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ON THE POINCARÉ-VON ZEIPEL AND BROWN-SHOOK METHODS OF THE ELIMINATION OF THE SHORT PERIOD TERMS FROM A HAMILTONIAN

by P. Musen

Goddard Space Flight Center
Greenbelt, Md.

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ABSTRACT

In this work we discuss and compare the methods of Poincaré-von Zeipel and Brown-Shook for the elimination of the short period terms from a Hamiltonian by means of canonical transformations. We develop the general theory, and also supply a table of operators which serve both to eliminate short period effects of any order and to invert the canonical transformation, i.e., to express the original elements in terms of the elements affected only by the long period perturbations. The generating functions which produce these operators are the Taylor and the Lagrange operators. The Lagrange operator is a generalization of the classical operator to the case of several independent variables. In every attempt to solve the problem of the general perturbations using the electronic machines, the table of differential operators and the recurrence relations between them constitute an essential part of the programming.

The elimination of the short period terms seems to be somewhat simpler in the Brown-Shook method than in Poincaré-von Zeipel method. However, the process of determination of the original, osculating, elements in terms of the elements affected by the long period perturbations only is more complicated in the Brown-Shook method than in von Zeipel's method, because the inversion of the canonical transformation in the Brown-Shook method requires the application of a chain of Lagrange operators. In the Poincaré-von Zeipel method, the use of only one Lagrange operator is required for the process of the inversion. In this respect the Brown-Shook method resembles the classical method of Delaunay. Of course, if we are interested in the first order effects only, then both methods coincide, and the statement of Jeffreys about the identity of both methods remains valid.

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ON THE POINCARÉ-VON ZEIPEL AND BROWN-SHOOK METHODS OF THE ELIMINATION OF THE SHORT PERIOD TERMS FROM A HAMILTONIAN

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P. Musen

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INTRODUCTION

In this work we discuss and compare the methods of Poincaré (1892)-von Zeipel (1916) and Brown-Shook (1933) for eliminating the short period terms from a Hamiltonian by means of canonical transformations. We also develop a system of differential operators to be used to perform this elimination up to any desired order of a small parameter. After the short period terms are eliminated, only secular, long period and critical terms, as caused by the near commensurability of the mean motions of celestial bodies, are left in the transformed Hamiltonian. We have in view applications to planetary or lunar problems, and the mathematical as well as physical importance of eliminating the short period terms in these two types of problem is quite evident. The behavior of a planetary orbit over a very long interval of time depends mainly upon the transformed Hamiltonian. In connection with this topic, the recent work by Meffroy (1967) deserves special attention.

In the lunar problem, the transformed Hamiltonian serves to determine the mean motion of the perigee and of the node. In addition, the Poincaré-von Zeipel and Brown-Shook methods because of their concise form, represent excellent tools in understanding the effect of the propagation of the influence of lower-order effects into the effects of higher orders.

The elimination of all significant short period terms, if performed by hand, can be extremely time consuming. It took Delaunay (1860) about twenty years to complete his lunar theory. In the recent work by Hori (1963), based on the application of von Zeipel's method, a considerable portion of Delaunay's theory is obtained in a shorter and faster manner. The application of von Zeipel's theory to the motion of the artificial satellite by Brouwer (1959) belongs to the most elegant results obtained in recent times.

At the present time, the idea of applying electronic machines to obtain the analytical expansion of the general perturbations is becoming very popular. We are still far from solving the problem of the literal expansion in a complete form by machines, but successful experiments are being performed at several institutes and the future looks promising. Thus, it is quite possible that in the very near future we shall witness significant progress in celestial mechanics, which will be comparable only to the progress in numerical methods since electronic machines were first applied on a large scale nearly twenty years ago.

In every attempt to solve the problem of the general perturbations using the machines, the table of differential operators and the recurrence relations between them will constitute the essential part of the programming. The "generating functions" which produce the necessary operators are the Taylor and Lagrange operators. The Lagrange operator is associated with a generalization of the classical Lagrange expansion to the case of several variables.

In this work, we give a general theory and a table and formulas for operators which serve to eliminate short period effects up to any order, and to invert the canonical transformation, i.e., to express the original elements in terms of the elements affected by the long period perturbations only.

In any particular problem, the selection of significant operators and the programming can be facilitated on the basis of the formalized procedure for the von Zeipel and Brown-Shook methods.

A part of the exposition given here is from lectures on von Zeipel's method given by the author at the University of Maryland. An approach different from that presented in this work was given by Giacaglia (1964). The recent interesting results obtained by Hori (1966) using Lie series deserve to be mentioned.

TAYLOR AND LAGRANGE OPERATORS

The basic operators which we use in this work are

$$\mathbf{h}(\mathbf{x}) \cdot \nabla_{\mathbf{x}} = \sum_{\alpha=1}^{n} h_\alpha \frac{\partial}{\partial x_\alpha} \tag{1}$$

and

$$\nabla_{\mathbf{x}} \cdot \mathbf{h}(\mathbf{x}) = \sum_{\alpha=1}^{n} \frac{\partial}{\partial x_\alpha} h_\alpha , \qquad \alpha = 1, 2, \cdots, n , \tag{2}$$

where

$$\mathbf{h} = \left(h_1, h_2, \cdots, h_n\right)$$

is a vector function of the position vector

$$\mathbf{x} = \left(x_1, x_2, \cdots, x_n\right)$$

and

$$\nabla_{\mathbf{x}} = \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \ldots, \frac{\partial}{\partial x_n}\right) \tag{3}$$

is the gradient operator relative to x. We have by definition

$$(\mathbf{h} \cdot \nabla_{\mathbf{x}})\,\phi(\mathbf{x}) = \mathbf{h} \cdot (\nabla_{\mathbf{x}}\phi) = \sum_{\alpha=1}^{n} h_\alpha \frac{\partial\phi}{\partial x_\alpha} \tag{4}$$

and

$$(\nabla_{\mathbf{x}} \cdot \mathbf{h})\,\phi(\mathbf{x}) = \nabla_{\mathbf{x}} \cdot (\mathbf{h}\phi) = \sum_{\alpha=1}^{n} \frac{\partial}{\partial x_\alpha}(h_\alpha \phi) . \tag{5}$$

In particular, ϕ can be a vector or a polyadic.

With a given set of vectors

$$\mathbf{g}_j(\mathbf{x}) = (g_{j1}, g_{j2}, \cdots, g_{jn}) , \qquad j = 1, 2, 3, \cdots$$

we can associate two sets of operators:

$$\mathbf{g}_j \cdot \nabla_{\mathbf{x}} = \sum_{\alpha=1}^{n} g_{j\alpha} \frac{\partial}{\partial x_\alpha} , \tag{6}$$

and

$$\nabla_{\mathbf{x}} \cdot \mathbf{g}_j = \sum_{\alpha=1}^{n} \frac{\partial}{\partial x_\alpha} g_{j\alpha} . \tag{7}$$

Furthermore, for the purpose of elimination of the short period terms from a Hamiltonian, we introduce the products of operators (6) and (7). By definition:

$$(\mathbf{g}_1 \cdot \nabla_{\mathbf{x}}) \otimes (\mathbf{g}_2 \cdot \nabla_{\mathbf{x}}) \otimes \cdots \otimes (\mathbf{g}_m \cdot \nabla_{\mathbf{x}}) = \prod_{\alpha=1}^{m} \otimes (\mathbf{g}_\alpha \cdot \nabla_{\mathbf{x}})$$

$$= \sum_{\alpha_1, \alpha_2, \ldots, \alpha_m} g_{1,\alpha_1} g_{2,\alpha_2} \cdots g_{m,\alpha_m} \frac{\partial^m}{\partial x_{\alpha_1} \partial x_{\alpha_2} \cdots \partial x_{\alpha_m}} , \tag{8}$$

$$(\nabla_{\mathbf{x}} \cdot \mathbf{g}_1) \otimes (\nabla_{\mathbf{x}} \cdot \mathbf{g}_2) \otimes \cdots \otimes (\nabla_{\mathbf{x}} \cdot \mathbf{g}_m) = \prod_{\alpha=1}^{m} \otimes (\nabla_{\mathbf{x}} \cdot \mathbf{g}_\alpha)$$

$$= \sum_{\alpha_1, \alpha_2, \ldots, \alpha_m} \frac{\partial^m}{\partial x_{\alpha_1} \partial x_{\alpha_2} \cdots \partial x_{\alpha_m}} g_{1,\alpha_1} g_{2,\alpha_2} \cdots g_{m,\alpha_m} , \qquad (9)$$

$$\alpha_1, \alpha_2, \cdots, \alpha_m = 1, 2, \cdots n .$$

Evidently these products resemble the direct Kronesker products from the theory of representations. It is important to emphasize that the products (8) and (9) do not mean that the operators (6) or (7) are being applied in succession. The operators are first being multiplied together, then applied to the function standing to the right. For example:

$$(\mathbf{g}_1 \cdot \nabla_{\mathbf{x}}) \otimes (\mathbf{g}_2 \cdot \nabla_{\mathbf{x}})\phi \neq (\mathbf{g}_1 \cdot \nabla_{\mathbf{x}})\{(\mathbf{g}_2 \cdot \nabla_{\mathbf{x}} \phi)\} ,$$

and

$$(\nabla_{\mathbf{x}} \cdot \mathbf{g}_1) \otimes (\nabla_{\mathbf{x}} \cdot \mathbf{g}_2)\phi \neq (\nabla_{\mathbf{x}} \cdot \mathbf{g}_1)\{\nabla_{\mathbf{x}} (\mathbf{g}_2 \phi)\} .$$

We can write (8) and (9) also in the form

$$\prod_{\alpha=1}^{m} \otimes (\mathbf{g}_\alpha \cdot \nabla_{\mathbf{x}}) = \left(\prod_{\alpha=1}^{m} \mathbf{g}_\alpha\right) * \nabla_{\mathbf{x}}^m , \qquad (10)$$

$$\prod_{\alpha=1}^{m} \otimes (\nabla_{\mathbf{x}} \cdot \mathbf{g}_\alpha) = \nabla_{\mathbf{x}}^m * \prod_{\alpha=1}^{m} \mathbf{g}_\alpha , \qquad (11)$$

where the powers of ∇ and the product $\mathbf{g}_1, \mathbf{g}_2, \cdots \mathbf{g}_n$ are polyadics and the asterisks mean a complete contraction, until a scalar is obtained. In particular

$$\mathbf{g}_j * \nabla_{\mathbf{x}} = \mathbf{g}_j \cdot \nabla_{\mathbf{x}} ,$$

$$\nabla_{\mathbf{x}} * \mathbf{g}_j = \nabla_{\mathbf{x}} \cdot \mathbf{g}_j .$$

The product of the identical operators appears in elementary calculus and can be written as a symbolic power,

$$\underbrace{(\mathbf{h} \cdot \nabla_{\mathbf{x}}) \otimes (\mathbf{h} \cdot \nabla_{\mathbf{x}}) \otimes \cdots \otimes (\mathbf{h} \cdot \nabla_{\mathbf{x}})}_{m \text{ times}} = (\mathbf{h} \cdot \nabla_{\mathbf{x}})^m = \mathbf{h}^m * \nabla_{\mathbf{x}}^m ,$$

and

$$\underbrace{(\nabla_{\mathbf{x}} \cdot \mathbf{h}) \otimes (\nabla_{\mathbf{x}} \cdot \mathbf{h}) \otimes \cdots \otimes (\nabla_{\mathbf{x}} \cdot \mathbf{h})}_{\text{m times}} = (\nabla_{\mathbf{x}} \cdot \mathbf{h})^m = \nabla_{\mathbf{x}}^m * \mathbf{h}^m .$$

