ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 37, Number 6, Winter 2007

ON THE PERMANENTS OF SOME TRIDIAGONAL MATRICES WITH APPLICATIONS TO THE FIBONACCI AND LUCAS NUMBERS

EMRAH KILIC AND DURSUN TAŞCI

ABSTRACT. In this paper, we derive some interesting relationships between the permanents of some tridiagonal matrices with applications to the negatively and positively subscripted usual Fibonacci and Lucas numbers. Also, we give a relation involving the generalized order-k Lucas number and permanent of a matrix.

1. Introduction. The Fibonacci sequence, $\{F_n\}$, is defined by the recurrence relation, for $n \geq 1$

$$F_{n+1} = F_n + F_{n-1} \tag{1.1}$$

where $F_0 = 0$, $F_1 = 1$. The Lucas sequence, $\{L_n\}$, is defined by the recurrence relation, for $n \geq 1$

$$L_{n+1} = L_n + L_{n-1} \tag{1.2}$$

where $L_0 = 2$, $L_1 = 1$.

Rules (1.1) and (1.2) can be used to extend the sequences backward, respectively, thus

$$\begin{aligned} F_{-1} &= F_1 - F_0, \qquad F_{-2} = F_0 - F_{-1} \\ L_{-1} &= L_1 - L_0, \qquad L_{-2} = L_0 - L_{-1}, \dots , \end{aligned}$$

and so on. Clearly,

$$F_{-n} = F_{-n+2} - F_{-n+1} = (-1)^{n+1} F_n, \tag{1.3}$$

$$L_{-n} = L_{-n+2} - L_{-n+1} = (-1)^{n} L_n. \tag{1.4}$$

2000 AMS *Mathematics subject classification.* Primary 15A15, 11B39, 15A36.
Keywords and phrases. Permanent, Fibonacci number, generalized order-k Lucas sequence
Received by the editors on December 8, 2004, and in revised form on August 18, 2005.

In [**2**] Er defined k sequences of the generalized *order-k* Fibonacci numbers as shown:

$$g_n^i = \sum_{j=1}^{k} g_{n-j}^i, \quad \text{for} \quad n > 0 \quad \text{and} \quad 1 < i \leq k, \tag{1.5}$$

with boundary conditions for $1 - k \leq n \leq 0$,

$$g_n^i = \begin{cases} 1 & \text{if } n = 1 - i, \\ 0 & \text{otherwise,} \end{cases}$$

where g_n^i is the nth term of the ith sequence. For example, if $k = 2$, then $\{g_n^2\}$ is the usual Fibonacci sequence, $\{F_n\}$, and, if $k = 4$, then the fourth sequence of the generalized *order*-4 Fibonacci number is

$$1, 1, 2, 4, 8, 15, 29, 56, 108, 208, 401, 773, 1490, \dots .$$

In [**6**] the authors defined k sequences of the generalized *order-k* Lucas numbers as shown:

$$l_n^i = \sum_{j=1}^{k} l_{n-j}^i, \quad \text{for} \quad n > 0 \quad \text{and} \quad 1 < i \leq k,$$

with boundary conditions for $1 - k \leq n \leq 0$,

$$l_n^i = \begin{cases} -1 & \text{if } n = 1 - i, \\ 2 & \text{if } n = 2 - i, \\ 0 & \text{otherwise,} \end{cases}$$

where l_n^i is the nth term of the ith sequence. For example, if $k = 2$, then $\{l_n^2\}$ is the usual Lucas sequence, $\{L_n\}$, and, if $k = 4$, then the fourth sequence of the generalized *order*-4 Lucas numbers is

$$1, 3, 4, 8, 16, 31, 59, 114, 220, 424, 817, 1575, 30636, \dots .$$

In [**3**], we gave the generalized Binet formula, the combinatorial representations and some relations involving the generalized order-k Fibonacci and Lucas numbers. In particular, we showed that, for $k \geq 2$

$$l_n^k = g_n^k + 2g_{n-1}^k \tag{1.6}$$

where l_n^k and g_n^k are the generalized *order-k* Lucas and Fibonacci numbers, respectively, for $i = k$. The above result is a well-known relation that, for $k = 2$,

$$L_n = F_n + 2F_{n-1} \quad \text{(see [\textbf{7}, page 176])}.$$

The *permanent* of an n-square matrix $A = (a_{ij})$ is defined by

$$\operatorname{per} A = \sum_{\sigma \in S_n} \prod_{i=1}^{n} a_{i\sigma(i)}$$

where the summation extends over all permutations σ of the symmetric group S_n.

