ON THE PERMANENT AND MAXIMAL CHARACTERISTIC ROOT OF A NONNEGATIVE MATRIX

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One of the many theorems that M. Marcus and M. Newman prove in [**3**] is, when it is restricted to nonnegative matrices, that: *if A is an $n\times n$ positive semidefinite symmetric irreducible nonnegative matrix, then*

$$\lim_{m\to\infty} (\operatorname{per}(A^m))^{1/m} = r^n$$

where r denotes the maximal (positive) characteristic root of A. Here per (A^m) denotes the *permanent* of A^m. We assume that the reader is familiar with the terminology and results of the classical Perron-Frobenius-Wielandt theory of nonnegative matrices, the requisite parts of which can be found in [**1**, Chapter XIII].

The purpose of this note is to prove the following extension of the result quoted above.

THEOREM. *Let A be an $n\times n$ nonnegative irreducible matrix and suppose A has a nonzero permanent. Let r be the maximal characteristic root of A. Then*

$$\lim_{m\to\infty} (\operatorname{per}(A^m))^{1/m} = r^n. \tag{1}$$

We first prove two lemmas.

LEMMA 1. *Let B be an $n\times n$ nonnegative matrix with maximal characteristic root ρ. Then*

$$\operatorname{per}(B) \leqq \rho^n.$$

PROOF.[2] First assume A is positive (or irreducible). Then ρ is positive and has an associated characteristic vector $x=(x_1, \cdots, x_n)$, all of whose coordinates are positive. From $Bx=\rho x$, it follows that

$$\sum_{j=1}^{n} b_{ij}x_j = \rho x_i, \qquad i = 1, \cdots, n$$

and hence

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[2] This is a slight variation of a proof originally shown the author by Morris Newman.

$$\prod_{i=1}^{n}\left(\sum_{j=1}^{n} b_{ij}x_j\right) = \rho^n \prod_{i=1}^{n} x_i.$$

This may be written as

$$\text{per}\,(B)\prod_{i=1}^{n} x_i + \beta = \rho^n \prod_{i=1}^{n} x_i$$

where $\beta \geqq 0$ and the result follows. The complete lemma now follows by continuity.

LEMMA 2. *Let A be an $n\times n$ nonnegative primitive matrix with maximal characteristic root r. Then* (1) *holds.*

PROOF. If m is a positive integer, then using Lemma 1 and the fact that A^m is nonnegative, we may write

$$\prod_{i=1}^{n} a_{ii}^{(m)} \leqq \text{per}\,(A^m) \leqq (r^m)^n$$

where $a_{ij}^{(m)}$ denotes the (i, j) entry of A^m. Thus we have

$$\prod_{i=1}^{n} (a_{ii}^{(m)})^{1/m} \leqq \text{per}\,(A^m)^{1/m} \leqq r^n. \tag{2}$$

But it is well known (see [**2**, p. 128] or [**1**, p. 81]) that for a primitive matrix A,

$$\lim_{m\to\infty} (a_{ij}^{(m)})^{1/m} = r.$$

Taking limits in (2), we obtain (1).

PROOF OF THEOREM. Let h denote the index of imprimitivity of A, that is, the number of characteristic roots of A of modulus r. Then [**1**, p. 81–82] there is a permutation matrix P such that

$$P^T A^h P = \text{diag}\,(A_1, A_2, \cdots, A_h)$$

where each A_i, $1 \leqq i \leqq h$, is an $n_i \times n_i$ primitive matrix with maximal characteristic root r^h. Observe that if k is a positive integer, then

$$(P^T A^h P)^k = P^T A^{hk} P = \text{diag}\,(A_1^k, A_2^k, \cdots, A_h^k).$$

Since the permanent of a matrix is invariant under permutations, we have

$$(\text{per}\,(A^{hk}))^{1/hk} = \left[(\text{per}\,(A_1^k))^{1/k} \cdots (\text{per}\,(A_h^k))^{1/k}\right]^{1/h}.$$

Since each A_i is a primitive matrix, it follows by Lemma 2, that

$$\lim_{k\to\infty} (\operatorname{per}(A^{hk}))^{1/hk} = \left(\prod_{i=1}^{h} (r^h)^{n_i}\right)^{1/h} = r^n. \tag{3}$$