Decomposing $\mathbf{h}$ along the fixed axes,

$$\mathbf{h} = (h_1, h_2, \cdots, h_n) ;$$

we have

$$(\mathbf{h} \cdot \nabla_{\mathbf{x}})^m = \sum_{\alpha_1, \alpha_2, \cdots, \alpha_m} \frac{m!}{\alpha_1!\,\alpha_2!\cdots\alpha_m!} h_1^{\alpha_1} h_2^{\alpha_2} \cdots h_m^{\alpha_m} \frac{\partial^m}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \cdots \partial x_m^{\alpha_m}} ,$$

$$(\nabla_{\mathbf{x}} \cdot \mathbf{h})^m = \sum_{\alpha_1, \alpha_2, \cdots, \alpha_m} \frac{m!}{\alpha_1!\,\alpha_2!\cdots\alpha_m!} \frac{\partial^m}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \cdots \partial x_m^{\alpha_m}} h_1^{\alpha_1} h_2^{\alpha_2} \cdots h_m^{\alpha_m} ,$$

$$\alpha_1 + \alpha_2 + \cdots + \alpha_m = m .$$

If $\mathbf{h}$ is expanded into a series in powers of a small parameter

$$\mathbf{h} = \sum_{\alpha=1}^{\infty} \mathbf{h}_\alpha ,$$

$$\mathbf{h}_j = (h_{j1}, h_{j2}, \cdots, h_{jn}) ,$$

then the Taylor operator

$$T = \exp(\mathbf{h} \cdot \nabla_{\mathbf{x}}) = \sum_{\alpha=0}^{\infty} \frac{(\mathbf{h} \cdot \nabla_{\mathbf{x}})^\alpha}{\alpha!} = \sum_{\alpha=0}^{\infty} \frac{\mathbf{h}^\alpha * \nabla_{\mathbf{x}}^\alpha}{\alpha!} ,$$

and the Lagrange operator

$$\Lambda = \exp(\nabla_{\mathbf{x}} \cdot \mathbf{h}) = \sum_{\alpha=0}^{\infty} \frac{(\nabla_{\mathbf{x}} \cdot \mathbf{h})^\alpha}{\alpha!} = \sum_{\alpha=0}^{\infty} \frac{\nabla_{\mathbf{x}}^\alpha * \mathbf{h}^\alpha}{\alpha!} ,$$

can be expanded into series of the form

$$T = \sum_{\alpha=0}^{\infty} T_\alpha , \qquad \Lambda = \sum_{\alpha=0}^{\infty} \Lambda_\alpha , \tag{12}$$

where T_j and Λ_j are polynomials in $\mathbf{h}_k \cdot \nabla_{\mathbf{x}}$, and $\nabla_{\mathbf{x}} \cdot \mathbf{h}_k$ $(k = 1, 2, \cdots, j)$ respectively and of order j relative to the small parameter. We can represent the operators T_j as the sum of polynomials homogeneous in $\mathbf{h}_s \cdot \nabla_{\mathbf{x}}$ $(s = 1, 2, \cdots)$,

$$T_j = \sum_{k=1}^{j} T_{j,k} ,$$

where $T_{j,k}$ is of degree k relative to the operators $\mathbf{h}_s \cdot \nabla_{\mathbf{x}}$. The T_j-operators can be associated with the name of Faa de Bruno (1855). Making use of the table of Faa de Bruno operators (Duboshin, 1950) we obtain

$$T_0 = 1 ,$$

$$T_{1,1} = \mathbf{h}_1 \cdot \nabla_{\mathbf{x}} ,$$

$$T_{2,1} = \mathbf{h}_2 \cdot \nabla_{\mathbf{x}} ,$$

$$T_{2,2} = \frac{1}{2} (\mathbf{h}_1 \cdot \nabla_{\mathbf{x}})^2 = \frac{1}{2} \mathbf{h}_1^2 * \nabla_{\mathbf{x}}^2 ,$$

$$T_{3,1} = \mathbf{h}_3 \cdot \nabla_{\mathbf{x}} ,$$

$$T_{3,2} = (\mathbf{h}_1 \cdot \nabla_{\mathbf{x}}) \otimes (\mathbf{h}_2 \cdot \nabla_{\mathbf{x}}) = \mathbf{h}_1 \mathbf{h}_2 * \nabla_{\mathbf{x}}^2 ,$$

$$T_{3,3} = \frac{1}{6} (\mathbf{h}_1 \cdot \nabla_{\mathbf{x}})^3 = \frac{1}{6} \mathbf{h}_1^3 * \nabla_{\mathbf{x}}^3 ,$$

$$T_{4,1} = \mathbf{h}_4 \cdot \nabla_{\mathbf{x}} ,$$

$$T_{4,2} = (\mathbf{h}_1 \cdot \nabla_{\mathbf{x}}) \otimes (\mathbf{h}_3 \cdot \nabla_{\mathbf{x}}) + \frac{1}{2} (\mathbf{h}_2 \cdot \nabla_{\mathbf{x}})^2 = \left(\mathbf{h}_1 \mathbf{h}_3 + \frac{1}{2} \mathbf{h}_2^2\right) * \nabla_{\mathbf{x}}^2 ,$$

$$T_{4,3} = \frac{1}{2} (\mathbf{h}_1 \cdot \nabla_{\mathbf{x}})^2 \otimes (\mathbf{h}_2 \cdot \nabla_{\mathbf{x}}) = \frac{1}{2} \mathbf{h}_1^2 \mathbf{h}_2 * \nabla_{\mathbf{x}}^3 ,$$

$$T_{4,4} = \frac{1}{24} (\mathbf{h}_1 \cdot \nabla_{\mathbf{x}})^4 = \frac{1}{24} \mathbf{h}_1^4 * \nabla_{\mathbf{x}}^4 ,$$

$$T_{5,1} = \mathbf{h}_5 \cdot \nabla_{\mathbf{x}} ,$$

$$T_{5,2} = (\mathbf{h}_1 \cdot \nabla_\mathbf{x}) \odot (\mathbf{h}_4 \cdot \nabla_\mathbf{x}) + (\mathbf{h}_2 \cdot \nabla_\mathbf{x}) \otimes (\mathbf{h}_3 \cdot \nabla_\mathbf{x})$$

$$= (\mathbf{h}_1 \mathbf{h}_4 + \mathbf{h}_2 \mathbf{h}_3) * \nabla_\mathbf{x}^2 ,$$

$$T_{5,3} = \frac{1}{2} (\mathbf{h}_1 \cdot \nabla_\mathbf{x})^2 \otimes (\mathbf{h}_3 \cdot \nabla_\mathbf{x}) + \frac{1}{2} (\mathbf{h}_1 \cdot \nabla_\mathbf{x}) \otimes (\mathbf{h}_2 \cdot \nabla_\mathbf{x})^2$$

$$= \frac{1}{2} (\mathbf{h}_1^2 \mathbf{h}_3 + \mathbf{h}_1 \mathbf{h}_2^2) * \nabla_\mathbf{x}^3 ,$$

$$T_{5,4} = \frac{1}{6} (\mathbf{h}_1 \cdot \nabla_\mathbf{x})^3 \otimes (\mathbf{h}_2 \cdot \nabla_\mathbf{x}) = \frac{1}{6} \mathbf{h}_1^3 \mathbf{h}_2 * \nabla_\mathbf{x}^4 ,$$

$$T_{5,5} = \frac{1}{120} (\mathbf{h}_1 \cdot \nabla_\mathbf{x})^5 = \frac{1}{120} \mathbf{h}_1^5 * \nabla_\mathbf{x}^5 ,$$

$$T_{6,1} = \mathbf{h}_6 \cdot \nabla_\mathbf{x} ,$$

$$T_{6,2} = (\mathbf{h}_1 \cdot \nabla_\mathbf{x}) \otimes (\mathbf{h}_5 \cdot \nabla_\mathbf{x}) + (\mathbf{h}_2 \cdot \nabla_\mathbf{x}) \otimes (\mathbf{h}_4 \cdot \nabla_\mathbf{x}) + \frac{1}{2} (\mathbf{h}_3 \cdot \nabla_\mathbf{x})^2$$

$$= \left(\mathbf{h}_1 \mathbf{h}_5 + \mathbf{h}_2 \mathbf{h}_4 + \frac{1}{2} \mathbf{h}_3^2\right) * \nabla_\mathbf{x}^2 ,$$

$$T_{6,3} = \frac{1}{2} (\mathbf{h}_1 \cdot \nabla_\mathbf{x})^2 \otimes (\mathbf{h}_4 \cdot \nabla_\mathbf{x}) + (\mathbf{h}_1 \cdot \nabla_\mathbf{x})(\mathbf{h}_2 \cdot \nabla_\mathbf{x})(\mathbf{h}_3 \cdot \nabla_\mathbf{x}) + \frac{1}{6} (\mathbf{h}_2 \cdot \nabla_\mathbf{x})^3$$

$$= \left(\frac{1}{2} \mathbf{h}_1^2 \mathbf{h}_4 + \mathbf{h}_1 \mathbf{h}_2 \mathbf{h}_3 + \frac{1}{6} \mathbf{h}_2^3\right) * \nabla_\mathbf{x}^3 ,$$

$$T_{6,4} = \frac{1}{6} (\mathbf{h}_1 \cdot \nabla_\mathbf{x})^3 \otimes (\mathbf{h}_3 \cdot \nabla_\mathbf{x}) + \frac{1}{4} (\mathbf{h}_1 \cdot \nabla_\mathbf{x})^2 \otimes (\mathbf{h}_2 \cdot \nabla_\mathbf{x})^2$$

$$= \left(\frac{1}{6} \mathbf{h}_1^3 \mathbf{h}_3 + \frac{1}{4} \mathbf{h}_1^2 \mathbf{h}_2^2\right) * \nabla_\mathbf{x}^4 ,$$

$$T_{6,5} = \frac{1}{24} (\mathbf{h}_1 \cdot \nabla_\mathbf{x})^4 \otimes (\mathbf{h}_2 \cdot \nabla_\mathbf{x}) = \frac{1}{24} \mathbf{h}_1^4 \mathbf{h}_2 * \nabla_\mathbf{x}^5 ,$$

$$T_{6,6} = \frac{1}{720} (\mathbf{h}_1 \cdot \nabla_\mathbf{x})^6 = \frac{1}{720} \mathbf{h}_1^6 * \nabla_\mathbf{x}^6 ,$$

$$T_{7,1} = \mathbf{h}_7 \cdot \nabla_\mathbf{x} ,$$

$$T_{7,2} = (\mathbf{h}_1 \cdot \nabla_\mathbf{x}) \otimes (\mathbf{h}_6 \cdot \nabla_\mathbf{x}) + (\mathbf{h}_2 \cdot \nabla_\mathbf{x}) \otimes (\mathbf{h}_5 \cdot \nabla_\mathbf{x}) + (\mathbf{h}_3 \cdot \nabla_\mathbf{x}) \otimes (\mathbf{h}_4 \cdot \nabla_\mathbf{x})$$

