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On the Pipage Rounding Algorithm for Submodular Function Maximization —A View from Discrete Convex Analysis—

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We consider the problem of maximizing a nondecreasing submodular set function under a matroid constraint. Recently, Calinescu et al. (2007) proposed an elegant framework for the approximation of this problem, which is based on the pipage rounding technique by Ageev and Sviridenko (2004), and showed that this framework indeed yields a $(1 - 1/e)$-approximation algorithm for the class of submodular functions which are represented as the sum of weighted rank functions of matroids. This paper sheds a new light on this result from the viewpoint of discrete convex analysis by extending it to the class of submodular functions which are the sum of M$^\natural$-concave functions. M$^\natural$-concave functions are a class of discrete concave functions introduced by Murota and Shioura (1999), and contain the class of the sum of weighted rank functions as a proper subclass. Our result provides a better understanding for why the pipage rounding algorithm works for the sum of weighted rank functions. Based on the new observation, we further extend the approximation algorithm to the maximization of a nondecreasing submodular function over an integral polymatroid. This extension has an application in multi-unit combinatorial auctions.

Keywords: Submodular function; matroid; concave function; discrete convexity.

Mathematics Subject Classification 2000: 90C27

1. Introduction

1.1. *Main Results*

We consider the maximization of a nondecreasing submodular function under a matroid constraint. In the area of continuous optimization, the maximization of a concave function is recognized as a tractable problem while the maximization of a convex function is hard to solve. In discrete optimization, submodular function is often regarded as discrete convexity, and indeed the maximization of a submodular function is known to be NP-hard. On the other hand, some classes of submodular functions are deeply related to discrete concavity (cf. [9,18,22]). For example, a set function $f(X) = \varphi(|X|)$ given by a univariate concave function φ is a submodular function, and it is natural that such a function has discrete concavity. The objective of this paper is to shed a new light on the pipage rounding algorithm [3] from the viewpoint of discrete convex analysis by pointing out that discrete concavity plays an essential role in computing an approximate solution in the maximization of a submodular function.

Our problem is formulated as follows:

$$\text{(P)} \qquad \text{Maximize } f(X) \qquad \text{subject to } X \in \mathcal{F},$$

where $f : 2^N \to \mathbb{R}$ is a nondecreasing submodular set function on a finite set N with $f(\emptyset) = 0$, and $\mathcal{M} = (N, \mathcal{F})$ is a matroid with the family of independent sets $\mathcal{F}$. We assume that the membership oracle for $\mathcal{M}$ is available. A set function $f : 2^N \to \mathbb{R}$ is said to be submodular if it satisfies

$$f(X) + f(Y) \geq f(X \cap Y) + f(X \cup Y) \qquad (X, Y \in 2^N),$$

and nondecreasing if $f(X) \leq f(Y)$ for any $X, Y \in 2^N$ with $X \subseteq Y$.

In the literature, various problems related to (P), including the optimal allocation in combinatorial auctions to be explained later, have been discussed over decades [4,7,11,16,27,33,34]. Recently, Calinescu et al. [3] proposed an elegant framework for the approximation of the problem (P), which is based on the pipage rounding technique developed by Ageev and Sviridenko [1]. In their framework, they firstly consider a relaxation of the problem (P):

$$\text{(RP)} \qquad \text{Maximize } \tilde{f}(x) \qquad \text{subject to } x \in \overline{P}(\mathcal{M}),$$

where $\overline{P}(\mathcal{M})$ ($\subseteq \mathbb{R}^N$) is the matroid polytope of $\mathcal{M}$ and $\tilde{f} : [0,1]^N \to \mathbb{R}$ is an *extension* of f, i.e., a function such that $\tilde{f}(\chi_X) = f(X)$ for every $X \in 2^N$ and its characteristic vector $\chi_X \in \{0,1\}^N$. Then, an optimal (fractional) solution $x \in [0,1]^N$ of the relaxed problem (RP) is computed and rounded to a $\{0,1\}$-vector that corresponds to an independent set of $\mathcal{M}$ by using a potential function defined over $[0,1]^N$. The main result of Calinescu et al. [3] is described as follows, where e denotes the base of natural logarithm, and for a matroid $\mathcal{M}' = (N, \mathcal{F}')$ and a nonnegative vector $w \in \mathbb{R}^N_+$, a *weighted rank function* $f : 2^N \to \mathbb{R}$ is defined by

$$f(X) = \max\{w(Y) \mid Y \in \mathcal{F}',\ Y \subseteq X\} \qquad (X \in 2^N). \tag{1.1}$$

We note that any weighted rank function is a nondecreasing submodular function with $f(\emptyset) = 0$.

Theorem 1.1 ([3]). *Let $f : 2^N \to \mathbb{R}$ be a function given as the sum of weighted rank functions. Then, the pipage rounding algorithm (see Section 2.3) outputs a $(1 - 1/e)$-approximate solution of the problem (P) in polynomial time (if the extension $\tilde{f} : [0,1]^N \to \mathbb{R}$ of f is defined appropriately).*

A connection of this result to discrete concavity is made by the observation that a weighted rank function has discrete concavity called M$^\natural$-concavity. A set function $f : 2^N \to \mathbb{R} \cup \{-\infty\}$ is said to be M$^\natural$-concave if f satisfies the following property:

for every $X, Y \in 2^N$ with $f(X) > -\infty$, $f(Y) > -\infty$ and every $i \in X \setminus Y$, it holds that either

$$f(X) + f(Y) \leq f(X \setminus \{i\}) + f(Y \cup \{i\})$$

or

$$f(X) + f(Y) \leq \max_{j \in Y \setminus X} \{f(X \setminus \{i\} \cup \{j\}) + f(Y \cup \{i\} \setminus \{j\})\}.$$

The proof of the following fact will be given in Section 2.4.

Theorem 1.2. *Any weighted rank function is an M$^\natural$-concave function.*

The concepts of M$^\natural$-concavity/M$^\natural$-convexity are introduced by Murota and Shioura [24] as discrete concavity/convexity for functions defined over the integer lattice, and are variants of M-concavity/M-convexity due to Murota [21]. These concepts play primary roles in the theory of discrete convex analysis [22]. It is shown in [10,26] that M$^\natural$-concavity is equivalent to the gross substitutes property in mathematical economics. The class of M$^\natural$-concave functions properly contains that of weighted rank functions; for example, the set function $f(X) = \varphi(|X|)$ with concave φ is an M$^\natural$-concave function and not a weighted rank function. Therefore, the class of the sum of M$^\natural$-concave functions contains the class of the sum of weighted rank functions, but so far we do not know whether this is a proper inclusion or not. Although the two classes of functions might be the same, any function in the class can be represented by a smaller number of functions if we use M$^\natural$-concave functions instead of weighted rank functions. Indeed, the set function $f(X) = \varphi(|X|)$ with strictly concave φ can be represented by a single M$^\natural$-concave function, while it is the sum of $|N|$ weighted rank functions.

An M$^\natural$-concave function has a natural extension called the concave closure. For a set function $f : 2^N \to \mathbb{R}$, its *concave closure* $\overline{f} : [0,1]^N \to \mathbb{R}$ is given by

$$\overline{f}(x) = \max\left\{ \sum_{X \subseteq N} \lambda_X f(X) \;\middle|\; \sum_{X \subseteq N} \lambda_X \chi_X = x,\ \sum_{X \subseteq N} \lambda_X = 1,\ \lambda_X \geq 0 \right\}. \quad (1.2)$$

We will show that the maximization of the sum of the concave closures of M$^\natural$-concave functions can be solved (almost) optimally in polynomial time. We assume, without loss of generality, that $\{j\} \in \mathcal{F}$ and $f(\{j\}) > 0$ for every $j \in N$ since otherwise there exists an optimal solution $X^* \in \mathcal{F}$ of (P) with $j \notin X^*$. We also assume that the membership oracle for $\mathcal{M}$ and the function evaluation oracles for M$^\natural$-concave functions are available. We denote by n the cardinality of N.

Theorem 1.3. *Let $f_k : 2^N \to \mathbb{R}$ $(k = 1, 2, \ldots, m)$ be a family of nondecreasing M$^\natural$-concave functions with $f_k(\emptyset) = 0$, and denote by $\overline{f}_k : [0,1]^N \to \mathbb{R}$ the concave closure of f_k. Suppose that the function $\tilde{f}$ in the problem (RP) is given as $\tilde{f}(x) = \sum_{k=1}^{m} \overline{f}_k(x)$.*

(i) For any $\varepsilon > 0$*, a* $(1-\varepsilon)$*-approximate solution of (RP) can be computed in time polynomial in* n*,* m*,* $\log f(N)$*,* Λ*, and* $\log(1/\varepsilon)$*, where*

$$\Lambda = \max\left[0, \max_{j\in N} \log \frac{1}{f(\{j\})}\right].$$

(ii) If each f_k *is an integer-valued function, then an optimal solution of (RP) can be computed in time polynomial in* n*,* m*, and* $\log f(N)$*.*

Our algorithm used in the proof of Theorem 1.3 is based on the ellipsoid method combined with an algorithm for computing a subgradient of the concave function $\tilde{f}$. Since $\tilde{f}(x) = \sum_{k=1}^{m} \overline{f}_k(x)$, a subgradient of $\tilde{f}$ is given as the sum of subgradients of the functions $\overline{f}_k$ ($k = 1, 2, \ldots, m$), and subgradients of each $\overline{f}_k$ are computed in polynomial time by using the combinatorial structure of M$^\natural$-concave functions.