Now let m be an arbitrary positive integer and write

$$m = hp_m + q_m; \qquad p_m \geqq 0, \quad 0 \leqq q_m < h.$$

Then

$$A^m = A^{hp_m} A^{q_m}$$

where $A^{q_m} = I$ if $q_m = 0$. Thus since all matrices are nonnegative

$$\operatorname{per}(A^m) \geqq \operatorname{per}(A^{hp_m}) \operatorname{per}(A^{q_m}).$$

From (3) it follows immediately that

$$\lim_{m\to\infty} (\operatorname{per}(A^{hp_m}))^{1/hp_m} = r^n. \tag{4}$$

But then using (4) we obtain

$$\lim_{m\to\infty} (\ln(\operatorname{per}(A^{hp_m}))^{1/hp_m} - \ln(\operatorname{per}(A^{hp_m})^{1/m}))$$

$$= \lim_{m\to\infty} \frac{q_m}{m} \ln(\operatorname{per}(A^{hp_m}))^{1/hp_m} = 0.$$

Hence

$$\lim_{m\to\infty} \frac{(\operatorname{per}(A^{hp_m}))^{1/hp_m}}{(\operatorname{per}(A^{hp_m}))^{1/m}} = 1. \tag{5}$$

Now (4) and (5) together imply

$$\lim_{m\to\infty} (\operatorname{per}(A^{hp_m}))^{1/m} = r^n. \tag{6}$$

Now by assumption $\operatorname{per}(A) > 0$. If $\operatorname{per}(A) \geqq 1$, then

$$\operatorname{per}(A^{q_m}) \geqq (\operatorname{per}(A))^{q_m} \geqq 1, \quad \text{for all } m;$$

while if $\operatorname{per}(A) < 1$, then

$$\operatorname{per}(A^{q_m}) \geqq (\operatorname{per}(A))^{q_m} \geqq \operatorname{per}(A)^{h-1}, \quad \text{for all } m.$$

If $r \geqq 1$, then by Lemma 1

$$\operatorname{per}(A^{q_m}) \leqq (r^n)^{q_m} \leqq r^{n(h-1)}, \quad \text{for all } m;$$

while if $r \leqq 1$, then

$$\text{per}\,(A^{qm}) \leqq (r^n)^{qm} \leqq 1, \quad \text{for all } m.$$

Thus in any case there are *positive* constants c_1 and c_2 such that

$$c_1 \leqq \text{per}\,(A^{qm}) \leqq c_2$$

for all positive integers m. From this it now follows that

$$\lim_{m\to\infty} (\text{per}\,(A^{qm}))^{1/m} = 1. \tag{7}$$

Passing to the limit in the inequality

$$(\text{per}\,(A^{hpm}))^{1/m}(\text{per}\,(A^{qm}))^{1/m} \leqq (\text{per}\,(A^m))^{1/m} \leqq r^n,$$

and using (6) and (7), we obtain the desired result.

The condition that A have a nonzero permanent cannot in general be omitted. The irreducible matrix

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

has the property that per $(A^k)=0$ for k odd and per $(A^k)>0$ for k even. It is easy to verify that A has maximal characteristic root $2^{1/2}$ and index of imprimitivity 2. It follows from (3) that if $\lim_{m\to\infty}$ (per $(A^m))^{1/n}$ exists, it must be $2^{3/2}$. Hence the limit does not even exist.

Finally we remark that our theorem does constitute an extension of the result of Marcus and Newman mentioned in the beginning. For, if A is a positive semidefinite symmetric irreducible nonnegative matrix, then the entries on the main diagonal of A must be positive. Otherwise A has a zero row and thus is not irreducible. Hence the permanent of such a matrix is positive.

References

1. F. R. Gantmacher, *The theory of matrices*, Vol. II, Chelsea, New York, 1959.

2. M. Marcus and H. Minc, *A survey of matrix theory and matrix inequalities*, Allyn and Bacon, Boston, 1964.

3. M. Marcus and M. Newman, *Inequalities for the permanent function*, Ann. of Math. (2) **75** (1962), 47–62.

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