The permanent of a matrix is analogous to the determinant, where all of the signs used in the Laplace expansion of minors are positive.

A matrix is said to be a $(0,1)$-*matrix* if each of its entries is either 0 or 1.

In [**5**], Minc constructed the $n \times n$ $(0,1)$-*matrix* $F(n,k)$ as follows:

$$F(n,k) = \begin{bmatrix} 1 & 1 & \dots & 1 & 0 & 0 & 0 & \dots & 0 \\ 1 & 1 & \dots & 1 & 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & \dots & 1 & 1 & 1 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & \dots & 0 & 1 & 1 & \dots & 1 & 0 \\ 0 & \dots & \dots & 0 & 0 & 1 & \dots & 1 & 1 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & \dots & 0 & \dots & 0 & 1 & 1 & 1 \\ 0 & \dots & \dots & 0 & \dots & 0 & 0 & 1 & 1 \end{bmatrix} \tag{1.7}$$

where, $k \le n+1$, $F(n,k)$ denote the n-square $(0,1)$-*matrix* with 1 in the (i,j) position for $i - 1 \le j \le i + k - 1$ and 0 otherwise. Also, he showed that

$$\operatorname{per} F(n,k) = g_{n+1}^k \tag{1.8}$$

where g_n^k is the nth generalized order-k Fibonacci number, for $i = k$.

Also, Lee defined the matrix $\mathcal{L}_n$ as follows [4]:

$$\mathcal{L}_n = \begin{bmatrix} 1 & 0 & 1 & 0 & & & & \\ 1 & 1 & 1 & 0 & \ddots & & & \\ 0 & 1 & 1 & 1 & \ddots & 0 & & \\ & \ddots & \ddots & \ddots & \ddots & 0 & 0 & \\ & & 0 & 1 & 1 & 1 & 0 & 0 \\ & & & 0 & 1 & 1 & 1 & 0 \\ & & & & 0 & 1 & 1 & 1 \\ & & & & & 0 & 1 & 1 \end{bmatrix}$$

and showed that

$$\operatorname{per} \mathcal{L}_n = L_{n-1}$$

where L_n is the nth usual Lucas number.

A matrix is said to be a $(-1, 0, 1)$-*matrix* if each of its entries are $-1, 0$ or 1.

The purpose of this paper is to develop relations involving the positively and negatively subscripted Fibonacci and Lucas numbers and the permanents of some $(-1, 0, 1)$-tridiagonal matrices.

Let $A = [a_{ij}]$ be an $m \times n$ real matrix row vectors $\alpha_1, \alpha_2, \dots, \alpha_m$. We say A is *contractible on column*, respectively *row*, k *if column*, respectively *row*, k contains exactly two nonzero entries. Suppose A is contractible on column k with $a_{ik} \neq 0 \neq a_{jk}$ and $i \neq j$. Then the $(m-1) \times (n-1)$ matrix $A_{ij:k}$ obtained from A by replacing row i with $a_{jk}\alpha_i + a_{ik}\alpha_j$ and deleting row j and column k is called the *contraction of A on column k relative to rows i and j.* If A is contractible on row k with $a_{ki} \neq 0 \neq a_{kj}$ and $i \neq j$, then the matrix $A_{k:ij} = \left[A^T_{ij:k}\right]^T$ is called the *contraction of A on row k relative to columns i and j.* Every contraction used in this paper will be on the first column using the first and second rows.

We say that A can be contracted to a matrix B if either $B = A$ or there exist matrices $A_0, A_1, \dots A_t$, $t \geq 1$, such that $A_0 = A$, $A_t = B$ and A_r is a contraction of A_{r-1} for $r = 1, 2, \dots, t$.