As a corollary of Theorem 1.3, we see that the pipage rounding algorithm of Calinescu et al. [3] also works for the sum of M$^\natural$-concave functions.

Theorem 1.4. *Suppose that the function* f *is given as* $f(X) = \sum_{k=1}^{m} f_k(X)$ *with a family of nondecreasing M*$^\natural$*-concave functions* $f_k : 2^N \to \mathbb{R}$ *with* $f_k(\emptyset) = 0$ ($k = 1, 2, \ldots, m$)*.*
(i) For any $\varepsilon > 0$*, a* $(1 - 1/e - \varepsilon)$*-approximate solution of the problem (P) can be obtained in time polynomial in* n*,* m*,* $\log f(N)$*,* Λ*, and* $\log(1/\varepsilon)$*.*
(ii) If each f_k *is an integer-valued function, then a* $(1 - 1/e)$*-approximate solution of (P) can be obtained in time polynomial in* n*,* m*, and* $\log f(N)$*.*

Our results show that the success of the pipage rounding algorithm for the sum of weighted rank functions (Theorem 1.1) can be understood as a special case of Theorem 1.4. It should be emphasized that our algorithm uses only the value oracle for functions f_k, while the original algorithm in [3] requires an explicit representation of each weighted rank function f_k.

1.2. *Extension*

Based on the observation above, we can further extend the pipage rounding algorithm to a more general problem than (P).

Let $v \in \mathbb{Z}_+^N$, and $[\mathbf{0}, v]_\mathbb{Z} (\subseteq \mathbb{Z}^N)$ denotes the set of integral vectors in the interval $[\mathbf{0}, v]$ $(\subseteq \mathbb{R}^N)$. Let $h : [\mathbf{0}, v]_\mathbb{Z} \to \mathbb{R}$ be a nondecreasing submodular function with $h(\mathbf{0}) = 0$. We assume that h is submodular, i.e., h satisfies

$$h(x) + h(y) \geq h(x \vee y) + h(x \wedge y) \qquad (\forall x, y \in \mathrm{dom}_\mathbb{Z}\, h),$$

and that h is "concave" in the sense that the function $\varphi(\alpha) = h(x+\alpha\chi_j)$ is a concave function in $\alpha \in \mathbb{Z}$ for all $x \in [\mathbf{0}, v]_\mathbb{Z}$ and $j \in N$. Here, the vectors $x \vee y, x \wedge y \in \mathbb{R}^N$ for $x, y \in \mathbb{R}^N$ are defined by

$$(x \vee y)(j) = \max\{x(j), y(j)\}, \quad (x \wedge y)(j) = \min\{x(j), y(j)\} \qquad (j \in N).$$

We consider the following problem:

$$\text{(GP)} \quad \text{Maximize } h(x) \quad \text{subject to } x \in P,$$

where P is an integral polymatroid with $P \subseteq [\mathbf{0}, v]_{\mathbb{Z}}$. It is easy to see that the problem (P) is a special case of (GP) with $v = \mathbf{1}$. On the other hand, the problem (GP) can be easily reduced to the problem (P) associated with a certain nondecreasing submodular set function $f : 2^{\hat{N}} \to \mathbb{R}$ and a certain matroid $\mathcal{M} = (\hat{N}, \mathcal{F})$, but the size of the ground set $\hat{N}$ is polynomial in $\sum_{j=1}^{n} v(j)$, i.e., pseudopolynomial in the input size of (GP) (see, e.g., [30, Section 44.6b]). We assume, without loss of generality, that $\chi_j \in P$ and $h(\chi_j) > 0$ for every $j \in N$.

We also consider the relaxation of the problem (GP):

$$\text{(RGP)} \quad \text{Maximize } \tilde{h}(x) \quad \text{subject to } x \in \overline{P},$$

where $\overline{P} \subseteq \mathbb{R}^N$ is the convex hull of the integral polymatroid P and $\tilde{h} : [\mathbf{0}, v] \to \mathbb{R}$ is an *extension* of h, i.e., a function such that $\tilde{h}(x) = h(x)$ for all $x \in [\mathbf{0}, v]_{\mathbb{Z}}$. We note that for every M$^\natural$-concave function $h : [\mathbf{0}, v]_{\mathbb{Z}} \to \mathbb{R}$, its *concave closure* $\overline{h} : \mathbb{R}^N \to \mathbb{R} \cup \{-\infty\}$ defined by

$$\overline{h}(x) = \max \left\{ \sum_{y \in [\mathbf{0},v]_{\mathbb{Z}}} \lambda_y h(y) \;\middle|\; \sum_{y \in [\mathbf{0},v]_{\mathbb{Z}}} \lambda_y y = x, \sum_{y \in [\mathbf{0},v]_{\mathbb{Z}}} \lambda_y = 1, \right.$$
$$\left. \lambda_y \geq 0 \ (y \in [\mathbf{0}, v]_{\mathbb{Z}}) \right\} \quad (x \in [\mathbf{0}, v]) \qquad (1.3)$$

satisfies $\overline{h}(x) = h(x)$ for every $x \in [\mathbf{0}, v]_{\mathbb{Z}}$ (see Theorem 2.10).

Theorem 1.5. *Let $h_k : [\mathbf{0}, v]_{\mathbb{Z}} \to \mathbb{R}$ $(k = 1, 2, \ldots, m)$ be a family of nondecreasing M$^\natural$-concave functions with $h_k(\mathbf{0}) = 0$, and denote by $\overline{h}_k : [0, v] \to \mathbb{R}$ the concave closure of h_k. Suppose that the function $\tilde{h}$ in the problem (RGP) is given as $\tilde{h}(x) = \sum_{k=1}^{m} \overline{h}_k(x)$.*
(i) For any $\varepsilon > 0$, a $(1-\varepsilon)$-approximate solution of (RGP) can be computed in time polynomial in n, m, $\log h(v)$, $\widetilde{\Lambda}$, $\log ||v||_\infty$, and $\log(1/\varepsilon)$, where $\widetilde{\Lambda}$ is given by

$$\widetilde{\Lambda} = \max \left[0, \max_{j \in N} \log \frac{1}{h(\chi_j)} \right].$$

(ii) If each h_k is an integer-valued function, then an optimal solution of (RGP) can be computed in time polynomial in n, m, $\log h(v)$, and $\log ||v||_\infty$.

Theorem 1.6. *Suppose that h is given as the sum of nondecreasing M$^\natural$-concave functions $h_k : [\mathbf{0}, v]_{\mathbb{Z}} \to \mathbb{R}$ $(k = 1, 2, \ldots, m)$ with $h_k(\mathbf{0}) = 0$.*
(i) For any $\varepsilon > 0$, a $(1 - 1/e - \varepsilon)$-approximate solution of the problem (GP) can be obtained in time polynomial in n, m, $\log h(v)$, $\widetilde{\Lambda}$, $\log ||v||_\infty$, and $\log(1/\varepsilon)$.
(ii) If each h_k is an integer-valued function, then a $(1 - 1/e)$-approximate solution of (GP) can be obtained in time polynomial in n, m, $\log h(v)$, and $\log ||v||_\infty$.

Application to combinatorial auctions The problem (GP) contains as a special case the optimal allocation problem in multi-unit combinatorial auctions (see, e.g., [2,17]). Multi-unit combinatorial auctions are those in which some of the items for sale are identical. We assume that there are n goods and $u(j)$ denotes the number of available units of goods $j \in \{1, 2, \ldots, n\}$. We also assume that there are m bidders and the k-th bidder has a valuation $g_k : [\mathbf{0}, u]_{\mathbb{Z}} \to \mathbb{R}$ that is a nondecreasing submodular function with $g_k(\mathbf{0}) = 0$ such that $\varphi(\alpha) = g_k(x + \alpha\chi_i)$ is a concave function in $\alpha \in \mathbb{Z}$ for all $x \in [\mathbf{0}, u]_{\mathbb{Z}}$ and $i \in N$. Then, the optimal allocation problem is formulated as

$$\begin{array}{lll} \text{(OAP)} & \text{Maximize} & \displaystyle\sum_{k=1}^{m} g_k(x_k) \\ & \text{subject to} & \displaystyle\sum_{k=1}^{m} x_k(j) = u(j) \quad (j = 1, 2, \ldots, n), \\ & & x_k \in \mathbb{Z}^n_+ \quad (k = 1, 2, \ldots, m), \end{array}$$

which can be easily reduced to the problem (GP) as follows:

$$\text{Maximize } h(y) \quad \text{subject to } y \in P \ (\subseteq \mathbb{Z}^E_+),$$

where $E = \{(k, j) \mid k = 1, 2, \ldots, m,\ j = 1, 2, \ldots, n\}$, $v \in \mathbb{Z}^E_+$ is a vector given by $v(k, j) = u(j)$ $((k, j) \in E)$, $h : [\mathbf{0}, v]_{\mathbb{Z}} \to \mathbb{R}$ $(k = 1, 2, \ldots, m)$ is a function defined by

$$h(y) = \sum_{k=1}^{m} g_k(y(k, 1), y(k, 2), \ldots, y(k, n)) \qquad (y \in [\mathbf{0}, v]_{\mathbb{Z}}), \tag{1.4}$$

and $P \subseteq \mathbb{Z}^E_+$ is an integral polymatroid defined by

$$P = \{y \in \mathbb{Z}^E_+ \mid \sum_{k=1}^{m} y(k, j) \leq u(j) \ (j = 1, 2, \ldots, n)\}.$$

While the single-unit case (i.e., $u(j) = 1$ for all j) has been discussed in the literature (see, e.g., [6,15,16,34]), no polynomial-time approximation algorithm with theoretical error bound has been proposed for the multi-unit case, as far as the present author knows. Note that the multi-unit case can be easily reduced to the single-unit case with a pseudopolynomial number of goods, and therefore the previous approximation algorithms for the single-unit case can be applied to the multi-unit case, but they require pseudopolynomial time.