$$= (\mathbf{h}_1 \mathbf{h}_6 + \mathbf{h}_2 \mathbf{h}_5 + \mathbf{h}_3 \mathbf{h}_4) * \nabla_\mathbf{x}^2 ,$$

$$T_{7,3} = \frac{1}{2}(\mathbf{h}_1 \cdot \nabla_{\mathbf{x}})^2 \otimes (\mathbf{h}_5 \cdot \nabla_{\mathbf{x}}) + (\mathbf{h}_1 \cdot \nabla_{\mathbf{x}}) \otimes (\mathbf{h}_2 \cdot \nabla_{\mathbf{x}}) \otimes (\mathbf{h}_4 \cdot \nabla_{\mathbf{x}})$$

$$+ \frac{1}{2}(\mathbf{h}_1 \cdot \nabla_{\mathbf{x}}) \otimes (\mathbf{h}_3 \cdot \nabla_{\mathbf{x}})^2 + \frac{1}{2}(\mathbf{h}_2 \cdot \nabla_{\mathbf{x}})^2 \otimes (\mathbf{h}_3 \cdot \nabla_{\mathbf{x}})$$

$$= \left(\frac{1}{2}\mathbf{h}_1^2 \mathbf{h}_5 + \mathbf{h}_1 \mathbf{h}_2 \mathbf{h}_4 + \frac{1}{2}\mathbf{h}_1 \mathbf{h}_3^2 + \frac{1}{2}\mathbf{h}_2^2 \mathbf{h}_3\right) * \nabla_{\mathbf{x}}^3 ,$$

$$T_{7,4} = \frac{1}{6}(\mathbf{h}_1 \cdot \nabla_{\mathbf{x}})^3 \otimes (\mathbf{h}_4 \cdot \nabla_{\mathbf{x}}) + \frac{1}{2}(\mathbf{h}_1 \cdot \nabla_{\mathbf{x}})^2 \otimes (\mathbf{h}_2 \cdot \nabla_{\mathbf{x}}) \otimes (\mathbf{h}_3 \cdot \nabla_{\mathbf{x}}) + \frac{1}{6}(\mathbf{h}_1 \cdot \nabla_{\mathbf{x}}) \otimes (\mathbf{h}_2 \cdot \nabla_{\mathbf{x}})^3$$

$$= \left(\frac{1}{6}\mathbf{h}_1^3 \mathbf{h}_4 + \frac{1}{2}\mathbf{h}_1^2 \mathbf{h}_2 \mathbf{h}_3 + \frac{1}{6}\mathbf{h}_1 \mathbf{h}_2^3\right) * \nabla_{\mathbf{x}}^4 ,$$

$$T_{7,5} = \frac{1}{24}(\mathbf{h}_1 \cdot \nabla_{\mathbf{x}})^4 \otimes (\mathbf{h}_3 \cdot \nabla_{\mathbf{x}}) + \frac{1}{12}(\mathbf{h}_1 \cdot \nabla_{\mathbf{x}})^3 \otimes (\mathbf{h}_2 \cdot \nabla_{\mathbf{x}})^2$$

$$= \left(\frac{1}{24}\mathbf{h}_1^4 \mathbf{h}_3 + \frac{1}{12}\mathbf{h}_1^3 \mathbf{h}_2^2\right) * \nabla_{\mathbf{x}}^5 ,$$

$$T_{7,6} = \frac{1}{120}(\mathbf{h}_1 \cdot \nabla_{\mathbf{x}})^5 \otimes (\mathbf{h}_2 \cdot \nabla_{\mathbf{x}}) = \frac{1}{120}\mathbf{h}_1^5 \mathbf{h}_2 * \nabla_{\mathbf{x}}^6 ,$$

$$T_{7,7} = \frac{1}{5040}(\mathbf{h}_1 \cdot \nabla_{\mathbf{x}})^7 = \frac{1}{5040}\mathbf{h}_1^7 * \nabla_{\mathbf{x}}^7 ,$$

$$T_{8,1} = \mathbf{h}_8 \cdot \nabla_{\mathbf{x}} ,$$

$$T_{8,2} = (\mathbf{h}_1 \cdot \nabla_{\mathbf{x}}) \otimes (\mathbf{h}_7 \cdot \nabla_{\mathbf{x}}) + (\mathbf{h}_2 \cdot \nabla_{\mathbf{x}}) \otimes (\mathbf{h}_6 \cdot \nabla_{\mathbf{x}}) + (\mathbf{h}_3 \cdot \nabla_{\mathbf{x}}) \otimes (\mathbf{h}_5 \cdot \nabla_{\mathbf{x}}) + \frac{1}{2}(\mathbf{h}_4 \cdot \nabla_{\mathbf{x}})^2$$

$$= \left(\mathbf{h}_1 \mathbf{h}_7 + \mathbf{h}_2 \mathbf{h}_6 + \mathbf{h}_3 \mathbf{h}_5 + \frac{1}{2}\mathbf{h}_4^2\right) * \nabla_{\mathbf{x}}^2 ,$$

$$T_{8,3} = \frac{1}{2}(\mathbf{h}_1 \cdot \nabla_{\mathbf{x}})^2 \otimes (\mathbf{h}_6 \cdot \nabla_{\mathbf{x}}) + (\mathbf{h}_1 \cdot \nabla_{\mathbf{x}}) \otimes (\mathbf{h}_2 \cdot \nabla_{\mathbf{x}}) \otimes (\mathbf{h}_5 \cdot \nabla_{\mathbf{x}}) + (\mathbf{h}_1 \cdot \nabla_{\mathbf{x}}) \otimes (\mathbf{h}_3 \cdot \nabla_{\mathbf{x}}) \otimes (\mathbf{h}_4 \cdot \nabla_{\mathbf{x}})$$

$$+ \frac{1}{2}(\mathbf{h}_2 \cdot \nabla_{\mathbf{x}})^2 \otimes (\mathbf{h}_4 \cdot \nabla_{\mathbf{x}}) + \frac{1}{2}(\mathbf{h}_2 \cdot \nabla_{\mathbf{x}}) \otimes (\mathbf{h}_3 \cdot \nabla_{\mathbf{x}})^2$$

$$= \left(\frac{1}{2}\mathbf{h}_1^2 \mathbf{h}_6 + \mathbf{h}_1 \mathbf{h}_2 \mathbf{h}_5 + \mathbf{h}_1 \mathbf{h}_3 \mathbf{h}_4 + \frac{1}{2}\mathbf{h}_2^2 \mathbf{h}_4 + \frac{1}{2}\mathbf{h}_2 \mathbf{h}_3^2\right) * \nabla_{\mathbf{x}}^3 ,$$

$$T_{8,4} = \frac{1}{6}(\mathbf{h}_1 \cdot \nabla_{\mathbf{x}})^3 \otimes (\mathbf{h}_5 \cdot \nabla_{\mathbf{x}}) + \frac{1}{2}(\mathbf{h}_1 \cdot \nabla_{\mathbf{x}})^2 \otimes (\mathbf{h}_2 \cdot \nabla_{\mathbf{x}}) \otimes (\mathbf{h}_4 \cdot \nabla_{\mathbf{x}}) + \frac{1}{4}(\mathbf{h}_1 \cdot \nabla_{\mathbf{x}})^2 \otimes (\mathbf{h}_3 \cdot \nabla_{\mathbf{x}})^2$$

$$+ \frac{1}{2}(\mathbf{h}_1 \cdot \nabla_{\mathbf{x}}) \otimes (\mathbf{h}_2 \cdot \nabla_{\mathbf{x}})^2 \otimes (\mathbf{h}_3 \cdot \nabla_{\mathbf{x}}) + \frac{1}{24}(\mathbf{h}_2 \cdot \nabla_{\mathbf{x}})^4$$

$$= \left(\frac{1}{6} h_1^3 h_5 + \frac{1}{2} h_1^2 h_2 h_4 + \frac{1}{4} h_1^2 h_3^2 + \frac{1}{2} h_1 h_2^2 h_3 + \frac{1}{24} h_2^4\right) * \nabla_x^4 \ ,$$

$$T_{8,5} = \frac{1}{24} (h_1 \cdot \nabla_x)^4 \otimes (h_4 \cdot \nabla_x) + \frac{1}{6} (h_1 \cdot \nabla_x)^3 \otimes (h_2 \cdot \nabla_x) \otimes (h_3 \cdot \nabla_x) + \frac{1}{12} (h_1 \cdot \nabla_x)^2 \otimes (h_2 \cdot \nabla_x)^3$$

$$= \left(\frac{1}{24} h_1^4 h_4 + \frac{1}{6} h_1^3 h_2 h_3 + \frac{1}{12} h_1^2 h_2^3\right) * \nabla_x^5 \ ,$$

$$T_{8,6} = \frac{1}{120} (h_1 \cdot \nabla_x)^5 \otimes (h_3 \cdot \nabla_x) + \frac{1}{48} (h_1 \cdot \nabla_x)^4 \otimes (h_2 \cdot \nabla_x)^2$$

$$= \left(\frac{1}{120} h_1^5 h_3 + \frac{1}{48} h_1^4 h_2^2\right) * \nabla_x^6 \ ,$$

$$T_{8,7} = \frac{1}{720} (h_1 \cdot \nabla_x)^6 \otimes (h_2 \cdot \nabla_x) = \frac{1}{720} h_1^6 h_2 * \nabla_x^7 \ ,$$

$$T_{8,8} = \frac{1}{40320} (h_1 \cdot \nabla_x)^8 = \frac{1}{40320} h_1^8 * \nabla_x^8 \ ,$$

. . . .

The recent work of Sconzo and Valenzuela (1967) on the computation of Faa de Bruno operators on the electronic machines must be mentioned.