2. On the contraction of some tridiagonal matrices. Let us consider the following Lemma, see [1].

Lemma 1. *Let A be a nonnegative integral matrix of order $n > 1$, and let B be a contraction of A. Then*

$$\operatorname{per} A = \operatorname{per} B.$$

Let $C = [c_{ij}]$ be an $n \times n$ tridiagonal matrix as follows:
(2.1)

$$C = \begin{bmatrix} c_{11} & -c_{12} & 0 & & & & \\ -c_{21} & c_{22} & -c_{23} & \ddots & & & \\ 0 & -c_{32} & c_{33} & \ddots & & & \\ & \ddots & \ddots & \ddots & & & \\ & & & & c_{n-2,n-2} & -c_{n-2,n-1} & 0 \\ & & & & -c_{n-1,n-2} & c_{n-1,n-1} & -c_{n-1,n} \\ & & & & 0 & -c_{n,n-1} & c_{n,n} \end{bmatrix}$$

where the c_{ij}s are arbitrary integers and $c_{ij} \neq 0$.

So give an extension of Lemma 1 for the matrix C, by the following Theorem.

Theorem 1. *Let C be as in* (2.1)*, and let B be a contraction of C. Then*

$$\operatorname{per} C = \operatorname{per} B.$$

Proof. It suffices to consider the case where B is the contraction of C on column 1 relative to rows 1 and 2. Thus, C and B have the form

$$C = \begin{bmatrix} c_{11} & -c_{12} \\ -c_{21} & c_{22} \\ 0 & D \end{bmatrix}, \qquad B = \begin{bmatrix} c_{11}c_{22} + c_{12}c_{21} \\ D \end{bmatrix}$$

where $c_{ij} \neq 0$.

Using the Laplace expansion of the permanent with respect to column 1, we obtain

$$\operatorname{per} C = c_{11} \operatorname{per} \begin{bmatrix} c_{22} \\ D \end{bmatrix} + c_{21} \operatorname{per} \begin{bmatrix} c_{12} \\ D \end{bmatrix}$$

Hence, by linearity of the permanent, $\operatorname{per} C = \operatorname{per} B$. □

Now we give an application of the above result. We introduce an n-square $(-1,0,1)$-tridiagonal Toeplitz matrix whose permanent and principal subpermanents are Fibonacci numbers of prescribed order.

Let A_n denote an $n \times n$ $(-1,0,1)$-tridiagonal Toeplitz matrix as follows: for $n \geq 3$

$$A_n = \begin{bmatrix} 1 & -1 & & & & 0 \\ -1 & 1 & -1 & & & \\ & -1 & 1 & \ddots & & \\ & & \ddots & \ddots & -1 & \\ & & & -1 & 1 & -1 \\ 0 & & & & -1 & 1 \end{bmatrix} \tag{2.2}$$

Corollary 1. *Let A_n be the $n \times n$ $(-1,0,1)$-tridiagonal Toeplitz matrix as in* (2.2)*. Then, for $n \geq 3$*

$$\operatorname{per} A_n = F_{n+1}$$

where F_n is the nth Fibonacci number.

Proof. If $n = 3$, then we have

$$A_3 = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

and hence $\operatorname{per} A_3 = 3 = F_4$.

Let A_n^p be the pth contraction of A_n, $1 \leq p \leq n-2$. From the definition of A_n, the matrix A_n can be contracted on column 1 so that

$$A_n^1 = \begin{bmatrix} 2 & -1 & & & & \\ -1 & 1 & -1 & & & \\ & -1 & 1 & \ddots & & \\ & & \ddots & \ddots & -1 & \\ & & & -1 & 1 & -1 \\ & & & & -1 & 1 \end{bmatrix}.$$

Since the matrix A_n^1 can be contracted on column 1 and $F_4 = 3$, $F_3 = 2$,

$$A_n^2 = \begin{bmatrix} 3 & -2 & & & & \\ -1 & 1 & -1 & & & \\ & -1 & 1 & \ddots & & \\ & & \ddots & \ddots & -1 & \\ & & & -1 & 1 & -1 \\ & & & & -1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} F_4 & -F_3 & & & & \\ -1 & 1 & -1 & & & \\ & -1 & 1 & \ddots & & \\ & & \ddots & \ddots & -1 & \\ & & & -1 & 1 & -1 \\ & & & & -1 & 1 \end{bmatrix}.$$