The special case of (GP) where each valuation g_k is M$^\natural$-concave (i.e., satisfies the gross substitutes property) is well studied in the literature, and Lehmann et al. [16] show that this case can be solved exactly in polynomial time in the single-unit case; this result extends to the multi-unit case by reduction to the M-convex submodular flow problem (cf. [22]).

We consider a more general case where each valuation g_k is given as the sum of M$^\natural$-concave functions. In such a case the function h defined by (1.4) can be also represented as the sum of M$^\natural$-concave functions (see, e.g., [22]), and therefore Theorem 1.6 implies that $(1-1/e)$-approximation is possible.

Corollary 1.7. *Suppose that each valuation g_k $(k = 1, 2, \ldots, m)$ in (OAP) is given as the sum of nondecreasing M$^\natural$-concave functions.*
(i) For any $\varepsilon > 0$, a $(1-1/e-\varepsilon)$-approximate solution of the problem (OAP) can be obtained in time polynomial in n, m, $\log \sum_{k=1}^m g_k(u)$, $\widehat{\Lambda}$, $\log ||u||_\infty$, and $\log(1/\varepsilon)$, where

$$\widehat{\Lambda} = \max \left[0, \max_{1 \le k \le m} \max_{j \in N} \log \frac{1}{g_k(\chi_j)} \right].$$

(ii) If each g_k is an integer-valued function, then a $(1-1/e)$-approximate solution of (OAP) can be obtained in time polynomial in n, m, $\log \sum_{k=1}^m g_k(u)$, and $\log ||u||_\infty$.

1.3. *Organization of the paper*

In Section 2, we review the pipage rounding framework of Calinescu et al. [3] as well as the definition and some fundamental properties of M$^\natural$-concavity. In Section 3, we present algorithms for computing a subgradient of the concave closure of an M$^\natural$-concave function. In Section 4, we prove Theorems 1.3, 1.4, and 1.5 by giving polynomial-time algorithm for the maximization of the sum of concave closures. Finally, we explain how to extend the pipage rounding algorithm to (GP) and give a proof of Theorem 1.6 in Section 5.

Remark 1.8. Quite recently, Vondrák [34] has shown that for any nondecreasing submodular set function f, a $(1-1/e-\varepsilon)$-approximate solution of the problem (P) can be obtained in polynomial time. The algorithm in [34] is randomized, and uses the pipage rounding technique, as in our algorithm. The major difference between Vondrák's algorithm and ours is in how to compute a fractional solution; Vondrák [34] obtains it by solving a nonconcave relaxation of the problem (P) approximately by using a randomized algorithm, while we solve a concave relaxation of (P) (almost) exactly by a deterministic algorithm. In addition, it is not clear how to extend Vondrák's algorithm to the problem (GP) so that it runs in polynomial time, although it is easy to extend the algorithm to a pseudopolynomial-time approximation algorithm for (GP).

2. Preliminaries

2.1. *Matroids*

We denote by $\mathbb{Z}_+$ (resp., by $\mathbb{R}_+$) the set of nonnegative integers (resp., nonnegative real numbers). Also, we denote $\mathbf{0} = (0, 0, \ldots, 0) \in \mathbb{Z}^N$ and $\mathbf{1} = (1, 1, \ldots, 1) \in \mathbb{Z}^N$. Throughout this paper, we assume that $\mathcal{M} = (N, \mathcal{F})$ is a matroid with the family of independent sets $\mathcal{F}$ $(\subseteq 2^N)$, which gives a constraint in the problem (P). A set system $\mathcal{M} = (N, \mathcal{F})$ is called a matroid if the set family $\mathcal{F} \subseteq 2^N$ is given as

$$\mathcal{F} = \{X \in 2^N \mid |X \cap Y| \le r_{\mathcal{M}}(Y) \ (Y \in 2^N)\},$$

by using a nondecreasing submodular set function $r_{\mathcal{M}} : 2^N \to \mathbb{Z}_+$ such that $r_{\mathcal{M}}(Y) \le |Y|$ $(Y \in 2^N)$ (see, e.g., [28] for other equivalent definitions of matroids). The function $r_{\mathcal{M}}$ is called the rank function of $\mathcal{M}$. Any maximal element in $\mathcal{F}$ is called a base, and we denote by $\mathcal{B}$ the family of bases in $\mathcal{M}$. The family of bases $\mathcal{B}$ can be represented as

$$\mathcal{B} = \{X \in 2^N \mid X \in \mathcal{F},\ |X| = r_{\mathcal{M}}(N)\}.$$

For any $X \in 2^N$, we denote by $\chi_X \in \{0,1\}^N$ the characteristic vector of X, i.e.,

$$(\chi_X)(j) = \begin{cases} 1 & (j \in X) \\ 0 & (j \in N \setminus X). \end{cases}$$

In particular, we denote $\chi_j = \chi_{\{j\}}$ for each $j \in N$. The matroid polytope $\overline{P}(\mathcal{M})$ (resp., the base polytope $\overline{B}(\mathcal{M})$) is defined as the convex hull of the set of $\{0,1\}$-vectors $\{\chi_X \mid X \in \mathcal{F}\}$ (resp., $\{\chi_X \mid X \in \mathcal{B}\}$). They are also given as

$$\begin{aligned} \overline{P}(\mathcal{M}) &= \{x \in \mathbb{R}^N_+ \mid x(Y) \le r_{\mathcal{M}}(Y)\ (Y \in 2^N)\}, \\ \overline{B}(\mathcal{M}) &= \{x \in \mathbb{R}^N_+ \mid x \in \overline{P}(\mathcal{M}),\ x(N) = r_{\mathcal{M}}(N)\}, \end{aligned}$$

where $x(Y) = \sum_{j \in Y} x(j)$ for $x = (x(1), x(2), \ldots, x(n)) \in \mathbb{R}^N$ and $Y \in 2^N$.

In the following, we assume that the matroid $\mathcal{M}$ is "full-dimensional" in the sense that the matroid polytope $\overline{P}(\mathcal{M})$ is full-dimensional. This is equivalent to the property that every singleton set $\{j\}$ $(j \in N)$ is an independent set of $\mathcal{M}$.

We also assume that the membership oracle for $\mathcal{F}$ is available. Since the function value of the matroid rank function $r_{\mathcal{M}}$ can be computed by using the membership oracle at most n times, the following oracles for $\mathcal{M}$ can be implemented to run in polynomial time by using submodular function minimization algorithms [12,14,29] (see [9]). All of the four oracles can be also realized by the combinatorial algorithm of Cunningham [5] for testing membership in a matroid polytope.

- [membership oracle]
  for $x \in \mathbb{R}^N$, check whether $x \in \overline{P}(\mathcal{M})$,
- [separation oracle]
  for $x \notin \overline{P}(\mathcal{M})$, find a set $X \in 2^N$ with $x(X) > r_{\mathcal{M}}(X)$,
- [saturation capacity oracle]
  for $x \in \overline{P}(\mathcal{M})$ and $i \in N$, compute the value

$$\hat{c}(x, i) = \max\{\eta \in \mathbb{R}_+ \mid x + \eta\chi_i \in \overline{P}(\mathcal{M})\},$$

- [exchange capacity oracle]
  for $x \in \overline{B}(\mathcal{M})$ and $i, j \in N$, compute the value

$$\hat{c}(x, i, j) = \max\{\eta \in \mathbb{R}_+ \mid x + \eta(\chi_i - \chi_j) \in \overline{B}(\mathcal{M})\}.$$

2.2. *Polymatroids*

Throughout this paper, we assume that $P \subseteq \mathbb{Z}_+^N$ is an integral polymatroid, which gives a constraint in the problem (GP). A set of nonnegative integral vectors P is called an integral polymatroid if it is represented as

$$P = \{x \in \mathbb{Z}_+^N \mid x(Y) \leq r_P(Y)\ (Y \in 2^N)\}$$

by using a nondecreasing submodular set function $r_P : 2^N \to \mathbb{Z}_+$ with $r_P(\emptyset) = 0$. The function r_P is called a (polymatroid) rank function associated with the integral polymatroid P. A maximal vector in P is called a base of P, and the set of bases of P is denoted by B ($\subseteq \mathbb{Z}_+^N$), which is represented as

$$B = \{x \in \mathbb{Z}_+^N \mid x \in P,\ x(N) = r_P(N)\}$$

in terms of the polymatroid rank function r_P. Moreover, the convex hull $\overline{P}$ (resp., $\overline{B}$) of P (resp., B) is represented as

$$\overline{P} = \{x \in \mathbb{R}_+^N \mid x(Y) \leq r_P(Y)\ (Y \in 2^N)\},$$
$$\overline{B} = \{x \in \mathbb{R}_+^N \mid x \in \overline{P},\ x(N) = r_P(N)\}$$

In the following, we assume that the polymatroid P is "full-dimensional" in the sense that its convex hull $\overline{P}$ is a full-dimensional polytope. This is equivalent to the property that every unit vector χ_j $(j \in N)$ is in P.