In the process of elimination of the short period terms from a Hamiltonian, we shall make use of the following form of the Taylor expansion: if

$$F(x) = \sum_{\alpha=0}^{\infty} F_\alpha(x)$$

and

$$h = \sum_{\alpha=1}^{\infty} h_\alpha(x) \ ,$$

then making use of (12) we obtain:

$$F(x+h) = TF(x) = \sum_{\alpha=0}^{\infty} \sum_{\beta=0}^{\alpha} T_{\alpha-\beta} F_\alpha(x) \ .$$

The following two important relations exist between the operators $T_{j,k}$ and $h_s \cdot \nabla_x$:

$$T_{j,k} = \sum \frac{(h_1 \cdot \nabla_x)^{\alpha_1} \otimes (h_2 \cdot \nabla_x)^{\alpha_2} \otimes \cdots \otimes (h_p \cdot \nabla_x)^{\alpha_p}}{\alpha_1! \ \alpha_2! \cdots \alpha_p!} \ , \qquad (13)$$

where

$$\sum_{s=1}^{p} \alpha_s = k , \qquad \sum_{s=1}^{p} s\,\alpha_s = j ,$$

$$kT_{j,k} = \sum_{\sigma=1}^{j-k+1} \mathbf{h}_\sigma \cdot \nabla_{\mathbf{x}} \otimes T_{j-\sigma,k-1} . \tag{14}$$

The expansion of the Lagrange operator

$$\Lambda = \exp\left(\nabla_{\mathbf{x}} \cdot \mathbf{h}\right)$$

is performed in a manner similar to the expansion of T. We obtain:

$$\Lambda = \sum_{\alpha=0}^{\infty} \Lambda_\alpha$$

and

$$\Lambda_j = \sum_{k=1}^{j} \Lambda_{j,k} ,$$

where $\Lambda_{j,k}$ are the homogeneous polynomials of degree k relative to the operators $\nabla_{\mathbf{x}} \cdot \mathbf{h}_s$. We have, for example:

$$\Lambda_0 = 1 ,$$

$$\Lambda_{1,1} = \nabla_{\mathbf{x}} \cdot \mathbf{h}_1 ,$$

$$\Lambda_{2,1} = \nabla_{\mathbf{x}} \cdot \mathbf{h}_2 ,$$

$$\Lambda_{2,2} = \frac{1}{2}\left(\nabla_{\mathbf{x}} \cdot \mathbf{h}_1\right)^2 = \frac{1}{2} \nabla_{\mathbf{x}}^2 * \mathbf{h}_1^{\,2}$$

$$\Lambda_{3,1} = \nabla_{\mathbf{x}} \cdot \mathbf{h}_3 ,$$

$$\Lambda_{3,2} = \left(\nabla_{\mathbf{x}} \cdot \mathbf{h}_1\right) \otimes \left(\nabla_{\mathbf{x}} \cdot \mathbf{h}_2\right) = \nabla_{\mathbf{x}}^2 * \mathbf{h}_1\,\mathbf{h}_2 ,$$

$$\Lambda_{3,3} = \frac{1}{6}\left(\nabla_{\mathbf{x}} \cdot \mathbf{h}_1\right)^3 = \frac{1}{6} \nabla_{\mathbf{x}}^3 * \mathbf{h}_1^{\,3} ,$$

$$\Lambda_{4,1} = \nabla_{\mathbf{x}} \cdot \mathbf{h}_4 \ ,$$

$$\Lambda_{4,2} = \left(\nabla_{\mathbf{x}} \cdot \mathbf{h}_1\right) \otimes \left(\mathbf{h}_3 \cdot \nabla_{\mathbf{x}}\right) + \frac{1}{2} \left(\nabla_{\mathbf{x}} \cdot \mathbf{h}_2\right)^2$$

$$= \nabla_{\mathbf{x}}^2 * \left(\mathbf{h}_1\, \mathbf{h}_3 + \frac{1}{2}\, \mathbf{h}_2^{\,2}\right) .$$

. . . .

The relations (13) and (14) exist also for $\Lambda_{j,k}$ operators. Let us assume that x and y, defined on the n-dimensional manifold M, are related by the equation

$$\mathbf{x} = \mathbf{y} - \mathbf{h}(\mathbf{y}) \ , \tag{15}$$

where $\mathbf{h}(\mathbf{y})$ is analytic on M. We assume that the transformation (15) is continuous on M and also non-singular, so that the representation of y as an analytic function of x is possible. Let $F(\mathbf{y})$ be a given analytic function of y, defined in M. Under this assumption a generalized Lagrange expansion is valid (Stieltjes, 1885), (Good, 1960), (Dederick and Chu, 1964):

$$F(\mathbf{y}) = \Lambda\left\{\mathbf{h}(\mathbf{x})\right\} \left\{J\left[\mathbf{h}(\mathbf{x})\right] F(\mathbf{x})\right\} \ , \tag{16}$$

where

$$\Lambda\left\{\mathbf{h}(\mathbf{x})\right\} = \exp\left\{\nabla_{\mathbf{x}} \cdot \mathbf{h}(\mathbf{x})\right\}$$

is the Lagrange operator and

$$J\left[\mathbf{h}(\mathbf{x})\right] = \det\left[I - \nabla_{\mathbf{x}}\, \mathbf{h}(\mathbf{x})\right] .$$

Taking (16) in the expanded form we have (Dederick and Chu, 1964)

$$F(\mathbf{y}) = \sum_{\alpha=0}^{\infty} \frac{1}{\alpha!}\, \nabla_{\mathbf{x}}^{\alpha} * \left\{\left[\mathbf{h}(\mathbf{x})\right]^{\alpha} J\left[\mathbf{h}(\mathbf{x})\right] F(\mathbf{x})\right\} .$$

In particular we have

$$\mathbf{h}(\mathbf{y}) = \sum_{\alpha=0}^{\infty} \frac{1}{\alpha!}\, \nabla_{\mathbf{x}}^{\alpha} * \left\{\left[\mathbf{h}(\mathbf{x})\right]^{\alpha+1} J\left[\mathbf{h}(\mathbf{x})\right]\right\}$$

and the inversion of (15)

$$\mathbf{y} = \mathbf{x} + \sum_{\alpha=0}^{\infty} \frac{1}{\alpha!} \nabla_{\mathbf{x}}^{\alpha} * \left\{ \left[\mathbf{h}(\mathbf{x})\right]^{\alpha+1} J\left[\mathbf{h}(\mathbf{x})\right] \right\}$$

$$= \mathbf{x} + \Lambda\left[\mathbf{h}(\mathbf{x})\right] J\left[\mathbf{h}(\mathbf{x})\right] \mathbf{h}(\mathbf{x}) \ .$$

An essential part in the application of Lagrange formula is the expansion of Jacobians of the form

$$J = \det\left\{I - \nabla_{\mathbf{x}} \mathbf{h}(\mathbf{x})\right\}$$

in powers of a small parameter.

The expansion of determinants of the type

$$\det (I - A)$$

for the n-dimensional matrix A with the sufficiently small norm can be accomplished by the method of Leverrier (1840), based on the representation of the determinant of $I - A$ in the form:

$$\det (I - A) = \exp \operatorname{tr} \log (I - A) = \exp\left(- \sum_{k=1}^{\infty} \frac{\chi_k}{k}\right) ,$$

where

$$\chi_k = \operatorname{tr}\left(A^k\right) \ .$$

Evidently the expansion of $\det (I - A)$ will be of the form

$$\det (I - A) = \sum_{k=0}^{n} (-1)^k G_k \ , \tag{17}$$

where

$$(-1)^k G_k = T_k\left(-\frac{\chi_1}{1}, -\frac{\chi_2}{2}, \dots -\frac{\chi_k}{k}\right) = \sum \frac{(-1)^{\lambda_1+\lambda_2+\cdots+\lambda_s}}{\left(1^{\lambda_1} 2^{\lambda_2} \cdots s^{\lambda_s}\right)\left(\lambda_1! \, \lambda_2! \cdots \lambda_s!\right)} \chi_1^{\lambda_1} \chi_2^{\lambda_2} \cdots \chi_s^{\lambda_s} \ ,$$

$$G_0 = 1 \tag{18}$$

and where the polynomials $(-1)^k G_k$ are obtained from T_k simply by replacing each $\mathbf{h}_j \cdot \nabla$ by $-(\chi_j/j)$. The polynomials G_k in Higher Algebra are associated with the name of Warring.

In the explicit form we have

$$\det(I-A) = 1 - \chi_1 + \frac{1}{2}\left(\chi_1^2 - \chi_2\right) - \frac{1}{6}\left(\chi_1^3 - 3\chi_1\chi_2 + 2\chi_3\right) + \frac{1}{24}\left(\chi_1^4 - 6\chi_1^2\chi_2 + 8\chi_1\chi_3 + 3\chi_2^2 - 6\chi_4\right)$$

$$- \frac{1}{120}\left(\chi_1^5 - 10\chi_1^3\chi_2 + 20\chi_1^2\chi_3 + 15\chi_1\chi_2^2 - 30\chi_1\chi_4 - 20\chi_2\chi_3 + 24\chi_5\right) + \cdots .$$

If A is expanded into a power series relative to a small parameter

$$A = \sum_{j=1}^{\infty} A_j ,$$

then the procedure should be slightly changed. We have

$$\log\left(1 - \sum_{j=1}^{\infty} x_j\right) = \sum_{k=1}^{\infty} \frac{c_k}{k} = -x_1 - \left(x_2 + \frac{1}{2}x_1^2\right)$$

$$- \left(x_3 + x_1 x_2 + \frac{1}{3}x_1^3\right) - \left(x_4 + x_1 x_3 + \frac{1}{2}x_2^2 + x_1^2 x_2 + \frac{1}{4}x_1^4\right) - \cdots . \qquad (19)$$

Taking into account that $\mathrm{tr}(MN) = \mathrm{tr}(NM)$, we conclude that in order to obtain the expansion of $\mathrm{tr}\log(I-A)$ we simply replace in the last equation each product of the form

$$x_{k_1}^{j_1} x_{k_2}^{j_2} \cdots x_{k_s}^{j_s}$$

by

$$\mathrm{tr}\left(A_{k_1}^{j_1} A_{k_2}^{j_2} \cdots A_{k_s}^{j_s}\right) .$$

Thus we obtain

$$\det(I-A) = \exp\left(\psi_1 + \psi_2 + \psi_3 + \cdots\right)$$

$$= \sum_{k=0}^{+\infty} T_k\left(\psi_1, \psi_2, \cdots, \psi_k\right) , \qquad (20)$$

where we set

$$\psi_1 = -\operatorname{tr} A_1 ,$$

$$\psi_2 = -\operatorname{tr}\left(A_2 + \frac{1}{2} A_1^2\right) ,$$

$$\psi_3 = -\operatorname{tr}\left(A_3 + A_1 A_2 + \frac{1}{3} A_1^3\right) ,$$

$$\psi_4 = -\operatorname{tr}\left(A_4 + A_1 A_3 + \frac{1}{2} A_2^2 + A_1^2 A_2 + \frac{1}{4} A_1^4\right) . \qquad (21)$$

. . . .

If necessary, the extension of expansion (19) and the table of traces given above can be easily accomplished by means of the recurrence formula

$$c_{n+1} = \sum_{k=1}^{n} c_{n-k+1} x_k - (n+1) x_{n+1} .$$

In the theory of the elimination of short period terms from a Hamiltonian, all matrices A_j are trigonometric series and the diagonalization of $I - A$ cannot be done easily. For this reason the Leverrier method, because it requires only the multiplication of matrices and the formation of traces, appears to be a very convenient way to obtain expansion (20).