Furthermore, the matrix A_n^2 can be contracted on column 1 so that

$$A_n^3 = \begin{bmatrix} 5 & -3 & & & & \\ -1 & 1 & -1 & & & \\ & -1 & 1 & \ddots & & \\ & & \ddots & \ddots & -1 & \\ & & & -1 & 1 & -1 \\ & & & & -1 & 1 \end{bmatrix}$$

where $F_5 = 5$, $F_4 = 3$. Continuing this process, we obtain

$$A_n^r = \begin{bmatrix} F_{r+2} & -F_{r+1} & & & & \\ -1 & 1 & -1 & & & \\ & -1 & 1 & \ddots & & \\ & & \ddots & \ddots & -1 & \\ & & & -1 & 1 & -1 \\ & & & & -1 & 1 \end{bmatrix}$$

for $3 \le r \le n-4$. Hence,

$$A_n^{n-3} = \begin{bmatrix} F_{n-1} & -F_{n-2} & 0 \\ -1 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

which, by contraction of A_n^{n-4} on column 1, gives

$$A_n^{n-2} = \begin{bmatrix} F_{n-2}+F_{n-1} & -F_{n-1} \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} F_n & -F_{n-1} \\ -1 & 1 \end{bmatrix}$$

By applying Theorem 1, we obtain

$$\operatorname{per} A_n = \operatorname{per} A_n^{n-2} = F_{n+1}.$$

So the proof is complete. □

As a different application of the contraction, we give the following Lemma for negatively subscripted Lucas numbers.

Lemma 2. *Let V_n be an $n \times n$ matrix of the form:*

$$V_n = \begin{bmatrix} -1 & 2 & 0 & & & & & \\ 1 & -1 & 1 & 0 & & & & \\ 0 & 1 & -1 & 1 & 0 & & & \\ & 0 & 1 & -1 & 1 & \ddots & & \\ & & \ddots & \ddots & \ddots & \ddots & 0 & \\ & & & 0 & 1 & -1 & 1 & 0 \\ & & & & 0 & 1 & -1 & 1 \\ & & & & & 0 & 1 & -1 \end{bmatrix},$$

and let V_n^m denote the mth contraction of the matrix V_n. Then

$$\operatorname{per} V_n^{n-2} = L_{-n}$$

where L_{-n} is the nth negatively subscripted Lucas number.

Proof. From the definition of the matrix V_n, the matrix V_n can be contracted on column 1, so that

$$V_n^1 = \begin{bmatrix} 3 & -1 & 0 & & & \\ 1 & -1 & 1 & \ddots & & \\ 0 & 1 & -1 & \ddots & 0 & \\ & \ddots & \ddots & \ddots & 1 & 0 \\ & & 0 & 1 & -1 & 1 \\ & & & 0 & 1 & -1 \end{bmatrix}$$

Since the matrix V_n^1 can be contracted on column 1 and $L_{-3} = -4$, $L_{-2} = 3$,

$$V_n^2 = \begin{bmatrix} -4 & 3 & 0 & & & \\ 1 & -1 & 1 & \ddots & & \\ 0 & 1 & -1 & \ddots & 0 & \\ & \ddots & \ddots & \ddots & 1 & 0 \\ & & 0 & 1 & -1 & 1 \\ & & & 0 & 1 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} L_{-3} & L_{-2} & 0 & & & \\ 1 & -1 & 1 & 0 & & \\ 0 & 1 & -1 & 1 & \ddots & \\ & \ddots & \ddots & \ddots & \ddots & 0 \\ & & 0 & 1 & -1 & 1 \\ & & & 0 & 1 & -1 \end{bmatrix}.$$