We also assume that the membership oracle for the integral polymatroid P is available. In the same way as in the case of matroids, membership oracle, separation oracle, and saturation capacity oracle for $\overline{P}$ and exchange capacity oracle for $\overline{B}$ can be implemented to run in polynomial time with the aid of binary search.

2.3. *Pipage rounding algorithm*

The pipage rounding algorithm [3] for the problem (P) consists of the following three steps:

1. Define a relaxed problem (RP) of the original problem (P).
2. Compute an (approximately) optimal solution x^* of the relaxed problem (RP).
3. Round the fractional vector x^* to obtain a $\{0, 1\}$-vector $\hat{x}$.

We explain the details of each step below.

To define a relaxation (RP) of the problem (P), we use an *extension* $\tilde{f} : [0, 1]^N \to \mathbb{R}$ of f which is a function satisfying $\tilde{f}(\chi_X) = f(X)$ $(X \in 2^N)$. For example, the concave closure $\overline{f}$ of f given by (1.2) can be used as an extension of f; in case where f is given as $f(x) = \sum_{k=1}^{m} f_k(x)$ with a family of set functions $f_k : 2^N \to \mathbb{R}$ $(k = 1, 2, \ldots, m)$, the sum of the concave closures $\sum_{k=1}^{m} \overline{f}_k(x)$ can be also used.

In the second step, we compute an (approximately) optimal solution x^* of the relaxed problem (RP). We may assume that $x^* \in \overline{B}(\mathcal{M})$, since otherwise we can find $x \in \overline{B}(\mathcal{M})$ with $\tilde{f}(x) \geq \tilde{f}(x^*)$ by computing the saturation capacities $\hat{c}(x^*, i)$ at most n times.

In the third step, we round the fractional vector $x^* \in \overline{B}(\mathcal{M})$ to a $\{0,1\}$-vector χ_X with $X \in \mathcal{B}$ by using a potential function $\Psi : [0,1]^N \to \mathbb{R}$ defined by

$$\Psi(x) = \sum_{X \subseteq N} \left(\prod_{j \in X} x(j) \right) \left(\prod_{j \in N \setminus X} (1 - x(j)) \right) f(X) \qquad (x \in [0,1]^N).$$

Note that $\Psi(\chi_X) = f(X)$ for any $X \in 2^N$. We assume that the function evaluation oracle for $\Psi(x)$ is available, as in [3]. We note that the function value of Ψ can be evaluated to any desired accuracy in polynomial time by taking sufficiently many independent samples (see [3]).

Rounding of a fractional vector is done by using the following procedure.

Procedure Rounding(x)
Input: a vector $x \in \overline{B}(\mathcal{M})$
Output: a set $X \in \mathcal{B}$ such that $\Psi(\chi_X) \geq \Psi(x)$
Step 1: If $x \in \{0,1\}^N$, then output the set $X \in 2^N$ with $\chi_X = x$, and stop.
Step 2: Let Y be a minimal set satisfying

$$x(Y) = r_{\mathcal{M}}(Y), \quad Y \cap \{j \in N \mid 0 < x(j) < 1\} \neq \emptyset.$$

Step 3: Choose any distinct elements i, i' in $Y \cap \{j \in N \mid 0 < x(j) < 1\}$.
Step 4: Put

$$x' = x + \hat{c}(x, i, i')(\chi_i - \chi_{i'}), \qquad x'' = x + \hat{c}(x, i', i)(\chi_{i'} - \chi_i).$$

If $\Psi(x') \geq \Psi(x'')$, then put $x := x'$; otherwise put $x := x''$. Go to Step 1.

Theorem 2.1 ([3]). *The procedure* Rounding *terminates in* $\mathrm{O}(n^2)$ *iterations. Given a function evaluation oracle for* Ψ *and a membership oracle for* $\overline{B}(\mathcal{M})$*, the procedure can be implemented to run in polynomial time.*

The correctness of the procedure Rounding follows from the following property of Ψ.

Proposition 2.2 ([3]). *For any* $x \in \overline{B}(\mathcal{M})$ *and distinct* $i, j \in N$*, the function* $\psi(\eta) = \Psi(x + \eta(\chi_i - \chi_j))$ *is convex in the interval* $\eta \in [-\hat{c}(x, j, i), \hat{c}(x, i, j)]$.

The quality of the solution obtained by the procedure Rounding depends on the choice of the extension $\tilde{f}$. We denote by OPT the optimal value of the problem (P).

Theorem 2.3 (cf. [3]). *Suppose that* $\Psi(y) \geq \alpha \tilde{f}(y)$ *holds for all* $y \in [0,1]^N$*. Given a* β*-approximate solution* $x \in [0,1]^N$ *of the problem (RP), the procedure* Rounding *outputs a subset* $X \in 2^N$ *satisfying* $f(X) \geq \alpha\beta\,\mathrm{OPT}$.

The following properties show that if the function $\overline{f}(x)$ (or $\sum_{k=1}^{m} \overline{f}_k(x)$) is used as an extension of f and we can solve (RP) exactly (i.e., $\beta = 1$ in Theorem 2.3) in polynomial time, then the pipage rounding algorithm is a $(1 - 1/e)$-approximation algorithm for (P).

Theorem 2.4 ([3]). *For any nondecreasing submodular function* $f : 2^N \to \mathbb{R}$ *with* $f(\emptyset) = 0$*, we have*

$$\Psi(x) \geq \left(1 - \frac{1}{e}\right)\overline{f}(x) \qquad (x \in [0,1]^N).$$

Corollary 2.5 (cf. [3]). *Suppose that* f *is given as* $f(X) = \sum_{k=1}^{m} f_k(X)$ *with a family of nondecreasing submodular functions* $f_k : 2^N \to \mathbb{R}$ *with* $f_k(\emptyset) = 0$ $(k = 1, 2, \ldots, m)$*. Then, we have*

$$\Psi(x) \geq \left(1 - \frac{1}{e}\right)\sum_{k=1}^{m}\overline{f}_k(x) \qquad (x \in [0,1]^N).$$

2.4. *M♮-concave functions*

We review the definition of M♮-concavity and show some fundamental properties.

A function $h : \mathbb{Z}^N \to \mathbb{R} \cup \{-\infty\}$ defined over the integer lattice is said to be M♮-concave if it satisfies the following property:

$$\forall x, y \in \mathrm{dom}_{\mathbb{Z}}\, h,\ \forall i \in \mathrm{supp}^+(x-y),\ \exists j \in \mathrm{supp}^-(x-y) \cup \{0\}:$$

$$h(x) + h(y) \leq h(x - \chi_i + \chi_j) + h(y + \chi_i - \chi_j),$$

where $\chi_0 = \mathbf{0} \in \mathbb{R}^N$, $\mathrm{dom}_{\mathbb{Z}}\, h = \{x \in \mathbb{Z}^N \mid h(x) > -\infty\}$, and

$$\mathrm{supp}^+(x) = \{i \in N \mid x(i) > 0\}, \quad \mathrm{supp}^-(x) = \{i \in N \mid x(i) < 0\} \qquad (x \in \mathbb{R}^N).$$

We note that for any M♮-concave function h and any $p \in \mathbb{R}^N$, the function $h(x) + p^\top x$ is also M♮-concave in x.

The following property shows that M♮-concave functions constitute a subclass of submodular functions.

Theorem 2.6 ([22]). *An M♮-concave function* $h : \mathbb{Z}^N \to \mathbb{R} \cup \{-\infty\}$ *is a submodular function.*

M♮-concavity for set functions can be naturally defined through the one-to-one correspondence between set functions $f : 2^N \to \mathbb{R} \cup \{-\infty\}$ and functions $h : \mathbb{Z}^N \to \mathbb{R} \cup \{-\infty\}$ with $\mathrm{dom}_{\mathbb{Z}}\, h \subseteq \{0,1\}^N$. That is, a set function $f : 2^N \to \mathbb{R} \cup \{-\infty\}$ is said to be M♮-concave if f satisfies the following property:

> for every $X, Y \in 2^N$ with $f(X) > -\infty$, $f(Y) > -\infty$ and every $i \in X \setminus Y$, it holds that either
>
> $$f(X) + f(Y) \leq f(X \setminus \{i\}) + f(Y \cup \{i\}) \qquad (2.1)$$
>
> or
>
> $$f(X) + f(Y) \leq \max_{j \in Y \setminus X}\{f(X \setminus \{i\} \cup \{j\}) + f(Y \cup \{i\} \setminus \{j\})\}. \qquad (2.2)$$

Maximization of an M$^\natural$-concave function can be done efficiently.

Theorem 2.7 (cf. [22,31]). *For any M$^\natural$-concave function $h : \mathbb{Z}^N \to \mathbb{R} \cup \{-\infty\}$, a maximizer of h can be computed in time polynomial in n and in* $\log \max\{x(i) - y(i) \mid i \in N,\ x, y \in \mathrm{dom}_{\mathbb{Z}}\, h\}$.

We now give a proof of Theorem 1.2 stating that any weighted rank function is M$^\natural$-concave. The original proof in [32] relies on the convolution theorem [22, Theorem 6.13 (8)] for M$^\natural$-concave functions. We here give an elementary proof by Murota [23].