If $\mathbf{h}(\mathbf{x})$ is expanded in powers of a small parameter,

$$\mathbf{h}(\mathbf{x}) = \sum_{\alpha=1}^{\infty} \mathbf{h}_\alpha (\mathbf{x}) ,$$

then setting

$$A_k = \nabla_{\mathbf{x}} \cdot \mathbf{h}_k (\mathbf{x}) ,$$

in (21) we deduce the expansion of $J\{\mathbf{h}(\mathbf{x})\}$

$$J = \sum_{\alpha=0}^{\infty} J_\alpha ,$$

where J_α ($\alpha = 0, 1, 2, \cdots$) are obtained from the formulas for the T_α-operators by replacing

$$\mathbf{h}_1 \cdot \nabla_{\mathbf{x}} , \qquad \mathbf{h}_2 \cdot \nabla_{\mathbf{x}} , \quad \cdots$$

by ψ_1, ψ_2, $\cdots$, respectively. If $F(y)$ is also given as an expansion in powers of the small parameter,

$$F(y) = \sum_{\alpha=0}^{\infty} F_\alpha (y) ,$$

then the expansion of $F(y)$ and in powers of the same small parameter has the form

$$F(y) = \Lambda\{h(x)\} \{J [h(x)] F(x)\}$$

$$= \sum_{\alpha=0}^{\infty} \sum_{\beta=0}^{\alpha} \sum_{\gamma=0}^{\beta} \Lambda_{\alpha-\beta} J_{\beta-\gamma} F_\gamma (x) . \tag{22}$$

In particular

$$y = x + \sum_{\alpha=0}^{\infty} \sum_{\beta=0}^{\alpha} \sum_{\gamma=0}^{\beta} \Lambda_{\alpha-\beta} J_{\beta-\gamma} h_\gamma (x) . \tag{23}$$

ON THE METHOD OF VON ZEIPEL

In developing von Zeipel's method of elimination of the short period terms, we follow the work briefly described previously by the author (Musen, 1965).

Let us consider the system of canonical equations

$$\frac{dx}{dt} = + \nabla_y F , \qquad \frac{dy}{dt} = - \nabla_x F ,$$

$$\frac{d\xi}{dt} = + \nabla_\eta F , \qquad \frac{d\eta}{dt} = - \nabla_\xi F , \tag{24}$$

where the vector x has the same number of components as y, and ξ has the same number of components as η.

We assume the Hamiltonian F to have the form

$$F(x, \xi; y, \eta) = F_0 (x) + F_1(x, \xi; y, \eta) + F_2 (x, \xi; y, \eta) + F_3 (x, \xi; y, \eta) + \cdots , \tag{25}$$

where

$$F_j\ (x,\ \xi;\ y,\ \eta) = \sum_{m,\mu} A_{m,\mu}^{(j)}\ (x,\ \xi) \exp i(m \cdot y + \mu \cdot \eta)\ , \tag{26}$$

and m and μ are the vectors whose components are integers.

The terms in (25) or (26) are classified:

as short periodic if $m \neq 0$,

as long periodic if $m = 0$, but $\mu \neq 0$

as purely secular if $m = 0$ and $\mu = 0$.

The basic idea of the Poincaré (1892) and von Zeipel (1916) methods of elimination of the short period terms from (25) consists of finding such a canonical transformation

$$(x,\ y;\ \xi,\ \eta) \to (x^*,\ y^*;\ \xi^*,\ \eta^*)\ ,$$

$$x = x^* + \nabla_y S(x^*,\ \xi^*;\ y,\ \eta)\ , \tag{27}$$

$$y^* = y + \nabla_{x^*} S(x^*,\ \xi^*;\ y,\ \eta)\ , \tag{28}$$

$$\xi = \xi^* + \nabla_\eta S(x^*,\ \xi^*;\ y,\ \eta)\ , \tag{29}$$

$$\eta^* = \eta + \nabla_{\xi^*} S(x^*,\ \xi^*;\ y,\ \eta)\ , \tag{30}$$

where S is of the order of perturbations, such that no new short period argument y^* is present in the transformed Hamiltonian F^*. Thus

$$F(x,\ \xi;\ y,\ \eta) = F^*(x^*,\ \xi^*;\ -,\ \eta^*)\ .$$

Substituting (27) through (30) into the last equation we obtain

$$F\{x^* + \nabla_y S(x^*,\ \xi^*;\ y,\ \eta)\ ,\ \xi^* + \nabla_\eta S(x^*,\ \xi^*;\ y,\ \eta)\ ;\ y,\ \eta\} = F^*\{x^*,\ \xi^*;\ -,\ \eta + \nabla_{\xi^*} S(x^*,\ \xi^*;\ y,\ \eta)\}\ , \tag{31}$$

which serves for the determination of S and the form of F^*. In the process of solving (31), the notation does not play any essential role and thus the asterisks can be omitted. We write (31) in the form

$$F\{x + \nabla_y S(x,\ \xi;\ y,\ \eta)\ ,\ \xi + \nabla_\eta S(x,\ \xi;\ y,\ \eta)\ ;\ y,\ \eta\} = F^*\{x,\ \xi;\ -,\ \eta + \nabla_\xi S(x,\ \xi;\ y,\ \eta)\}\ . \tag{32}$$

In this respect we follow in von Zeipel's footsteps.

At this point, for the sake of brevity, it is convenient to combine the subspaces of x and ξ, and of y and η, into two full spaces and to introduce the combined vectors

$$\mathbf{u} = \begin{pmatrix} x \\ \xi \end{pmatrix}, \qquad \mathbf{w} = \begin{pmatrix} y \\ \eta \end{pmatrix}.$$

The corresponding gradient operators are

$$\nabla_{\mathbf{u}} = \begin{pmatrix} \nabla_{\mathbf{x}} \\ \nabla_{\xi} \end{pmatrix}, \qquad \nabla_{\mathbf{w}} = \begin{pmatrix} \nabla_{\mathbf{y}} \\ \nabla_{\eta} \end{pmatrix}.$$

Then the last equation can be re-written in the form

$$\mathrm{TF}(\mathbf{u}, \mathbf{w}) = \mathrm{T}^* \mathrm{F}^* (\mathbf{x}, \xi, -, \eta) , \tag{33}$$

where we set

$$\mathrm{T} = \exp\left(\nabla_{\mathbf{w}} \mathrm{S} \cdot \nabla_{\mathbf{u}}\right) = \exp\left(\nabla_{\mathbf{y}} \mathrm{S} \cdot \nabla_{\mathbf{x}} + \nabla_{\eta} \mathrm{S} \cdot \nabla_{\xi}\right) , \tag{34}$$

and

$$\mathrm{T}^* = \exp\left(\nabla_{\xi} \mathrm{S} \cdot \nabla_{\eta}\right) . \tag{35}$$

Assuming the expansions

$$\mathrm{S} = \mathrm{S}_1 + \mathrm{S}_2 + \mathrm{S}_3 + \cdots , \tag{36}$$

$$\mathrm{F}^* = \mathrm{F}_0^* + \mathrm{F}_1^* + \mathrm{F}_2^* + \cdots , \tag{37}$$

in powers of a small parameter we have

$$\mathrm{T} = \exp \sum_{j=1}^{\infty} \nabla_{\mathbf{w}} \mathrm{S}_j \cdot \nabla_{\mathbf{u}} = \sum_{j=0}^{\infty} \mathrm{T}_j , \tag{38}$$

$$\mathrm{T}^* = \exp \sum_{j=1}^{\infty} \nabla_{\xi} \mathrm{S}_j \cdot \nabla_{\eta} = \sum_{j=0}^{\infty} \mathrm{T}_j^* , \tag{39}$$

It follows from (33) and (37) through (39):

$$\sum_{j=0}^{\infty} \sum_{k=0}^{j} \mathrm{T}_{j-k} \mathrm{F}_k = \sum_{j=0}^{\infty} \sum_{k=0}^{j} \mathrm{T}_{j-k}^* \mathrm{F}_k^* \tag{40}$$

and thus

$$\sum_{k=0}^{j} T_{j-k} F_k = \sum_{k=0}^{j} T^*_{j-k} F^*_k \tag{41}$$

Making use of the formulas for the expansion of the T-operators we have

$$T_0 = 1 ,$$

$$T_1 = \nabla_w S_1 \cdot \nabla_{\mathbf{u}} ,$$

$$T_2 = \nabla_w S_2 \cdot \nabla_{\mathbf{u}} + \frac{1}{2} (\nabla_w S_1 \cdot \nabla_{\mathbf{u}})^2$$

$$= \nabla_w S_2 * \nabla_{\mathbf{u}} + \frac{1}{2} (\nabla_w S_1)^2 * \nabla_{\mathbf{u}}^2 ,$$

$$T_3 = \nabla_w S_3 \cdot \nabla_{\mathbf{u}} + (\nabla_w S_1 \cdot \nabla_{\mathbf{u}})(\nabla_w S_2 \cdot \nabla_{\mathbf{u}}) + \frac{1}{6} (\nabla_w S_1 \cdot \nabla_{\mathbf{u}})^3$$

$$= \nabla_w S_3 * \nabla_{\mathbf{u}} + \nabla_w S_1 \nabla_w S_2 * \nabla_{\mathbf{u}}^2 + \frac{1}{6} (\nabla_w S_1)^3 * \nabla_{\mathbf{u}}^3 ,$$

$$T_4 = \nabla_w S_4 \cdot \nabla_{\mathbf{u}} + (\nabla_w S_1 \cdot \nabla_{\mathbf{u}})(\nabla_w S_3 \cdot \nabla_{\mathbf{u}}) + \frac{1}{2} (\nabla_w S_2 \cdot \nabla_{\mathbf{u}})^2 + \frac{1}{2} (\nabla_w S_1 \cdot \nabla_{\mathbf{u}})^2 (\nabla_w S_2 \cdot \nabla_{\mathbf{u}}) + \frac{1}{24} (\nabla_w S_1 \cdot \nabla_{\mathbf{u}})^4$$

$$= \nabla_w S_4 * \nabla_{\mathbf{u}} + \nabla_w S_1 \nabla_w S_3 * \nabla_{\mathbf{u}}^2 + \frac{1}{2} (\nabla_w S_2)^2 * \nabla_{\mathbf{u}}^2 + \frac{1}{2} (\nabla_w S_1)^2 \nabla_w S_2 * \nabla_{\mathbf{u}}^3 + \frac{1}{24} (\nabla_w S_1)^4 * \nabla_{\mathbf{u}}^4 ,$$

. . . .

In the expansions of the T_j-operators, the expressions $\nabla_w S_k$ ($k = 1, 2, \cdots$) are not affected by the operator $\nabla_{\mathbf{u}}$. In a similar manner we obtain

$$T_0^* = 1 ,$$

$$T_1^* = \nabla_\xi S_1 \cdot \nabla_\eta ,$$

$$T_2^* = \nabla_\xi S_2 \cdot \nabla_\eta + \frac{1}{2} (\nabla_\xi S_1 \cdot \nabla_\eta)^2$$

$$= \nabla_\xi S_2 * \nabla_\eta + \frac{1}{2} (\nabla_\xi S_1)^2 * \nabla_\eta^2 ,$$

$$T_3^* = \nabla_\xi S_3 \cdot \nabla_\eta + (\nabla_\xi S_1 \cdot \nabla_\eta)(\nabla_\xi S_2 \cdot \nabla_\eta) + \frac{1}{6} (\nabla_\xi S_1 \cdot \nabla_\eta)^3$$

$$= \nabla_\xi S_3 * \nabla_\eta + \nabla_\xi S_1 \nabla_\xi S_2 * \nabla_\eta^2 + \frac{1}{6} (\nabla_\xi S_1)^3 * \nabla_\eta^3 ,$$

$$T_4^* = \nabla_\xi S_4 \cdot \nabla_\eta + (\nabla_\xi S_1 \cdot \nabla_\eta)(\nabla_\xi S_3 \cdot \nabla_\eta) + \frac{1}{2}(\nabla_\xi S_2 \cdot \nabla_\eta)^2 + \frac{1}{2}(\nabla_\xi S_1 \cdot \nabla_\eta)^2 (\nabla_\xi S_2 \cdot \nabla_\eta) + \frac{1}{24}(\nabla_\xi S_1 \cdot \nabla_\eta)^4$$

$$= \nabla_\xi S_4 * \nabla_\eta + (\nabla_\xi S_1 \nabla_\xi S_3) * \nabla_\eta^2 + \frac{1}{2}(\nabla_\xi S_2)^2 * \nabla_\eta^2 + \frac{1}{2}(\nabla_\xi S_1)^2 \nabla_\xi S_2 * \nabla_\eta^3 + \frac{1}{24}(\nabla_\xi S_1)^4 * \nabla_\eta^4 \quad .$$

. . . .