Furthermore, the matrix V_n^2 can be contracted on column 1 and $L_{-4} = 7$ so that

$$V_n^3 = \begin{bmatrix} 7 & -4 & 0 & & & \\ 1 & -1 & 1 & \ddots & & \\ 0 & 1 & -1 & \ddots & 0 & \\ & \ddots & \ddots & \ddots & 1 & 0 \\ & & 0 & 1 & -1 & 1 \\ & & & 0 & 1 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} L_{-4} & L_{-3} & 0 & & & \\ 1 & -1 & 1 & \ddots & & \\ 0 & 1 & -1 & \ddots & 0 & \\ & \ddots & \ddots & \ddots & 1 & 0 \\ & & 0 & 1 & -1 & 1 \\ & & & 0 & 1 & -1 \end{bmatrix}.$$

Continuing this process, we reach

$$V_n^k = \begin{bmatrix} L_{-(k+1)} & L_{-k} & 0 & & & \\ 1 & -1 & 1 & \ddots & & \\ 0 & 1 & -1 & \ddots & 0 & \\ & \ddots & \ddots & \ddots & 1 & 0 \\ & & 0 & 1 & -1 & 1 \\ & & & 0 & 1 & -1 \end{bmatrix}$$

for $3 \leq k \leq n-4$. Hence,

$$V_n^{n-3} = \begin{bmatrix} L_{-(n-2)} & L_{-(n-3)} & 0 \\ 1 & -1 & 1 \\ 0 & 1 & -1 \end{bmatrix}$$

which, by contraction of V_n^{n-4}, gives

$$V_n^{n-2} = \begin{bmatrix} L_{-(n-1)} & L_{-(n-2)} \\ 1 & -1 \end{bmatrix}.$$

Then we calculate

$$\operatorname{per} V_n^{n-2} = L_{-n+2} - L_{-n+1}$$

and, using formula (1.4), we obtain

$$\operatorname{per} V_n^{n-2} = L_{-n}.$$

So the proof is complete. □

3. On the permanents relations of some tridiagonal matrices. In this section, we introduce the sequences of $n \times n$ tridiagonal matrices $\{C_1(n),\ n = 1, 2, \dots\}$ and $\{C_2(n),\ n = 1, 2, \dots\}$, where $C_1(n)$ and $C_2(n)$ matrices of the forms:

$$C_1(n) = \begin{bmatrix} c_{1,1} & c_{1,2} & & & & \\ c_{2,1} & c_{2,2} & c_{2,3} & & & \\ & c_{3,2} & c_{3,3} & \ddots & & \\ & & \ddots & \ddots & & \\ & & & & c_{n-1,n-1} & c_{n-1,n} \\ & & & & c_{n,n-1} & c_{n,n} \end{bmatrix} \tag{3.1}$$

and

$$
(3.2)\qquad C_2(n) = \begin{bmatrix} -c_{1,1} & c_{1,2} & & & & \\ c_{2,1} & -c_{2,2} & c_{2,3} & & & \\ & c_{3,2} & -c_{3,3} & \ddots & & \\ & & \ddots & \ddots & & \\ & & & & -c_{n-1,n-1} & c_{n-1,n} \\ & & & & c_{n,n-1} & -c_{n,n} \end{bmatrix}
$$

where the signs of the main diagonals of the matrices $C_1(n)$ and $C_2(n)$ are positive and negative, respectively.

Now we give some recursive formula for the permanents of the matrices $C_1(n)$ and $C_2(n)$. We start with the following Lemma.

Lemma 3. *Let the sequence* $\{C_1(n),\ n = 1, 2, \ldots\}$ *be as in* (3.1). *Then the successive permanents of* $C_1(n)$ *are given by the recursive formula*

$$
(3.3)\qquad \begin{aligned} \operatorname{per} C_1(1) &= c_{1,1}, \\ \operatorname{per} C_1(2) &= c_{1,1}c_{2,2} + c_{1,2}c_{2,1}, \\ \operatorname{per} C_1(n) &= c_{n,n} perC_1(n-1) + c_{n-1,n}c_{n,n-1} perC_1(n-2). \end{aligned}
$$