Proof of Theorem 1.2. Recall the definition of a weighted rank function $f : 2^N \to \mathbb{R}$ in (1.1). To prove the M$^\natural$-concavity of a weighted rank function, we use the simultaneous exchange property of the family $\mathcal{F}'$ of independent sets (cf. [24, Remark 5.2]):

> for every $I, J \in \mathcal{F}'$ and $i \in I \setminus J$, either $I \setminus \{i\}, J \cup \{i\} \in \mathcal{F}'$ or $I \setminus \{i\} \cup \{j\}, J \cup \{i\} \setminus \{j\} \in \mathcal{F}'$ for some $j \in J \setminus I$.

Take $X, Y \subseteq N$ and $i \in X \setminus Y$. Let $I, J \in \mathcal{F}'$ be independent subsets of X and Y respectively such that $f(X) = w(I)$ and $f(Y) = w(J)$.

If $i \notin I$, then

$$f(X \setminus \{i\}) \geq w(I) = f(X), \quad f(Y \cup \{i\}) \geq w(J) = f(Y),$$

which implies (2.1). So assume $i \in I$. If $J \cup \{i\} \in \mathcal{F}'$, then

$$f(X \setminus \{i\}) \geq w(I \setminus \{i\}) = f(X) - w(i), \quad f(Y \cup \{i\}) \geq w(J \cup \{i\}) = f(Y) + w(i),$$

which implies (2.1). So assume $J \cup \{i\} \notin \mathcal{F}'$. Then we must have the second case in the simultaneous exchange axiom for I, J, i. That is, there exists $j \in J \setminus I$ such that $I \setminus \{i\} \cup \{j\}, J \cup \{i\} \setminus \{j\} \in \mathcal{F}'$. If $j \in X$, then $I \setminus \{i\} \cup \{j\} \subseteq X \setminus \{i\}$, $J \cup \{i\} \setminus \{j\} \subseteq Y \cup \{i\}$, and hence

$$\begin{aligned} f(X \setminus \{i\}) &\geq w(I \setminus \{i\} \cup \{j\}) = f(X) - w(i) + w(j), \\ f(Y \cup \{i\}) &\geq w(J \cup \{i\} \setminus \{j\}) = f(Y) + w(i) - w(j), \end{aligned}$$

which implies (2.1). If $j \notin X$, then $j \in Y \setminus X$, and

$$\begin{aligned} f(X \setminus \{i\} \cup \{j\}) &\geq w(I \setminus \{i\} \cup \{j\}) = f(X) - w(i) + w(j), \\ f(Y \cup \{i\} \setminus \{j\}) &\geq w(J \cup \{i\} \setminus \{j\}) = f(Y) + w(i) - w(j), \end{aligned}$$

which implies (2.2). □

We give some other examples of M$^\natural$-concave set functions.

Example 2.8 (laminar concave function). Let $\mathcal{F} \subseteq 2^N$ be a laminar family, i.e., for any $X, Y \in \mathcal{F}$ we have $X \setminus Y = \emptyset$, $Y \setminus X = \emptyset$, or $X \cap Y = \emptyset$. For a family of univariate concave functions $\varphi_Y : \mathbb{Z} \to \mathbb{R}$ $(Y \in \mathcal{F})$, the function $f : 2^N \to \mathbb{R}$ defined by

$$f(X) = \sum_{Y \in \mathcal{F}} \varphi_Y(|X \cap Y|) \quad (X \in 2^N)$$

is an M$^\natural$-concave function. In particular, f is nondecreasing if each φ_Y is nondecreasing.

Example 2.9. Let $G = (U, V; E)$ be a complete bipartite graph with vertex set $U \cup V$ and edge set E, and let $w_e \in \mathbb{R}_+$ be the weight of edge $e \in E$. We define a function $f : 2^U \to \mathbb{R}$ by

$$f(X) = \max\left\{ \sum_{e \in F} w_e \;\middle|\; F : \text{matching of } G,\ \{\partial^+ e \mid e \in F\} = X \right\},$$

where $\partial^+ e \in U$ denotes the end vertex of edge $e \in E$ contained in U. Then, f is a nondecreasing M$^\natural$-concave function.

We also consider M$^\natural$-concavity for polyhedral concave functions. A polyhedral concave function $b : \mathbb{R}^N \to \mathbb{R} \cup \{-\infty\}$ is said to be M$^\natural$-concave if it satisfies the following property:

$\forall x, y \in \mathrm{dom}_{\mathbb{R}}\, b$, $\forall i \in \mathrm{supp}^+(x - y)$, $\exists j \in \mathrm{supp}^-(x - y) \cup \{0\}$, $\exists \eta_0 > 0$:

$$b(x) + b(y) \le b(x - \eta(\chi_i - \chi_j)) + b(y + \eta(\chi_i - \chi_j)) \qquad (\forall \eta \in [0, \eta_0]),$$

where $\mathrm{dom}_{\mathbb{R}}\, b = \{x \in \mathbb{R}^N \mid b(x) > -\infty\}$.

Theorem 2.10 ([22,25]). *Let $h : \mathbb{Z}^N \to \mathbb{R} \cup \{-\infty\}$ be an M$^\natural$-concave function, and $\overline{h} : \mathbb{R}^N \to \mathbb{R} \cup \{-\infty\}$ be its concave closure.*
(i) If $\mathrm{dom}_{\mathbb{Z}}\, h$ is bounded, then $\overline{h}$ is a polyhedral M$^\natural$-concave function.
(ii) For any $x \in \mathbb{R}^N$, it holds that

$$\overline{h}(x) = \max\left\{ \sum_{X \subseteq N} \lambda_X h(\lfloor x \rfloor + \chi_X) \;\middle|\; \lfloor x \rfloor + \sum_{X \subseteq N} \lambda_X \chi_X = x, \right.$$
$$\left. \sum_{X \subseteq N} \lambda_X = 1,\ \lambda_X \ge 0\ (X \in 2^N) \right\},$$

where $\lfloor x \rfloor \in \mathbb{Z}^N$ denotes the vector such that $(\lfloor x \rfloor)(i) = \lfloor x(i) \rfloor$ $(i \in N)$. In particular, we have $\overline{h}(x) = h(x)$ for all $x \in \mathbb{Z}^N$.

A nonempty set $S \subseteq \mathbb{R}^N$ is called a g-polymatroid [8] if S is given as

$$S = \{x \in \mathbb{R}^N \mid \mu(X) \le x(X) \le \rho(X)\ (X \in 2^N)\}$$

with a pair of a submodular set function $\rho : 2^N \to \mathbb{R} \cup \{+\infty\}$ and a supermodular set function $\mu : 2^N \to \mathbb{R} \cup \{-\infty\}$ such that

$$\rho(\emptyset) = \mu(\emptyset) = 0, \quad \rho(X) - \rho(X \setminus Y) \ge \mu(Y) - \mu(Y \setminus X) \quad (X, Y \subseteq N).$$

Theorem 2.11 ([22,25]). *Let $h : \mathbb{Z}^N \to \mathbb{R} \cup \{-\infty\}$ be an M$^\natural$-concave function, and let $\overline{h} : \mathbb{R}^N \to \mathbb{R} \cup \{-\infty\}$ be its concave closure. For any $p \in \mathbb{R}^N$, the set $\arg\max\{\overline{h}(x) - p^\top x \mid x \in \mathbb{R}^N\}$ is an integral g-polymatroid if it is not empty.*

Finally, we explain the concept of L$^\natural$-concavity, which is deeply related to the concept of M$^\natural$-concavity. A function $g : \mathbb{Z}^N \to \mathbb{R} \cup \{-\infty\}$ defined over the integer lattice is said to be L$^\natural$-concave if it satisfies

$$g(p) + g(q) \leq g((p - \lambda\mathbf{1}) \vee q) + g(p \wedge (q + \lambda\mathbf{1})) \quad (\forall p, q \in \mathbb{Z}^N,\ \forall \lambda \in \mathbb{Z}_+).$$

Maximization of an L$^\natural$-concave function over the integer lattice can be solved efficiently.

Theorem 2.12 ([22]). *For any L$^\natural$-concave function $g : \mathbb{Z}^N \to \mathbb{R} \cup \{-\infty\}$ with bounded $\mathrm{dom}_\mathbb{Z}\, g$, its maximizer can be computed in time polynomial in n and in $\log\max\{p(i) - q(i) \mid i \in N,\ p, q \in \mathrm{dom}_\mathbb{Z}\, g\}$.*

L$^\natural$-concavity is also defined for polyhedral concave functions. A polyhedral concave function $g : \mathbb{R}^N \to \mathbb{R} \cup \{-\infty\}$ is said to be L$^\natural$-concave if it satisfies

$$g(p) + g(q) \leq g((p - \lambda\mathbf{1}) \vee q) + g(p \wedge (q + \lambda\mathbf{1})) \quad (\forall p, q \in \mathbb{R}^N,\ \forall \lambda \in \mathbb{R}_+).$$

Theorem 2.13 ([22,25]). *Let $h : \mathbb{Z}^N \to \mathbb{R} \cup \{-\infty\}$ be an M$^\natural$-concave function with bounded $\mathrm{dom}_\mathbb{Z}\, h$, and define a function $h^\circ : \mathbb{R}^N \to \mathbb{R}$ by*

$$h^\circ(p) = \min\{p^\top x - h(x) \mid x \in \mathbb{Z}^N\} \qquad (p \in \mathbb{R}^N). \tag{2.3}$$

Then, h° is a polyhedral L$^\natural$-concave function.