Here, $\nabla_\xi S_k$ ($k = 1, 2, \cdots$) are not affected by the operator ∇_η. The continuation of these tables does not present any difficulty.

Let us assume that x lies in a given domain D and let

$$m_1, m_2, m_3, \cdots$$

be a sequence of vectors where components are integers. If all the numbers

$$m_j \cdot \nabla_x F_0(x) \,, \qquad x \in D$$

can be considered small, then all the arguments of the form

$$m_j \cdot y + \mu_k \cdot \eta_k \,,$$

for

$$j = 1, 2, 3, \cdots$$

and any μ_k, are called the critical ones. In planetary problems they are associated with the commensurability of the mean motions of celestial bodies. The near commensurability conditions produce the small divisors in the process of integration, and thus solving the problem in terms of trigonometric series with the arguments of the form $m \cdot y + \mu \cdot \eta$ can become impossible. We can keep in the expansion only a finite number of the critical arguments and, normally, we do not keep more than one value of m_j, say m_1.

We introduce now the averaging operator M which performs the extraction from a given function of the purely secular, long periodic and the critical terms. In addition to operator M, it is convenient to introduce the operator P, which extracts only the short period terms from a given function. Thus, if a function Φ has the form

$$\Phi = \sum_{m,\mu} \Phi_{m,\mu}(x, \xi) \exp i(m \cdot y + \mu \cdot \eta) \,,$$

then

$$M\Phi = \sum_{\mu} \Phi_{0,\mu} \exp i \mu \cdot \eta + \sum_{j,\mu} \Phi_{m_j,\mu} \exp i \left(m_j \cdot y + \mu \cdot \eta\right) . \tag{42}$$

We make use of the operator P, which leaves in the functions of the form (40) the short period terms only, and we obtain

$$P\Phi = \sum_{m,\mu} \Phi_{m,\mu}(x, \xi) \exp i (m \cdot y + \mu \cdot \eta) , \tag{42'}$$

where m satisfies the conditions

$$m \neq 0 \qquad m \neq m_j \ (j = 1, 2, 3, \cdots) .$$

From (41), and using the representation of T-operators in terms of the partial derivatives of $S_1, S_2, \cdots$, we deduce:

$$F_0 = F_0^*$$

and

$$\nabla_\eta F_0 = 0 , \qquad T_j^* F_0 = 0 \quad (j = 1, 2, 3, \cdots) ,$$

and consequently,

$$\mathbf{n} \cdot \nabla_y S_1 = F_1 - F_1^* ,$$

$$\mathbf{n} \cdot \nabla_y S_2 = \frac{1}{2} (\nabla_y S_1 \cdot \nabla_x)^2 F_0 + \left(T_1 F_1 - T_1^* F_1^*\right) + \left(F_2 - F_2^*\right) ,$$

$$\mathbf{n} \cdot \nabla_y S_3 = \left[(\nabla_y S_1 \cdot \nabla_x)(\nabla_y S_2 \cdot \nabla_x) + \frac{1}{6} (\nabla_y S_1 \cdot \nabla_x)^3\right] F_0$$
$$+ \left(T_2 F_1 - T_2^* F_1^*\right) + \left(T_1 F_2 - T_1^* F_2^*\right) + \left(F_3 - F_3^*\right) ,$$

$$\mathbf{n} \cdot \nabla_y S_4 = \left[(\nabla_y S_1 \cdot \nabla_x)(\nabla_y S_3 \cdot \nabla_x) + \frac{1}{2} (\nabla_y S_2 \cdot \nabla_x)^2 + \frac{1}{2} (\nabla_y S_1 \cdot \nabla_x)^2 (\nabla_y S_2 \cdot \nabla_x) + \frac{1}{24} (\nabla_y S_1 \cdot \nabla_x)^4\right] F_0$$
$$+ \left(T_3 F_1 - T_3^* F_1^*\right) + \left(T_2 F_2 - T_2^* F_2^*\right) + \left(T_1 F_3 - T_1^* F_3^*\right) + \left(F_4 - F_4^*\right) ,$$

. . . .

where we set

$$\mathbf{n} = - \nabla_x F_0 .$$

The functions $S_1, S_2, \cdots$ must contain the short period terms only, but $F_1^*, F_2^*, \cdots$ must contain only the purely secular, the long periodic and the critical terms. Thus we can set

$$F_1^* = M F_1 ,$$

$$n \cdot \nabla_y S_1 = P F_1 ,$$

$$F_2^* = M\left\{\frac{1}{2} (\nabla_y S_1 \cdot \nabla_x)^2 F_0 + T_1 F_1 + F_2 - T_1^* F_1^*\right\} ,$$

$$n \cdot \nabla_y S_2 = P\left\{\frac{1}{2} (\nabla_y S_1 \cdot \nabla_x)^2 F_0 + T_1 F_1 + F_2\right\} ,$$

$$F_3^* = M\Big\{\Big[(\nabla_y S_1 \cdot \nabla_x)(\nabla_y S_2 \cdot \nabla_x) + \frac{1}{6} (\nabla_y S_1 \cdot \nabla_x)^3\Big] F_0$$

$$+ T_2 F_1 + T_1 F_2 + F_3 - \left(T_2^* F_1^* + T_1^* F_2^*\right)\Big\} ,$$

$$n \cdot \nabla_y S_3 = P\Big\{\Big[(\nabla_y S_1 \cdot \nabla_x)(\nabla_y S_2 \cdot \nabla_x) + \frac{1}{6} (\nabla_y S_1 \cdot \nabla_x)^3\Big] F_0 + \left(T_2 F_1 + T_1 F_2 + F_3\right)\Big\} ,$$

$$F_4^* = M\Big\{\Big[(\nabla_y S_1 \cdot \nabla_x)(\nabla_y S_3 \cdot \nabla_x) + \frac{1}{2} (\nabla_y S_2 \cdot \nabla_x)^2 + \frac{1}{2} (\nabla_y S_1 \cdot \nabla_x)^2 (\nabla_y S_2 \cdot \nabla_x) + \frac{1}{24} (\nabla_y S_1 \cdot \nabla_x)^4\Big] F_0$$

$$+ \left(T_3 F_1 + T_2 F_2 + T_1 F_3 + F_4\right) - \left(T_3^* F_1^* + T_2^* F_2^* + T_1^* F_3^*\right)\Big\} ,$$

$$n \cdot \nabla_y S_4 = P\Big\{\Big[(\nabla_y S_1 \cdot \nabla_x)(\nabla_y S_3 \cdot \nabla_x) + \frac{1}{2} (\nabla_y S_2 \cdot \nabla_x)^2 + \frac{1}{2} (\nabla_y S_1 \cdot \nabla_x)^2 (\nabla_y S_2 \cdot \nabla_x) + \frac{1}{24} (\nabla_y S_1 \cdot \nabla_x)^4\Big] F_0$$

$$+ \left(T_3 F_1 + T_2 F_2 + T_1 F_3\right) + F_4\Big\} ,$$

. . . .

where we took into consideration that the product of P and M is a zero-operator

$$PM \equiv MP \equiv 0$$

and that

$$M^2 = M .$$

The problem of integration of the linear partial differential equations determining $S_1, S_2, \cdots$ evidently is reducible to the integration of the set of differential equations of the type

$$\mathbf{n} \cdot \nabla_{\mathbf{y}} \phi = A \exp i (\mathbf{m} \cdot \mathbf{y} + \mu \cdot \eta) ,$$

then in this particular case

$$\phi = \frac{A}{i\, \mathbf{m} \cdot \mathbf{n}} \exp i (\mathbf{m} \cdot \mathbf{y} + \mu \cdot \eta) .$$

Returning to Equations (27) through (30), which can be written in the form

$$\begin{aligned} \mathbf{w}^* &= \mathbf{w} + \nabla_{\mathbf{u}^*} S(\mathbf{u}^*, \mathbf{w}) , \\ \mathbf{u} &= \mathbf{u}^* + \nabla_{\mathbf{w}} S(\mathbf{u}^*, \mathbf{w}) , \end{aligned} \tag{43}$$

we can determine $\mathbf{u}$, $\mathbf{w}$ in terms of $\mathbf{u}^*$, $\mathbf{w}^*$; as the final result we have

$$\mathbf{w} = \mathbf{w}^* - \Lambda\{\mathbf{w}^*, -\nabla_{\mathbf{u}^*} S(\mathbf{u}^*, \mathbf{w}^*)\} \, J\{-\nabla_{\mathbf{u}^*} S(\mathbf{u}^*, \mathbf{w}^*)\} \, \nabla_{\mathbf{u}^*} S(\mathbf{u}^*, \mathbf{w}^*)$$

and

$$\mathbf{u} = \mathbf{u}^* + \Lambda\{\mathbf{w}^*, -\nabla_{\mathbf{u}^*} S(\mathbf{u}^*, \mathbf{w}^*)\} \, J\{-\nabla_{\mathbf{u}^*} S(\mathbf{u}^*, \mathbf{w}^*)\} \, \nabla_{\mathbf{w}^*} S(\mathbf{u}^*, \mathbf{w}^*) ,$$

where we are in agreement with the previous notations we set

$$\Lambda\{\mathbf{w}^*; -\nabla_{\mathbf{u}^*} S(\mathbf{u}^*, \mathbf{w}^*)\} = \exp\{-\nabla_{\mathbf{w}^*} \cdot \nabla_{\mathbf{u}^*} S(\mathbf{u}^*, \mathbf{w}^*)\} ,$$

and

$$J\{-\nabla_{\mathbf{u}^*} S(\mathbf{u}^*, \mathbf{w}^*)\} = \det \{I + \nabla_{\mathbf{w}^*} \nabla_{\mathbf{u}^*} S(\mathbf{u}^*, \mathbf{w}^*)\} .$$