Proof. We prove Lemma 3 by the second principle of finite induction, computing all permanents by the Laplace expansion of the permanent with respect to the last column. For the basis step, we have:

$$
\begin{aligned} \operatorname{per} C_1(1) &= c_{1,1}, \\ \operatorname{per} C_1(2) &= \operatorname{per} \begin{bmatrix} c_{1,1} & c_{1,2} \\ c_{2,1} & c_{2,2} \end{bmatrix} = c_{1,1}c_{2,2} + c_{1,2}c_{2,1}, \\ \operatorname{per} C_1(3) &= \operatorname{per} \begin{bmatrix} c_{1,1} & c_{1,2} & 0 \\ c_{2,1} & c_{2,2} & c_{2,3} \\ 0 & c_{3,2} & c_{3,3} \end{bmatrix} \\ &= c_{3,3} \operatorname{per} C_1(2) + c_{2,3} \operatorname{per} \begin{bmatrix} c_{1,1} & c_{1,2} \\ 0 & c_{3,2} \end{bmatrix} \\ &= c_{3,3} \operatorname{per} C_1(2) + c_{2,3}c_{3,2} \operatorname{per} C_1(1). \end{aligned}
$$

For the inductive step, we assume that

$$\operatorname{per} C_1(k) = c_{k,k}\operatorname{per} C_1(k-1) + c_{k-1,k}c_{k,k-1}\operatorname{per} C_1(k-2)$$

for $3 \le k \le n$. Now we show that the equation is true for $k+1$. Then

$$\begin{aligned}
&\operatorname{per} C_1(k+1)\\
&= \operatorname{per}\begin{bmatrix} c_{1,1} & c_{1,2} & & & & \\ c_{2,1} & c_{2,2} & c_{2,3} & & & \\ & c_{3,2} & c_{3,3} & \ddots & & \\ & & \ddots & \ddots & & \\ & & & c_{k-1,k-1} & c_{k-1,k} & \\ & & & c_{k,k-1} & c_{k,k} & c_{k,k+1} \\ & & & & c_{k+1,k} & c_{k+1,k+1} \end{bmatrix}\\
&= c_{k+1,k+1}\operatorname{per} C_1(k)\\
&\quad + c_{k,k+1}\operatorname{per}\begin{bmatrix} c_{1,1} & c_{1,2} & 0 & & & \\ c_{2,1} & c_{2,2} & c_{2,3} & & & \\ 0 & c_{3,2} & c_{3,3} & \ddots & & \\ & & \ddots & \ddots & c_{k-2,k-1} & 0 \\ & & & & c_{k-1,k-1} & c_{k-1,k} \\ & & & & 0 & c_{k+1,k} \end{bmatrix}\\
&= c_{k+1,k+1}\operatorname{per} C_1(k) + c_{k,k+1}c_{k+1,k}\operatorname{per} C_1(k-1).
\end{aligned}$$

So the proof is complete. □

If we take $c_{ij} = 1$, $i-1 \le j \le i+1$, in the above Lemma, then $\operatorname{per} C_1(1) = 1$, $\operatorname{per} C_1(2) = 2$ and $\operatorname{per} C_1(n) = \operatorname{per} C_1(n-1) + \operatorname{per} C_1(n-2)$, which is exactly the Fibonacci recurrence.

Lemma 4. *Let the sequence* $\{C_2(n),\ n = 1, 2, \ldots\}$ *be as in* (3.2). *Then the successive permanents of* $C_2(n)$ *is given by the recursive formula*

$$\begin{aligned}
&\operatorname{per} C_2(1) = -c_{1,1},\\
&\operatorname{per} C_2(2) = c_{1,1}c_{2,2} + c_{1,2}c_{2,1}, \qquad (3.4)\\
&\operatorname{per} C_2(n) = -c_{n,n}\operatorname{per} C_2(n-1) + c_{n-1,n}c_{n,n-1}\operatorname{per} C_2(n-2).
\end{aligned}$$

Proof. By the similar method in Lemma 3, the proof is readily seen. □

If we take $c_{ij} = 1$, $i-1 \le j \le i+1$, in the above Lemma, then $\operatorname{per} C_2(1) = -1$, $\operatorname{per} C_2(2) = 2$ and $\operatorname{per} C_2(n) = -\operatorname{per} C_2(n-1) + \operatorname{per} C_2(n-2)$, which is exactly the negatively subscripted Fibonacci recurrence.