3. Approximation Algorithms for Concave Closure

For a concave function $b : \mathbb{R}^N \to \mathbb{R} \cup \{-\infty\}$, a vector $p \in \mathbb{R}^N$ is called a subgradient of b at x if p satisfies

$$b(y) - b(x) \leq p^\top (y - x) \qquad (y \in \mathbb{R}^N),$$

and the set of subgradients of b at x is denoted by $\partial b(x)$ ($\subseteq \mathbb{R}^N$). In this section, we consider the concave closure $\overline{h}$ of a nondecreasing M$^\natural$-concave function h and show that an approximate subgradient of $\overline{h}$ can be computed efficiently.

Theorem 3.1. *Let $v \in \mathbb{Z}^N$ and $h : \mathbb{Z}^N \to \mathbb{R} \cup \{-\infty\}$ be a nondecreasing M$^\natural$-concave function satisfying $h(\mathbf{0}) = 0$ and $\mathrm{dom}_\mathbb{Z}\, h = [\mathbf{0}, v]_\mathbb{Z}$.*
(i) For any $x \in [\mathbf{0}, v]$ and any $\delta > 0$, we can compute a vector $p \in \mathbb{R}^N$ and a real number $\alpha \in \mathbb{R}$ satisfying

$$\overline{h}(y) - \overline{h}(x) \leq p^\top (y - x) + \delta \quad (\forall y \in [\mathbf{0}, v]), \qquad \overline{h}(x) \leq \alpha \leq \overline{h}(x) + \delta \tag{3.1}$$

in time polynomial in n, $\log\max_{i \in N} h(\chi_i)$, $\log ||v||_\infty$, and $\log(1/\delta)$.
(ii) Suppose that h is an integer-valued function. Then, for any $x \in [\mathbf{0}, v]$ we can compute a subgradient $p \in \partial\overline{h}(x) \cap \mathbb{Z}^N$ and the exact value of $\overline{h}(x)$ in time polynomial in n, $\log\max_{i \in N} h(\chi_i)$, and $\log ||v||_\infty$.

In the following, we give a proof of Theorem 3.1. By the definition of the concave closure (1.3) and LP duality, we have

$$\begin{aligned}\overline{h}(x) &= \min\{p^\top x + \gamma \mid p^\top y + \gamma \geq h(y)\ (y \in [\mathbf{0}, v]_{\mathbb{Z}}),\ p \in \mathbb{R}^N,\ \gamma \in \mathbb{R}\}\\ &= \min\{p^\top x - h^\circ(p) \mid p \in \mathbb{R}^N\} \qquad (3.2)\end{aligned}$$

for any $x \in [\mathbf{0}, v]$, where $h^\circ : \mathbb{R}^N \to \mathbb{R}$ is given by (2.3). We define

$$g_x(p) = h^\circ(p) - p^\top x \qquad (p \in \mathbb{R}^N).$$

Then, Eq. (3.2) is rewritten as

$$\overline{h}(x) = -\max\{g_x(p) \mid p \in \mathbb{R}^N\}. \qquad (3.3)$$

The following properties show that finding a subgradient (resp., an approximate subgradient) of $\overline{h}$ can be reduced to the problem of finding a maximizer (resp., an approximate maximizer) of the polyhedral concave function g_x.

Lemma 3.2 ([22,25]). *Let $x \in [\mathbf{0}, v]$.*
(i) $\partial\overline{h}(x) = \arg\max\{g_x(p) \mid p \in \mathbb{R}^N\}$.
(ii) $\partial\overline{h}(x)$ is a polyhedron.
(iii) Suppose that h is an integer-valued function. For any vectors $u, u' \in \mathbb{Z}^N$ with $u \leq u'$, the set $\partial\overline{h}(x) \cap [u, u']$ is an integral polyhedron if it is nonempty.

We note that the set $\partial\overline{h}(x)$ has a nice combinatorial structure called L$^\natural$-convexity (see [22,25] for the definition of L$^\natural$-convex set), from which the statements (ii) and (iii) of Lemma 3.2 follow.

Lemma 3.3. *Let $p \in \mathbb{R}^N$ be a vector such that*

$$\min\{||p - p^*||_\infty \mid p^* \in \arg\max g_x\} \leq \frac{\delta}{n||v||_\infty}.$$

Then, the vector p and the value $\alpha = -g_x(p)$ satisfy the inequalities in (3.1).

Proof. Let p^* be a vector in $\arg\max g_x$ minimizing the value $||p - p^*||_\infty$. Since $p^* \in \partial\overline{h}(x)$ by Lemma 3.2 (i), we have

$$\overline{h}(y) - \overline{h}(x) \leq (p^*)^\top(y - x) = p^\top(y - x) + (p^* - p)^\top(y - x) \leq p^\top(y - x) + \delta$$

for all $y \in [\mathbf{0}, v]$, i.e., the former inequality in (3.1) holds.

We have $\overline{h}(x) = -g_x(p^*)$ by (3.3). Let $y_p \in [\mathbf{0}, v]_{\mathbb{Z}}$ be a vector such that $h^\circ(p) = p^\top y_p - h(y_p)$. Then, we have

$$\begin{aligned}g_x(p^*) = h^\circ(p^*) - (p^*)^\top x &\leq \{(p^*)^\top y_p - h(y_p)\} - (p^*)^\top x\\ &= (h^\circ(p) - p^\top x) + (p^* - p)^\top(y_p - x) \ \leq\ g_x(p) + \delta.\end{aligned}$$

On the other hand, we have $g_x(p^*) \geq g_x(p)$ since $p^* \in \arg\max g_x$. Hence, the value $\alpha = -g_x(p)$ satisfies the latter inequality in (3.1). $\square$

To complete the proof of Theorem 3.1, we show that a maximizer (or an approximate maximizer) of g_x can be computed in polynomial time. Define a vector

$u \in \mathbb{R}^N$ by $u(i) = h(\chi_i)$ $(i \in N)$. Although the effective domain of the function g_x is unbounded, the next lemma shows that it suffices to consider the bounded interval $[\mathbf{0}, u]$ when maximizing g_x.

Lemma 3.4. *For every $x \in [\mathbf{0}, v]$, there exists a subgradient $p \in \partial\overline{h}(x)$ such that $\mathbf{0} \leq p \leq u$.*

Proof. To prove Lemma 3.4, we use the following properties of concave closure $\overline{h}$.

Claim 1: For every $x, y \in [\mathbf{0}, v]$ with $x \geq y$ and for every $i \in \mathrm{supp}^+(x - y)$, there exists $\eta_0 > 0$ such that

$$\overline{h}(x) + \overline{h}(y) \leq \overline{h}(x - \eta\chi_i) + \overline{h}(y + \eta\chi_i) \qquad (\forall \eta \in [0, \eta_0]).$$

Claim 2: For every $x \in [\mathbf{0}, v]_{\mathbb{Z}}$ and $i \in N$, we have

$$\overline{h}(x + \eta\chi_i) - \overline{h}(x) = \eta\{\overline{h}(x + \chi_i) - \overline{h}(x)\} \qquad (\forall \eta \in [0, 1]).$$

Claim 1 follows from Theorem 2.10 (i) and the definition of polyhedral M$^\natural$-concave functions, and Claim 2 is by Theorem 2.10 (ii).

Let $x \in [\mathbf{0}, v]$. Since $\overline{h}$ is a polyhedral concave function such that $\mathrm{dom}_{\mathbb{R}}\,\overline{h}$ is a full-dimensional polytope, there exists a subgradient $p \in \partial\overline{h}(x)$ such that the set

$$S = \{y \in [\mathbf{0}, v] \mid \overline{h}(y) - \overline{h}(x) = p^\top(y - x)\}$$

is a full-dimensional polytope. We show that such a subgradient p satisfies $\mathbf{0} \leq p \leq u$.

Let $x_0 \in \mathbb{R}^N$ be a vector in the interior of S. Then, there exists $\varepsilon_0 > 0$ such that

$$\varepsilon p(i) = \overline{h}(x_0 + \varepsilon\chi_i) - \overline{h}(x_0) = \overline{h}(x_0) - \overline{h}(x_0 - \varepsilon\chi_i) \qquad (\forall i \in N,\ 0 \leq \forall\varepsilon \leq \varepsilon_0). \quad (3.4)$$

Since $\mathbf{0} < x_0 < v$, Claim 1 implies that

$$\overline{h}(\varepsilon\chi_i) - \overline{h}(\mathbf{0}) \geq \overline{h}(x_0) - \overline{h}(x_0 - \varepsilon\chi_i) \qquad (\forall i \in N), \quad (3.5)$$

$$\overline{h}(v) - \overline{h}(v - \varepsilon\chi_i) \leq \overline{h}(x_0 + \varepsilon\chi_i) - \overline{h}(x_0) \qquad (\forall i \in N) \quad (3.6)$$

for a sufficiently small $\varepsilon > 0$. By Claim 2 and Theorem 2.10 (ii), we have

$$\overline{h}(\varepsilon\chi_i) - \overline{h}(\mathbf{0}) = \varepsilon\{\overline{h}(\chi_i) - \overline{h}(\mathbf{0})\} = \varepsilon\{h(\chi_i) - h(\mathbf{0})\} = \varepsilon u(i), \quad (3.7)$$

$$\overline{h}(v) - \overline{h}(v - \varepsilon\chi_i) = \varepsilon\{\overline{h}(v) - \overline{h}(v - \chi_i)\} = \varepsilon\{h(v) - h(v - \chi_i)\} \geq 0 \quad (3.8)$$

for every $i \in N$, where the last inequality in (3.8) is by the monotonicity of h. Combining (3.4), (3.5), and (3.7), we have $p(i) \leq u(i)$ for all $i \in N$. Similarly, (3.4), (3.6), and (3.8) imply $p(i) \geq 0$ for all $i \in N$. □

By Theorem 2.13, g_x is a polyhedral L$^\natural$-concave function, and its function value can be computed in time polynomial in n and $\log ||v||_\infty$ by Theorem 2.7. Let $\delta' = \delta/(n^2||v||_\infty)$, and define a function $g_{\mathbb{Z}} : \mathbb{Z}^N \to \mathbb{R} \cup \{-\infty\}$ by

$$g_{\mathbb{Z}}(p) = \begin{cases} g_x(\delta' p) & \text{if } \delta' p \in [0, u], \\ -\infty & \text{otherwise} \end{cases} \qquad (p \in \mathbb{Z}^N).$$

Then, $g_{\mathbb{Z}}$ is an $\mathrm{L}^\natural$-concave function over the integer lattice. The following theorem states that any maximizer of $g_{\mathbb{Z}}$ is sufficiently close to a maximizer of g_x.

