4={UOO}, η_5={UOOO}, η_6={UOOOO}	None	{ϕ,η_2,η_3,η_4,η_5,η_6;OOC}
	IRR2	η_{16}= {O}	Λ_1={E}, Λ_2={ADOOOA}, Λ_3={AODOOA}, Λ_4={AOODOA}, Λ_5={AOOODA}, Λ_6={AOOOOA}, …, Λ_{14}={ADA}, Λ_{15}={AOA}, Λ_{16}={AA}, Λ_{17}={DD}, Λ_{18}={DOD}, Λ_{19}={DAD}, …, Λ_{27}={DOOOOD}, Λ_{28}={DOOOAD}, Λ_{29}={DOOAOD}, Λ_{30}={DOAOOD}, Λ_{31}={DAOOOD}	None	η_1={ADOOO}, η_2={AODOO}, η_3={AOODO}, η_4={AOOOD}, η_6={AOOOO}, …, η_{13}={AD}, η_{14}={AO}, η_{15}={A}, η_{17}={D}, η_{18}={DO}, η_{19}={DA}, …, η_{27}={DOOOO}, η_{28}={DOOOA}, η_{29}={DOOAO}, η_{30}={DOAOO}, η_{31}={DAOOO}	None	{η_1,…,η_{15},ϕ,η_{17},…,η_{31};OOC}
	SC2			Ψ_1={±A}, Ψ_2={±D}, Ψ_3={±OA}, Ψ_4={±OD}, Ψ_5={±OOA}, Ψ_6={±OOD}, Ψ_7={±OOOA}, Ψ_8={±OOOD}, Ψ_9={±OOOOA}, Ψ_{10}={±OOOOD}		ψ_1={±}, ψ_2={±O}, ψ_3={±OO}, ψ_4={±OOO}, ψ_5={±OOOO}	{η_1,…,η_{15},ϕ,η_{17},…,η_{31};ψ_1,…,ψ_5;OOC}
	IRR3	η_6= {O}	Λ_1={E}, Λ_2={AOOOOA}, Λ_3={AOOOA}, Λ_4={AOOA}, Λ_5={AOA}, Λ_6={AA}, Λ_7={DD}, Λ_8={DOD}, Λ_9={DOOD}, Λ_{10}={DOOOD}, Λ_{11}={DOOOOD}	None	η_1={AOOOO}, η_2={AOOO}, η_3={AOO}, η_4={AO}, η_5={A}, η_7={D}, η_8={DO}, η_9={DOO}, η_{10}={DOOO}, η_{11}={DOOOO}	None	{η_1,…,η_5,ϕ,η_7,…,η_{11};OOC}
	SC3			Ψ_1={±A}, Ψ_2={±D}, Ψ_3={±OA}, Ψ_4={±OD}, Ψ_5={±OOA}, Ψ_6={±OOD}, Ψ_7={±OOOA}, Ψ_8={±OOOD}, Ψ_9={±OOOOA}, Ψ_{10}={±OOOOD}		ψ_1={±}, ψ_2={±O}, ψ_3={±OO}, ψ_4={±OOO}, ψ_5={±OOOO}	{η_1,…,η_5,ϕ,η_7,…,η_{11};ψ_1,…,ψ_5;OOC}
	IRR4	η_6= {B,C}	Λ_1={E}, Λ_2={ABBBBA}, Λ_3={ABBBA}, Λ_4={ABBA}, Λ_5={ABA}, Λ_6={AA}, Λ_7={DD}, Λ_8={DCD}, Λ_9={DCCD}, Λ_{10}={DCCCD}, Λ_{11}={DCCCCD}	None	η_1={ABBBB}, η_2={ABBB}, η_3={ABB}, η_4={AB}, η_5={A}, η_7={D}, η_8={DC}, η_9={DCC}, η_{10}={DCCC}, η_{11}={DCCCC}	None	{η_1,…,η_5,ϕ,η_7,…,η_{11};OOC}
	SC4			Ψ_1={±A}, Ψ_2={±D}, Ψ_3={±BA}, Ψ_4={±CD}, Ψ_5={±BBA}, Ψ_6={±CCD}, Ψ_7={±BBBA}, Ψ_8={±CCCD}, Ψ_9={±BBBBA}, Ψ_{10}={±CCCCD}		ψ_1={±}, ψ_2={±B}, ψ_3={±C}, ψ_4={±BB}, ψ_5={±CC}, ψ_6={±BBB}, ψ_7={±CCC}, ψ_8={±BBBB}, ψ_9={±CCCC}	{η_1,…,η_5,ϕ,η_7,…,η_{11};ψ_1,…,ψ_9;OOC}