Combining Lemmas 3 and 4, we give the following Theorem.

Theorem 2. *Let the sequences*

$$\{C_1(n),\ n = 1, 2, \ldots\} \quad \textit{and} \quad \{C_2(n),\ n = 1, 2, \ldots\}$$

be as in (3.1) *and* (3.2)*, respectively. Then, for* $n \ge 1$,

$$(-1)^n \operatorname{per} C_2(n) = \operatorname{per} C_1(n).$$

Proof. We will use induction method to prove that $(-1)^n \operatorname{per} C_2(n) = \operatorname{per} C_1(n)$. If $n = 1$, then

$$(-1)^1 \operatorname{per} C_2(1) = c_{1,1} = \operatorname{per} C_1(1).$$

Suppose that the equation holds for n. So we have

$$(-1)^n \operatorname{per} C_2(n) = \operatorname{per} C_1(n). \tag{3.5}$$

Now we show that the equation is true for $n+1$. From equation (3.4), we write that

$$\begin{aligned} &(-1)^{n+1} \operatorname{per} C_2(n+1) \\ &= (-1)^{n+1} \left(-c_{n+1,n+1} \operatorname{per} C_2(n) + c_{n,n+1} c_{n+1,n} \operatorname{per} C_2(n-1)\right) \\ &= (-1)^n c_{n+1,n+1} \operatorname{per} C_2(n) + (-1)^{n+1} c_{n,n+1} c_{n+1,n} \operatorname{per} C_2(n-1) \end{aligned}$$

and by using equation (3.5), we may write

$$(-1)^{n+1} \operatorname{per} C_2(n+1) = c_{n+1,n+1} \operatorname{per} C_1(n) + c_{n,n+1} c_{n+1,n} \operatorname{per} C_1(n-1).$$

Also, from Lemma 3, the last equation can be written as

$$(-1)^{n+1}\operatorname{per} C_2\,(n+1) = \operatorname{per} C_1(n+1).$$

So the proof is complete. □

In the above Theorem, we obtain the relation $(-1)^{n+1}\operatorname{per} C_2\,(n+1) = \operatorname{per} C_1\,(n+1)$ which is exactly the relation between the negatively and positively subscripted Fibonacci numbers.

As a result of Theorem 2, we give following Corollary for the negatively subscripted Fibonacci numbers.

Corollary 2. *Let $\{T\,(n)\,,\ n = 1, 2, \dots\}$ be a sequence of $n \times n$ tridiagonal matrices of the form:*

$$T\,(n) = \begin{bmatrix} -1 & 1 & 0 & 0 & & & \\ 1 & -1 & 1 & 0 & & & \\ 0 & 1 & -1 & 1 & \ddots & & \\ & & & \ddots & \ddots & & \\ & & & & 1 & -1 & 1 \\ & & & & 0 & 1 & -1 \end{bmatrix}. \tag{3.6}$$

Then the permanent of the matrix $T\,(n)$ is $F_{-(n+1)}$ where F_{-n} is the nth negatively subscripted Fibonacci number.

Proof. When $k = 2$ in (1.7), then $F\,(n,k)$ is reduced to the matrix

$$F\,(n,2) = \begin{bmatrix} 1 & 1 & 0 & & & & \\ 1 & 1 & 1 & 0 & & & \\ 0 & 1 & 1 & 1 & \ddots & & \\ & \ddots & \ddots & \ddots & \ddots & 0 & \\ & & 0 & 1 & 1 & 1 & 0 \\ & & & 0 & 1 & 1 & 1 \\ & & & & 0 & 1 & 1 \end{bmatrix}.$$