Theorem 3.5 ([19]). *Let $p_{\mathbb{Z}} \in \mathbb{Z}^N$ be a maximizer of $g_{\mathbb{Z}}$. Then, there exists a maximizer p^* of g_x with $p^* \in [\mathbf{0}, u]$ such that*

$$||p^* - \delta' p_{\mathbb{Z}}||_\infty \leq n\delta' = \frac{\delta}{n||v||_\infty}.$$

Since $g_{\mathbb{Z}}$ is an $\mathrm{L}^\natural$-concave function, Theorem 2.12 implies that a maximizer of $g_{\mathbb{Z}}$ can be computed in time polynomial in n and $\log \max_{i \in N}(u(i)/\delta')$. This concludes the proof of Theorem 3.1 (i).

In case where h is integer-valued, Lemmas 3.2 (iii) and 3.4 imply that an optimal solution of the problem $\max\{g_x(p) \mid p \in \mathbb{Z}^N,\ p \in [\mathbf{0}, u]\}$ is a subgradient of $\overline{h}$ at x, and such an optimal solution can be obtained in polynomial time by Theorem 2.12. Hence, Theorem 3.1 (ii) is proved.

4. Solving the Relaxed Problem

We prove Theorem 1.5 by providing polynomial-time algorithms for the relaxed problem (RGP). Theorem 1.3 is an immediate corollary of Theorem 1.5, and Theorem 1.4 follows from Theorems 1.3, 2.1, and 2.3, and Corollary 2.5.

4.1. *Algorithm for real-valued functions*

We first prove Theorem 1.5 (i). Let α^* be the optimal value of the problem (RGP), i.e.,

$$\alpha^* = \max\{\tilde{h}(x) \mid x \in \overline{P}\}.$$

It suffices to show that for every $\varepsilon > 0$, we can find a vector $x \in \overline{P}$ with $\tilde{h}(x) \geq \alpha^* - \varepsilon$ in time polynomial in n, m, $\log h(v)$, $\widetilde{\Lambda}$, $\log ||v||_\infty$, and $\log(1/\varepsilon)$. If we put $\varepsilon = \varepsilon' h(\chi_j)$ for $\varepsilon' > 0$ and an arbitrarily chosen $j \in N$, then we obtain a $(1 - \varepsilon')$-approximate solution of (RGP) since

$$\frac{\tilde{h}(x)}{\alpha^*} \geq \frac{\alpha^* - \varepsilon' h(\chi_j)}{\alpha^*} \geq 1 - \varepsilon'.$$

For every $\underline{\alpha} \in \mathbb{R}$, we define a set by

$$\mathrm{L}(\underline{\alpha}) = \{(x, \alpha) \in \mathbb{R}^N \times \mathbb{R} \mid \underline{\alpha} \leq \alpha \leq \tilde{h}(x)\},$$

which satisfies $\mathrm{L}(\underline{\alpha}) \neq \emptyset$ if and only if $\underline{\alpha} \leq \alpha^*$. Given a real number $\underline{\alpha}$, our algorithm described below checks the nonemptyness of the set $\mathrm{L}(\underline{\alpha})$; more precisely, our algorithm either asserts $\underline{\alpha} > \alpha^* - (\varepsilon/4)$ or finds a point $x \in \overline{P}$ such that $\underline{\alpha} \leq \tilde{h}(x) + (\varepsilon/2)$. By combining this algorithm with binary search w.r.t. $\underline{\alpha}$, we can find a real number $\underline{\alpha}$ and a point $x \in \overline{P}$ such that

$$\underline{\alpha} > \alpha^* - \frac{\varepsilon}{2}, \qquad \underline{\alpha} \leq \tilde{h}(x) + \frac{\varepsilon}{2}.$$

This implies that $\tilde{h}(x) \geq \alpha^* - \varepsilon$, i.e., x is a desired approximate solution of (RGP).

Our algorithm for checking the nonemptyness of the set $\mathrm{L}(\underline{\alpha})$ is based on the ellipsoid method [13]. Recall that $\overline{P}$ is assumed to be a full-dimensional polytope; this assumption is needed for the correctness of the ellipsoid method since we will use approximate separating hyperplanes for $\mathrm{L}(\underline{\alpha})$ (see [13, Remark 6.3.3]).

The ellipsoid method always maintains an ellipsoid containing the set $\mathrm{L}(\underline{\alpha})$; initially, we can use a sufficiently large ellipsoid containing the following set:

$$\{(x, \alpha) \in \mathbb{R}^N \times \mathbb{R} \mid x \in [\mathbf{0}, v],\ 0 \leq \alpha \leq h(v)\}.$$

In each iteration, the algorithm checks whether the set $\mathrm{L}(\underline{\alpha})$ approximately contains the point (x_c, α_c) which is the center of the current ellipsoid E ($\subseteq \mathbb{R}^N \times \mathbb{R}$); if not, it computes a hyperplane which almost separates the point (x_c, α_c) and the set $\mathrm{L}(\underline{\alpha})$ in the following way, where $\delta > 0$ is a constant given by $\delta = \varepsilon/2m$.

Case 1: If $\alpha_c < \underline{\alpha}$, then we output the inequality $\alpha \geq \underline{\alpha}$ as a separating hyperplane.
Case 2: If $x_c \notin \overline{P}$, we compute a separating hyperplane for $\overline{P}$ and x_c and output it.
Case 3: Suppose that $\alpha_c \geq \underline{\alpha}$ and $x_c \in \overline{P}$. For each $k = 1, 2, \ldots, m$, we compute a real number β_k satisfying

$$\overline{h}_k(x_c) \leq \beta_k \leq \overline{h}_k(x_c) + \delta$$

(cf. Theorem 3.1 (i)) and put $\beta = \sum_{k=1}^{m} \beta_k$.
Case 3-1: If $\alpha_c \leq \beta$, then we output the point $x_c \in \overline{P}$ and stop. (The point x_c satisfies $\underline{\alpha} \leq \tilde{h}(x_c) + \varepsilon/2$.)
Case 3-2: Suppose that $\alpha_c > \beta$. For each $k = 1, 2, \ldots, m$, we compute a vector $p_k \in \mathbb{R}^N$ satisfying

$$\overline{h}_k(x) - \overline{h}(x_c) \leq p_k^\top (x - x_c) + \delta \qquad (\forall x \in [\mathbf{0}, v])$$

(cf. Theorem 3.1 (i)), and put $p = \sum_{k=1}^{m} p_k$. We output the inequality

$$\alpha - \beta \leq p^\top (x - x_c) + 2m\delta\ (= p^\top (x - x_c) + \frac{\varepsilon}{2})$$

as a separating hyperplane.

After computing a separating hyperplane $q^\top y + q_0 \alpha \leq \xi$ in this way, we compute a new ellipsoid E' such that

$$E' \supseteq E \cap \{(y, \alpha) \mid q^\top y + q_0 \alpha \leq \xi\}$$

and the ratio of the volumes of E and E' is bounded by a constant less than one, where the constant is dependent only on n (cf. [13, Lemma 3.2.10]).

We now show that a polynomial number of iterations is sufficient to check the nonemptyness of $\mathrm{L}(\underline{\alpha})$. It is noted that the inequality

$$\alpha - \beta \leq p^\top (x - x_c) + \frac{\varepsilon}{2}$$

obtained in Case 3-2 is satisfied by all $(x, \alpha) \in \mathrm{L}(\underline{\alpha})$, implying that the ellipsoid E always contains the set $\mathrm{L}(\underline{\alpha})$ in each iteration. This fact, together with the next lemma, implies that if the volume of the current ellipsoid E is sufficiently small, then $\mathrm{L}(\underline{\alpha})$ is almost empty.