The Lagrange operator Λ can also be expanded into a power series with respect to the small parameter. Using the system of formulas developed in the previous chapter we obtain:

$$\Lambda\{\mathbf{w}^*, -\nabla_{\mathbf{u}^*} S\} = \sum_{j=0}^{\infty} \Lambda_j ,$$

where

$$\Lambda_0 = 1 \ ,$$

$$\Lambda_1 = -\nabla_{\mathbf{u}*} \cdot \nabla_{\mathbf{w}*} S_1 \ ,$$

$$\Lambda_2 = -\nabla_{\mathbf{u}*} \cdot \nabla_{\mathbf{w}*} S_2 + \frac{1}{2} \left(\nabla_{\mathbf{u}*} \cdot \nabla_{\mathbf{w}*} S_1\right)^2$$

$$= -\nabla_{\mathbf{u}*} \cdot \nabla_{\mathbf{w}*} S_2 + \frac{1}{2} \left(\nabla_{\mathbf{w}*} S_1\right)^2 * \nabla_{\mathbf{u}*}^2 \ ,$$

$$\Lambda_3 = -\nabla_{\mathbf{u}*} \cdot \nabla_{\mathbf{w}*} S_3 + \left(\nabla_{\mathbf{u}*} \cdot \nabla_{\mathbf{w}*} S_1\right)\left(\nabla_{\mathbf{u}*} \cdot \nabla_{\mathbf{w}*} S_2\right) - \frac{1}{6} \left(\nabla_{\mathbf{u}*} \cdot \nabla_{\mathbf{w}*} S_1\right)^3$$

$$= -\nabla_{\mathbf{u}*} \cdot \nabla_{\mathbf{w}*} S_3 + \nabla_{\mathbf{u}*}^2 * \left(\nabla_{\mathbf{w}*} S_1 \nabla_{\mathbf{w}*} S_2\right) - \frac{1}{6} \nabla_{\mathbf{u}*}^3 * \left(\nabla_{\mathbf{w}*} S_1\right)^3 \ ,$$

$$\Lambda_4 = -\nabla_{\mathbf{u}*} \cdot \nabla_{\mathbf{w}*} S_4 + \left(\nabla_{\mathbf{u}*} \cdot \nabla_{\mathbf{w}*} S_1\right)\left(\nabla_{\mathbf{u}*} \cdot \nabla_{\mathbf{w}*} S_3\right)$$

$$+ \frac{1}{2} \left(\nabla_{\mathbf{u}*} \cdot \nabla_{\mathbf{w}*} S_2\right)^2 - \frac{1}{2} \left(\nabla_{\mathbf{u}*} \cdot \nabla_{\mathbf{w}*} S_1\right)^2 \left(\nabla_{\mathbf{u}*} \cdot \nabla_{\mathbf{w}*} S_2\right) + \frac{1}{24} \left(\nabla_{\mathbf{u}*} \cdot \nabla_{\mathbf{w}*} S_1\right)^4$$

$$= -\nabla_{\mathbf{u}*} * \nabla_{\mathbf{w}*} S_4 + \nabla_{\mathbf{u}*}^2 * \nabla_{\mathbf{w}*} S_1 \nabla_{\mathbf{w}*} S_3$$

$$+ \frac{1}{2} \nabla_{\mathbf{u}*}^2 * \left(\nabla_{\mathbf{w}*} S_2\right)^2 - \frac{1}{2} \nabla_{\mathbf{u}*}^3 * \left(\nabla_{\mathbf{w}*} S_1\right)^2 \left(\nabla_{\mathbf{w}*} S_2\right) + \frac{1}{24} \nabla_{\mathbf{u}*}^4 * \left(\nabla_{\mathbf{w}*} S_1\right)^4 \ ,$$

. . . .

and where

$$S_j = S_j\,(\mathbf{u}*, \mathbf{w}*) \ , \qquad (j = 1, 2, 3, \cdots) \ .$$

The expansion of $J\{-\nabla_{\mathbf{u}*} S\}$ can be obtained using (20) and (21). Setting in these equations

$$A_j = -\nabla_{\mathbf{w}*} \nabla_{\mathbf{u}*} S_j \ ,$$

$$\mathrm{tr}\left\{\left(\nabla_{\mathbf{w}*} \nabla_{\mathbf{u}*} S_{k_1}\right)^{j_1}\left(\nabla_{\mathbf{w}*} \nabla_{\mathbf{u}*} S_{k_2}\right)^{j_2} \cdots \left(\nabla_{\mathbf{w}*} \nabla_{\mathbf{u}*} S_{k_s}\right)^{j_3}\right\} = \chi_{k_1, k_2, \dots, k_s}^{(j_1, j_2, \cdots, j_s)}$$

$$\psi_1 = \chi_1^{(1)} \ ,$$

$$\psi_2 = \chi_2^{(1)} - \frac{1}{2} \chi_1^{(2)} \ ,$$

$$\psi_3 = \chi_3^{(1)} - \chi_{1,2}^{(1,1)} + \frac{1}{3}\chi_1^{(3)} ,$$

$$\psi_4 = \chi_4^{(1)} - \chi_{1,3}^{(1,1)} - \frac{1}{2}\chi_2^{(2)} + \chi_{1,2}^{(2,1)} - \frac{1}{4}\chi_1^{(4)} ,$$

. . . .

we obtain the following expansion in powers of a small parameter

$$J\{-\nabla_{u*} S\} = 1 + \psi_1 + \left(\psi_2 + \frac{1}{2}\psi_1^2\right) + \left(\psi_3 + \psi_1\psi_2 + \frac{1}{6}\psi_1^3\right)$$

$$+ \left(\psi_4 + \psi_1\psi_3 + \frac{1}{2}\psi_2^2 + \frac{1}{2}\psi_1^2\psi_2 + \frac{1}{24}\psi_1^4\right) + \cdots = J_0 + J_1 + J_2 + \cdots ,$$

and the expansion of the original variables u and w in powers of the small parameter and in terms of the variables u^*, w^* in accordance with (23) becomes

$$w = w^* - \sum_{\alpha=0}^{\infty}\sum_{\beta=0}^{\alpha}\sum_{\gamma=0}^{\beta} \Lambda_{\alpha-\beta} J_{\beta-\gamma} \nabla_{u*} S_\gamma (u^*, w^*) ,$$

$$u = u^* + \sum_{\alpha=0}^{\infty}\sum_{\beta=0}^{\alpha}\sum_{\gamma=0}^{\beta} \Lambda_{\alpha-\beta} J_{\beta-\gamma} \nabla_{w*} S_\gamma (u^*, w^*) .$$

THE METHOD OF BROWN AND SHOOK

In von Zeipel's method, the transition from a given set of canonical variables $(x, \xi; y, \eta)$ and the Hamiltonian $F(x, \xi; y, \eta)$ to a new set $(x^*, \xi^*; y^*, \eta^*)$ and the Hamiltonian $F^*(x^*, \xi^*; -, \eta^*)$, which are not affected by the short period effect, is accomplished by a single canonical transformation.

In the method of Brown-Shook (1933) such a transition is accomplished by a chain of canonical transformations. The first transformation eliminates the short period effects of the first order from the Hamiltonian; the second transformation eliminates the short period effects of the second order, etc. We continue this process till all the significant short period effects are eliminated.

In this chapter we discuss the first and the successive steps of the elimination procedure and the type of operators associated with it.

Let the original set of canonical variables be

$$(x_0, \xi_0; y_0, \eta_0) ,$$

and the corresponding Hamiltonian expanded in powers of a small parameter be

$$F_0 = F_0(x_0, \xi_0; y_0, \eta_0) = F_{00}(x_0) + F_{01}(x_0, \xi_0; y_0, \eta_0) + F_{02}(x_0, \xi_0; y_0, \eta_0) + \cdots .$$

Evidently y_0 is the short period argument, and η_0 is the long period argument. We determine the canonical transformation

$$
\begin{aligned}
&\left(x_0, \xi_0; y_0, \eta_0\right) \rightarrow \left(x_1, \xi_1; y_1, \eta_1\right) , \\
x_1 &= x_0 - \nabla_{y_1} S_1\left(x_0, \xi_0; y_1, \eta_1\right) , \\
\xi_1 &= \xi_0 - \nabla_{\eta_1} S_1\left(x_0 \ \ \xi_0; y_1, \eta_1\right) . \\
y_0 &= y_1 - \nabla_{x_0} S_1\left(x_0, \xi_0; y_1, \eta_1\right) , \\
\eta_0 &= \eta_1 - \nabla_{\xi_0} S_1\left(x_0, \xi_0; y_1, \eta_1\right) ,
\end{aligned}
\tag{44}
$$

in such a way that the Hamiltonian takes the form

$$
\begin{aligned}
F_0\left(x_0, \xi_0, y_0, \eta_0\right) &= F_1\left(x_1, \xi_1; y_1, \eta_1\right) = F_{10}\left(x_1\right) + F_{11}\left(x_1, \xi_1 -, \eta_1\right) \\
&\quad + F_{12}\left(x_1, \xi_1; y_1, \eta_1\right) + F_{13}\left(x_1, \xi_1; y_1, \eta_1\right) + \cdots ,
\end{aligned}
\tag{45}
$$

with no first order short period terms present.

Substituting (44) into (45) and changing the notation we obtain the functional equation

$$
\begin{aligned}
&F_0\left[x, \xi; y - \nabla_x S_1 (x, \xi; y, \eta) , \eta - \nabla_\xi S_1 (x, \xi; y, \eta)\right] \\
&\quad = F_1\left[x - \nabla_y S_1 (x, \xi; y, \eta) , \xi - \nabla_\eta S_1 (x, \xi; y, \eta) ; y, \eta\right] ,
\end{aligned}
$$

which also can be written as

$$
\begin{aligned}
&\exp\left(-\nabla_x S_1 \cdot \nabla_y - \nabla_\xi S_1 \cdot \nabla_\eta\right) F_0 (x, \xi; y, \eta) \\
&\quad = \exp\left(-\nabla_y S_1 \cdot \nabla_x - \nabla_\eta S_1 \cdot \nabla_\xi\right) F_1 (x, \xi; y, \eta) ,
\end{aligned}
\tag{46}
$$

and which serves to determine the form of S_1 and F_1.