Table XII: TPMs of the Synthetic-$\bar{X}$ and IRR (by removing the ψ states) charts when H = 1 and 5

H=1 — SC1 and IRR1

	ϕ	η_2	OOC
ϕ	θ_O	θ_U	θ_E
η_2	θ_O		1-θ_O
OOC			1

H=1 — SC2, SC3 and SC4 (removing $_1$ yields IRR2, IRR3 and IRR4)

	η_1	ϕ	η_3	$_1$	OOC
η_1		θ_O	θ_D		θ_A+θ_E
ϕ	θ_A	θ_O	θ_D		θ_E
η_3	θ_A	θ_O			θ_D+θ_E
$_1$		θ_O			θ_A+θ_D+θ_E
OOC					1

H=5 — SC1 and IRR1

	ϕ	η_2	η_3	η_4	η_5	η_6	OOC
ϕ	θ_O	θ_U					θ_E
η_2			θ_O				1-θ_O
η_3				θ_O			1-θ_O
η_4					θ_O		1-θ_O
η_5						θ_O	1-θ_O
η_6	θ_O						1-θ_O
OOC							1

SC3 (removing $_1$, $_2$,…, $_5$ yields IRR3)

	η_1	η_2	η_3	η_4	η_5	ϕ	η_7	η_8	η_9	η_{10}	η_{11}	$_1$	$_2$	$_3$	$_4$	$_5$	OOC
η_1						θ_O	θ_D										θ_A+θ_E
η_2	θ_O						θ_D										θ_A+θ_E
η_3		θ_O					θ_D										θ_A+θ_E
η_4			θ_O				θ_D										θ_A+θ_E
η_5				θ_O			θ_D										θ_A+θ_E
ϕ					θ_A	θ_O	θ_D										θ_E
η_7					θ_A			θ_O									θ_D+θ_E
η_8					θ_A				θ_O								θ_D+θ_E
η_9					θ_A					θ_O							θ_D+θ_E
η_{10}					θ_A						θ_O						θ_D+θ_E
η_{11}					θ_A	θ_O											θ_D+θ_E
$_1$													θ_O				θ_A+θ_D+θ_E
$_2$														θ_O			θ_A+θ_D+θ_E
$_3$															θ_O		θ_A+θ_D+θ_E
$_4$																θ_O	θ_A+θ_D+θ_E
$_5$						θ_O											θ_A+θ_D+θ_E
OOC																	1

SC4 (removing $_1$, $_2$,…, $_9$ yields IRR4)