Also, when $k = 2$ in the definition of the k-generalized Fibonacci sequence $\{g_n^k\}$, then the sequence is reduced to the usual Fibonacci sequence $\{F_n\}$. Then by (1.8) we have

$$\operatorname{per} F\,(n,2) = g_{n+1}^2 = F_{n+1}. \tag{3.7}$$

It is also seen that the matrix $T(n)$ and $F(n,2)$ are elements of the sequences $\{C_2(n)\}$ and $\{C_1(n)\}$, respectively. Using Theorem 2, we have that

$$\operatorname{per} F(n,2) = (-1)^n \operatorname{per} T(n)$$

or

$$\operatorname{per} T(n) = (-1)^n \operatorname{per} F(n,2).$$

From equation (3.7), we write

$$\operatorname{per} T(n) = (-1)^n F_{n+1}.$$

Thus, we obtain

$$\operatorname{per} T(n) = F_{-(n+1)}.$$

So the proof is complete. □

4. On generalized order-k Lucas numbers. Let $H(n+1,k) = [h_{ij}]$ be a $(n+1)\times(n+1)$ matrix as the form:

$$H(n+1,k) = \begin{bmatrix} 1 & 2 & 0 & \dots & 0 \\ 1 & & & & \\ 0 & & F(n,k) & & \\ \vdots & & & & \\ 0 & & & & \end{bmatrix} \tag{4.1}$$

where $F(n,k)$ is the $n\times n$ $(0,1)$-*matrix* as in (1.7).

Now we give a relation between the generalized order-k Lucas number, l_n^i, for $i=k$ and permanent of the matrix $H(n+1,k)$ by the following theorem.

Theorem 3. *Let the matrix $H(n+1,k)$ be as in (4.1). Then*

$$\operatorname{per} H(n+1,k) = l_{n+1}^k$$

where l_n^k is the nth element of kth sequence of the generalized order-k Lucas numbers.

Proof. Using the Laplace expansion of the permanent for the matrix $H\,(n+1,k)$ with respect to row 1, we have

$$\operatorname{per} H\,(n+1,k) = \operatorname{per} F\,(n,k) + 2\operatorname{per} \begin{bmatrix} 1 & 1 & \dots & 1 & 0 & \dots & 0 \\ 0 & F(n-1,k) & & & & & \\ \vdots & & & & & & \\ 0 & & & & & & \end{bmatrix}.$$

Let

$$C = \begin{bmatrix} 1 & 1 & \dots & 1 & 0 & \dots & 0 \\ 0 & F(n-1,k) & & & & & \\ \vdots & & & & & & \\ 0 & & & & & & \end{bmatrix};$$

then we may write

$$\operatorname{per} C = \operatorname{per} F(n-1,k).$$

Thus,

$$\operatorname{per} H\,(n+1,k) = \operatorname{per} F\,(n,k) + 2\operatorname{per} F(n-1,k).$$

From equations (1.8) and (1.6), we obtain

$$\operatorname{per} H\,(n+1,k) = g_{n+1}^k + 2g_n^k = l_{n+1}^k.$$

So the proof is complete. □

Acknowledgments. The authors would like to thank the referee for a number of helpful suggestions.

REFERENCES

1. R.A. Brualdi and P.M. Gibson, *convex polyhedra of doubly stochastic matrices I: Applications of the permanent function*, J. Combin. Theory **22** (1977), 194–230.

2. M.C. Er, *Sums of Fibonacci numbers by matrix methods*, Fibonacci Quart. **22** (1984), 204–207.

3. E. Kilic and D. Taşci, *On the Generalized order-k Fibonacci and Lucas numbers*, Rocky Mountain J. Math. **36** (2006), 1915–1926.

4. G.Y. Lee, *k-Lucas numbers and associated bipartite graphs*, Linear Algebra Appl. **320** (2000), 51–61.

5. H. Minc, *Permanents of* $(0,1)$*-circulants*, Canad. Math. Bull. **7** (1964), 253–263.

6. D. Taşci and E. Kilic, *On the order-k generalized Lucas numbers*, Appl. Math. Comp. **155** (2004), 637–641.

7. S. Vajda, *Fibonacci & Lucas numbers and the golden section*, Ellis Horwood Ltd., New York, 1989.

TOBB University of Economics and Technology, Mathematics Department, 06560 Ankara, Turkey
Email address: ekilic@etu.edu.tr

Gazi University, Department of Mathematics, Teknikokullar 06500, Ankara, Turkey
Email address: dtasci@gazi.edu.tr