Lemma 4.1. *For any $\underline{\alpha} \in [0, \alpha^*]$, the volume of $\mathrm{L}(\underline{\alpha})$ is at least*

$$\frac{\alpha^*}{(n+1)!}\left(\frac{\alpha^* - \underline{\alpha}}{\alpha^*}\right)^{n+1}.$$

Proof. We first consider the case where $\underline{\alpha} = 0$. We denote by C_0 ($\subseteq \mathbb{R}^N \times \mathbb{R}$) the convex hull of the set

$$S = \{(y, 0) \in \mathbb{R}^N \times \mathbb{R} \mid y \text{ is a vertex of } \overline{P}\} \cup \{(x^*, \alpha^*)\},$$

where $x^* \in \mathbb{R}^N$ is an optimal solution of (RGP). Then, we have $C_0 \subseteq \mathrm{L}(0)$ since $\mathrm{L}(0)$ is a convex set and all of the vectors in S are contained in $\mathrm{L}(0)$. Since $\overline{P}$ is a full-dimensional integral polytope, its volume is at least $1/n!$. Hence, the volume of C_0 is at least $\alpha^*/(n+1)!$.

We then consider the general case. For any $\underline{\alpha} \in [0, \alpha^*]$, we define a set $C(\underline{\alpha}) \subseteq \mathbb{R}^N \times \mathbb{R}$ by

$$C(\underline{\alpha}) = C_0 \cap \{(y, \alpha) \mid \alpha \geq \underline{\alpha}\}.$$

Then, we have $C(0) = C_0$ and

$$\begin{aligned}(\text{the volume of } C(\underline{\alpha})) &= (\text{the volume of } C_0) \times \left(\frac{\alpha^* - \underline{\alpha}}{\alpha^*}\right)^{n+1} \\ &\geq \frac{\alpha^*}{(n+1)!}\left(\frac{\alpha^* - \underline{\alpha}}{\alpha^*}\right)^{n+1}.\end{aligned}$$

Since $C(\underline{\alpha}) \subseteq \mathrm{L}(\underline{\alpha})$, we obtain a desired result. □

Let

$$\Delta = \frac{\alpha^*}{(n+1)!}\left(\frac{\varepsilon/4}{\alpha^*}\right)^{n+1}.$$

We see from Lemma 4.1 that if the volume of the current ellipsoid E can be less than Δ, then it holds that $\alpha^* - \underline{\alpha} < \varepsilon/4$. This implies that after a polynomial number of iterations we can find a point $x_c \in \overline{P}$ with $\underline{\alpha} \leq \tilde{h}(x_c) + \varepsilon/2$ or discern $\underline{\alpha} > \alpha^* - \varepsilon/4$. Hence, we obtain a desired algorithm for checking the nonemptyness of $\mathrm{L}(\underline{\alpha})$.

4.2. *Algorithm for integer-valued functions*

We then prove Theorem 1.5 (ii). When each h_k is integer-valued, we use the ellipsoid method in a different way; the ellipsoid method is used to find a vector in the set $S^* = \arg\max\{\tilde{h}(x) \mid x \in \overline{B}\}$, where $\overline{B}$ is the convex hull of the set B of bases in P (see Section 2.2). We note that there exists an optimal solution x^* of (RGP) with $x^* \in \overline{B}$ since the objective function $\tilde{h}$ is nondecreasing. For the correctness and polynomial-time termination of the ellipsoid method, it suffices to prove the following properties (see [13, Theorem 6.4.1]):

(a) S^* is a rational polytope such that the encoding length of each facet is bounded by a polynomial in the input size,
(b) a separating hyperplane for the set S^* and a given point $x \in [\mathbf{0}, v]$ can be computed in time polynomial in the input size.

For any $k = 1, 2, \ldots, m$ and any $p \in \mathbb{R}^N$, the set $\arg\max\{\overline{h}_k(x) - p^\top x \mid x \in [\mathbf{0}, v]\}$ is an integral g-polymatroid by Theorem 2.11. Hence, S^* is given as the intersection of m integral g-polymatroids and the polytope $\overline{B}$. Therefore, S^* can be represented by the inequalities of the form $x(X) \leq \gamma_X$ or $x(X) \geq \gamma_X$ with $X \in 2^N$ and an integer γ_X with $0 \leq \gamma_X \leq n||v||_\infty$. This fact shows that S^* is a rational polytope such that the encoding length of each facet is bounded by a polynomial in n and in $\log ||v||_\infty$.

We then explain how to compute a separating hyperplane for S^* and x. We first check whether $x \in \overline{B}$ or not. If $x \notin \overline{B}$, then we compute a separating hyperplane for $\overline{B}$ and x, and output it. If $x \in \overline{B}$, then we compute a subgradient $p_k \in \partial\overline{h}_k(x)$ for all $k = 1, 2, \ldots, m$ in polynomial time, as shown in Theorem 3.1 (ii). Since $\tilde{h} = \sum_{k=1}^m \overline{h}_k$, the vector $p = \sum_{k=1}^m p_k$ is a subgradient of $\tilde{h}$ at x. Therefore, we have

$$0 \leq \tilde{h}(x^*) - \tilde{h}(x) \leq p^\top(x^* - x) \qquad (\forall x^* \in S^*),$$

i.e., $p^\top y \geq p^\top x$ is a separating hyperplane for S^* and x. This concludes the proof of Theorem 1.5 (ii).

5. Extension of Pipage Rounding Algorithm

We give a proof of Theorem 1.6 by extending the pipage rounding algorithm of Calinescu et al. [3] to the generalized problem (GP).

We firstly compute an (approximate) optimal solution of the relaxed problem (RGP) with the objective function $\tilde{h}(x) = \sum_{k=1}^m \overline{h}_k(x)$. The problem (RGP) can be solved optimally (or approximately) in polynomial time by Theorem 1.5.

Suppose that $x^* \in \overline{B}$ is an optimal (or an approximate) solution of (RGP). We consider the restriction of the problem (GP) over the hypercube $[\lfloor x^* \rfloor, \lfloor x^* \rfloor + \mathbf{1}]$. That is, we consider the problem:

$$\text{Maximize } \sum_{k=1}^m f_k(X) \quad \text{subject to } X \in \mathcal{F}, \tag{5.1}$$

where $f_k : 2^N \to \mathbb{R}$ $(k = 1, 2, \ldots, m)$ and $\mathcal{F} \subseteq 2^N$ are defined by

$$f_k(X) = h_k(\lfloor x^* \rfloor + \chi_X) - h_k(\lfloor x^* \rfloor) \qquad (X \in 2^N),$$
$$\mathcal{F} = \{X \in 2^N \mid \lfloor x^* \rfloor + \chi_X \in P\}.$$

Since the functions f_k are nondecreasing M$^\natural$-concave set functions with $f_k(\emptyset) = 0$ and $\mathcal{F}$ is the family of independent sets of a matroid $\mathcal{M} = (N, \mathcal{F})$, the problem (5.1) is of the form (P). We apply the pipage rounding algorithm for (P) to the problem (5.1), and compute an approximate solution $X_0 \in 2^N$. Finally, we output an integral vector $x_0 = \lfloor x^* \rfloor + \chi_{X_0}$ as an approximate solution of (GP).

We show that the vector x_0 obtained in this way is a $(1 - 1/e)$-approximate solution of (GP). It follows from Theorem 2.10 (ii) that

$$\begin{aligned}
&\overline{h}_k(\lfloor x^* \rfloor + y) \\
&= \max\left\{ \sum_{X \subseteq N} \lambda_X h_k(\lfloor x^* \rfloor + \chi_X) \,\middle|\, \sum_{X \subseteq N} \lambda_X \chi_X = y,\ \sum_{X \subseteq N} \lambda_X = 1, \lambda_X \geq 0\ (X \in 2^N) \right\} \\
&= h_k(\lfloor x^* \rfloor) \\
&\qquad + \max\left\{ \sum_{X \subseteq N} \lambda_X f_k(X) \,\middle|\, \sum_{X \subseteq N} \lambda_X \chi_X = y,\ \sum_{X \subseteq N} \lambda_X = 1, \lambda_X \geq 0\ (X \in 2^N) \right\} \\
&= h_k(\lfloor x^* \rfloor) + \overline{f}_k(y)
\end{aligned}$$

for every $y \in [\mathbf{0}, \mathbf{1}]$. Hence, it holds that

$$\sum_{k=1}^{m} \overline{h}_k(y + \lfloor x^* \rfloor) = \sum_{k=1}^{m} h_k(\lfloor x^* \rfloor) + \sum_{k=1}^{m} \overline{f}_k(y) \qquad (\forall y \in [\mathbf{0}, \mathbf{1}]). \tag{5.2}$$

This equation shows that the vector $d^* = x^* - \lfloor x^* \rfloor \in [\mathbf{0}, \mathbf{1}]$ is an optimal solution of (RP) associated with (5.1). Therefore, Theorem 1.4 implies that

$$\sum_{k=1}^{m} f_k(X_0) \geq \left(1 - \frac{1}{e}\right) \sum_{k=1}^{m} \overline{f}_k(d^*).$$

Using this inequality and (5.2), we have

$$\begin{aligned}
\sum_{k=1}^{m} h_k(x_0) &= \sum_{k=1}^{m} f_k(X_0) + \sum_{k=1}^{m} h_k(\lfloor x^* \rfloor) \\
&\geq \left(1 - \frac{1}{e}\right) \sum_{k=1}^{m} \overline{f}_k(d^*) + \left(1 - \frac{1}{e}\right) \sum_{k=1}^{m} h_k(\lfloor x^* \rfloor) \\
&= \left(1 - \frac{1}{e}\right) \sum_{k=1}^{m} \overline{h}_k(x^*).
\end{aligned}$$

This shows that the vector x_0 is a $(1 - 1/e)$-approximate solution of (GP).

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