	η_1	η_2	η_3	η_4	η_5	ϕ	η_7	η_8	η_9	η_{10}	η_{11}	$_1$	$_2$	$_3$	$_4$	$_5$	$_6$	$_7$	$_8$	$_9$	OOC
η_1						θ_B+θ_C	θ_D														θ_A+θ_E
η_2	θ_B					θ_C	θ_D														θ_A+θ_E
η_3		θ_B				θ_C	θ_D														θ_A+θ_E
η_4			θ_B			θ_C	θ_D														θ_A+θ_E
η_5				θ_B		θ_C	θ_D														θ_A+θ_E
ϕ					θ_A	θ_B+θ_C	θ_D														θ_E
η_7					θ_A	θ_B		θ_C													θ_D+θ_E
η_8					θ_A	θ_B			θ_C												θ_D+θ_E
η_9					θ_A	θ_B				θ_C											θ_D+θ_E
η_{10}					θ_A	θ_B					θ_C										θ_D+θ_E
η_{11}					θ_A	θ_B+θ_C															θ_D+θ_E
$_1$													θ_B	θ_C							θ_A+θ_D+θ_E
$_2$						θ_C	θ_D								θ_B						θ_A+θ_E
$_3$					θ_A	θ_B										θ_C					θ_D+θ_E
$_4$						θ_C	θ_D										θ_B				θ_A+θ_E
$_5$					θ_A	θ_B												θ_C			θ_D+θ_E
$_6$						θ_C	θ_D												θ_B		θ_A+θ_E
$_7$					θ_A	θ_B														θ_C	θ_D+θ_E
$_8$						θ_B+θ_C	θ_D														θ_A+θ_E
$_9$					θ_A	θ_B+θ_C															θ_D+θ_E
OOC																					1

Table XII: (continued)

SC2 (removing $_1$, $_2$,…, $_5$ yields IRR2)																																					
	η_1	η_2	η_3	η_4	η_5	η_6	η_7	η_8	η_9	η_{10}	η_{11}	η_{12}	η_{13}	η_{14}	η_{15}	ϕ	η_{17}	η_{18}	η_{19}	η_{20}	η_{21}	η_{22}	η_{23}	η_{24}	η_{25}	η_{26}	η_{27}	η_{28}	η_{29}	η_{30}	η_{31}	$_1$	$_2$	$_3$	$_4$	$_5$	OOC
η_1																											θ_O										θ_A+θ_D+θ_E
η_2																							θ_O														θ_A+θ_D+θ_E
η_3																				θ_O																	θ_A+θ_D+θ_E
η_4																		θ_O																			θ_A+θ_D+θ_E
η_5																θ_O	θ_D																				θ_A+θ_E
η_6	θ_O																																				θ_A+θ_D+θ_E
η_7		θ_O																																			θ_A+θ_D+θ_E
η_8			θ_O																																		θ_A+θ_D+θ_E
η_9				θ_D	θ_O																																θ_A+θ_E
η_{10}						θ_O																															θ_A+θ_D+θ_E
η_{11}							θ_O																														θ_A+θ_D+θ_E
η_{12}								θ_D	θ_O																												θ_A+θ_E
η_{13}										θ_O																											θ_A+θ_D+θ_E
η_{14}											θ_D	θ_O																									θ_A+θ_E
η_{15}													θ_D	θ_O																							θ_A+θ_E
ϕ															θ_A	θ_O	θ_D																				θ_E
η_{17}																		θ_O	θ_A																		θ_D+θ_E
η_{18}																				θ_O	θ_A																θ_D+θ_E
η_{19}																						θ_O															θ_A+θ_D+θ_E
η_{20}																							θ_O	θ_A													θ_D+θ_E
η_{21}																									θ_O												θ_A+θ_D+θ_E
η_{22}																										θ_O											θ_A+θ_D+θ_E
η_{23}																											θ_O	θ_A									θ_D+θ_E
η_{24}																													θ_O								θ_A+θ_D+θ_E
η_{25}																														θ_O							θ_A+θ_D+θ_E
η_{26}																															θ_O						θ_A+θ_D+θ_E
η_{27}															θ_A	θ_O																					θ_D+θ_E
η_{28}														θ_O																							θ_A+θ_D+θ_E
η_{29}												θ_O																									θ_A+θ_D+θ_E
η_{30}									θ_O																												θ_A+θ_D+θ_E
η_{31}					θ_O																																θ_A+θ_D+θ_E
$_1$																																	θ_O				θ_A+θ_D+θ_E
$_2$																																		θ_O			θ_A+θ_D+θ_E
$_3$																																			θ_O		θ_A+θ_D+θ_E
$_4$																																				θ_O	θ_A+θ_D+θ_E
$_5$																θ_O																					θ_A+θ_D+θ_E
OOC																																					1

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