Introducing vectors

$$
\begin{pmatrix} x \\ \xi \end{pmatrix} = u , \qquad \begin{pmatrix} y \\ \eta \end{pmatrix} = w
$$

we can write (46) as

$$T_0^{(1)} F_0 = T_1^{(1)} F_1 , \tag{47}$$

where

$$T_0^{(1)} = \exp\left(-\nabla_{\mathbf{u}} S_1 \cdot \nabla_{\mathbf{w}}\right) = \sum_{\alpha=0}^{\infty} T_{0,\alpha}^{(1)} ,$$

$$T_1^{(1)} = \exp\left(-\nabla_{\mathbf{w}} S \cdot \nabla_{\mathbf{u}}\right) = \sum_{\alpha=0}^{\infty} T_{1,\alpha}^{(1)} ,$$

and $T_{0,\alpha}^{(1)}$, $T_{1,\alpha}^{(1)}$ are Faa de Bruno operators, which in this case take a simple form:

$$T_{0,\alpha}^{(1)} = \frac{(-1)^\alpha}{\alpha!} \left(\nabla_{\mathbf{u}} S_1 \cdot \nabla_{\mathbf{w}}\right)^\alpha , \tag{48}$$

$$T_{1,\alpha}^{(1)} = \frac{(-1)^\alpha}{\alpha!} \left(\nabla_{\mathbf{w}} S_1 \cdot \nabla_{\mathbf{u}}\right)^\alpha . \tag{49}$$

It follows from (47)

$$\sum_{\beta=0}^{\alpha} T_{0,\alpha-\beta}^{(1)} F_{0,\beta} = \sum_{\beta=0}^{\alpha} T_{1,\alpha-\beta}^{(1)} F_{1,\beta} , \qquad \alpha = 0, 1, 2, \cdots . \tag{50}$$

The first two Equations (50) give

$$F_{1,0} = F_{0,0}(x) , \tag{51}$$

and

$$\mathbf{n}_0 \cdot \nabla_{\mathbf{y}} S_1 + F_{1,1} = F_{0,1} , \tag{52}$$

where

$$\mathbf{n}_0 = -\nabla_{\mathbf{x}} F_{0,0}(\mathbf{x}) .$$

Introducing again the M and P operators and designating by Q the corresponding integration operator we obtain

$$S_1 = QPF_{0,1} \quad \text{and} \quad F_{1,1} = MF_{0,1} .$$

The remaining Equations (50) serve to determine

$$F_{1,2} ; \; F_{1,3} ; \; \cdots \; .$$

Taking into account

$$T^{(1)}_{0,\alpha} F_{0,0} = 0 , \qquad T^{(1)}_{0,0} F_{0,\alpha} = F_{0,\alpha} , \qquad T^{(1)}_{1,0} F_{1,\alpha} = F_{1,\alpha} ,$$

we obtain from (50) for $\alpha \geq 1$

$$F_{1,\alpha} = F_{0,\alpha} + \frac{(-1)^{\alpha+1}}{\alpha!} \left(\nabla_y S_1 \cdot \nabla_x\right)^{\alpha} F_{0,0} + \sum_{\beta=1}^{\alpha-1} \left(T^{(1)}_{0,\alpha-\beta} F_{0,\beta} - T^{(1)}_{1,\alpha-\beta} F_{1,\beta}\right) . \tag{53}$$

$F_{1,1}$ does not contain y. Thus the short period effects of the first order are eliminated, but F_{12}, F_{13}, $\cdots$ contain such effects. After $F_{1,j}$ ($j = 1, 2, \cdots$) are determined, we can return to the variables x_1, ξ_1, y_1, η_1 by means of the substitution

$$\begin{pmatrix} x & \xi & y & \eta \\ x_1 & \xi_1 & y_1 & \eta_1 \end{pmatrix}$$

into $F_{1,j}$. However, this is not actually necessary. The whole procedure of successive elimination of the short period effects of second and higher orders can be continued in terms of x, ξ, y, η.

The elimination of the second order short period effects from F_1 is achieved by means of the canonical transformation:

$$(x_1, \xi_1; y_1, \eta_1) \rightarrow (x_2, \xi_2; y_2, \eta_2) ,$$

$$x_2 = x_1 - \nabla_{y_2} S_2 (x_1, \xi_1; y_2, \eta_2) ,$$

$$\xi_2 = \xi_1 - \nabla_{\eta_2} S_2 (x_1, \xi_1; y_2, \eta_2) ,$$

$$y_1 = y_2 - \nabla_{x_1} S_2 (x_1, \xi_1; y_2, \eta_2) ,$$

$$\eta_1 = \eta_2 - \nabla_{\xi_1} S_2 (x_1, \xi_1; y_2, \eta_2) .$$

with the properly chosen S_2. We must have

$$F_1\left(x_1, \xi_1; y_1, \eta_1\right) = F_2\left(x_2, \xi_2; y_2, \eta_2\right)$$

$$= F_{20}\left(x_2\right) + F_{21}\left(x_2, \xi_2, -, \eta_2\right)$$

$$+ F_{22}\left(x_2, \xi_2, -, \eta_2\right) + F_3\left(x_2, \xi_2; y_2, \eta_2\right) + \cdots ,$$

and consequently,

$$T_1^{(2)} F_1(u, w) = T_2^{(2)} F_2(u, w) ,$$

where

$$T_1^{(2)} = \exp\left(-\nabla_u S_2 \cdot \nabla_w\right) = \sum_{\alpha=1}^{\infty} T_{1,\alpha}^{(2)} ,$$

$$T_2^{(2)} = \exp\left(-\nabla_w S_2 \cdot \nabla_u\right) = \sum T_{2,\alpha}^{(2)} ,$$

and

$$T_{1,\alpha}^{(2)} = \begin{cases} \dfrac{(-1)^{\alpha/2}}{\left(\frac{\alpha}{2}\right)!} \left(\nabla_u S_2 \cdot \nabla_w\right)^{\alpha/2} , & \text{for } \alpha \text{ even} \\ 0 & \text{for } \alpha \text{ odd} \end{cases}$$

$$T_{2,\alpha}^{(2)} = \begin{cases} \dfrac{(-1)^{\alpha/2}}{\left(\frac{\alpha}{2}\right)!} \left(\nabla_w S_2 \cdot \nabla_u\right)^{\alpha/2} , & \text{for } \alpha \text{ even} \\ 0 & \text{for } \alpha \text{ odd} . \end{cases}$$

The function S_2 is of the second order and, in fact, the development of the Taylor operators $T_1^{(2)}$ and $T_2^{(2)}$ is performed in squares of the small parameter, and the operators of Faa de Bruno with the odd second index can be set equal to zero.

As before, we have

$$\sum_{\beta=0}^{\alpha} T_{1,\alpha-\beta}^{(2)} F_{1,\beta} = \sum_{\beta=0}^{\alpha} T_{2,\alpha-\beta}^{(2)} F_{2,\beta} , \qquad \beta = 0, 1, \cdots, \alpha ; \qquad \alpha = 0, 1, 2, \cdots . \tag{54}$$

The first three equations give

$$F_{1,0} = F_{2,0} = F_{00}(x) ,$$

$$F_{1,1}(u, w) = F_{2,1} ,$$

$$n_0 \cdot \nabla_w S_2 + F_{2,2} = F_{1,2} .$$

From the last equation we obtain

$$S_2 = QPF_{1,2} ,$$

$$F_{2,2} = MF_{1,2} .$$

Taking into account

$$T_{1,\alpha}^{(2)} F_{1,0} = 0 ,$$

Equations (54) become

$$F_{2,\alpha} = F_{1,\alpha} - T_{1,\alpha}^{(2)} F_{0,0} + \sum_{\beta=1}^{\alpha-1} \left(T_{1,\alpha-\beta}^{(2)} F_{1,\beta} - T_{2,\alpha-\beta}^{(2)} F_{2,\beta} \right) , \qquad (55)$$

$$\alpha = 3, 4, 5, \cdots \qquad T_{1,\alpha}^{(2)} F_{0,0} = \frac{(-1)^{\alpha/2}}{\left(\frac{\alpha}{2}\right)!} \left(\nabla_y S_2 \cdot \nabla_x \right) F_{0,0}$$

for α even and is equal to zero for α odd. The Equations (55) serve for the successive determination of $F_{2,3}$, $F_{2,4}$, $\cdots$. Hence, F_2 is obtained in the form

$$F_2 = F_{20}(x) + F_{21}(x, \xi; -, \eta) + F_{22}(x, \xi; -, \eta) + F_{23}(x, \xi; y, \eta) + \cdots ,$$

with the first and second order short period terms removed. The general problem consists of removing the short period terms from

$$F_k = F_{k,0}(x_k) + F_{k,1}(x_k, \xi_k; - \eta_k) + \cdots + F_{k,k}(x_k, \xi_k; -, \eta_k) + F_{k,k+1}(x_k, \xi_k; y_k, \eta_k) ,$$

$$k = 0, 1, 2 \cdots ,$$

by means of the properly chosen canonical transformation

$$(\mathbf{x}_k, \xi_k, \mathbf{y}_k, \eta_k) \to (\mathbf{x}_{k+1}, \xi_{k+1}, \mathbf{y}_{k+1}, \eta_{k+1}) ,$$

$$\mathbf{x}_{k+1} = \mathbf{x}_k - \nabla_{\mathbf{y}_{k+1}} S_{k+1} (\mathbf{x}_k, \xi_k; \mathbf{y}_{k+1}, \eta_{k+1}) ,$$

$$\xi_{k+1} = \xi_k - \nabla_{\eta_{k+1}} S_{k+1} (\mathbf{x}_k, \xi_k; \mathbf{y}_{k+1}, \eta_{k+1}) ,$$

$$\mathbf{y}_k = \mathbf{y}_{k+1} - \nabla_{\mathbf{x}_k} S_{k+1} (\mathbf{x}_k, \xi_k, \mathbf{y}_{k+1}, \eta_{k+1}) ,$$

$$\eta_k = \eta_{k+1} - \nabla_{\xi_k} S_{k+1} (\mathbf{x}_k, \xi_k; \mathbf{y}_{k+1}, \eta_{k+1}) ,$$

which leads to the Hamiltonian of the form

$$F_{k+1} = F_{k+1}(\mathbf{x}_{k+1}) + F_{k+1,1}(\mathbf{x}_{k+1}, \xi_{k+1}; -, \eta_{k+1}) + \cdots + F_{k+1,k+1}(\mathbf{x}_{k+1}, \xi_{k+1}, -, \eta_{k+1}) + F_{k+1,k+2}(\mathbf{x}_{k+1}, \xi_{k+1}; \mathbf{y}_{k+1}, \eta_{k+1}) + \cdots$$

expanded in powers of the small parameter. The condition

$$T_k^{(k+1)} F_k (\mathbf{u}, \mathbf{w}) = T_k^{(k+1)} F_{k+1} (\mathbf{u}, \mathbf{w}) ,$$

must be satisfied, where

$$T_k^{(k+1)} = \exp\left(-\nabla_{\mathbf{u}} S_{k+1} \cdot \nabla_{\mathbf{w}}\right) = \sum_{\alpha=1}^{\infty} T_{k,\alpha}^{(k+1)} ,$$

$$T_{k+1}^{(k+1)} = \exp\left(-\nabla_{\mathbf{w}} S_{k+1} \cdot \nabla_{\mathbf{u}}\right) = \sum_{\alpha=1}^{\infty} T_{k+1,\alpha}^{(k+1)} ,$$

and

$$T_{k,\alpha}^{(k+1)} = \begin{cases} \dfrac{(-1)^{\alpha/k}}{\left(\frac{\alpha}{k}\right)!} \left(\nabla_{\mathbf{u}} S_{k+1} \cdot \nabla_{\mathbf{w}}\right), & \text{for } \alpha \equiv 0 \pmod{k} \\ 0 & \text{for } \alpha \not\equiv 0 \pmod{k} \end{cases}$$

$$T_{k+1,\alpha}^{(k+1)} = \begin{cases} \dfrac{(-1)^{\alpha/k}}{\left(\frac{\alpha}{k}\right)!} \left(\nabla_{\mathbf{w}} S_{k+1} \cdot \nabla_{\mathbf{u}}\right), & \text{for } \alpha \equiv 0 \pmod{k} \\ 0 & \text{for } \alpha \not\equiv 0 \pmod{k} . \end{cases}$$

We deduce as before

$$F_{k+1,\alpha} = F_{k,\alpha} \qquad \alpha = 0, 1, 2, \cdots k$$

$$\mathbf{n}_0 \cdot \nabla_w S_{k+1} + F_{k+1,k+1} = F_{k,k+1} ,$$

and

$$\mathbf{n}_0 \cdot \nabla_w S_{k+1} = P F_{k,k+1} ,$$

$$S_{k+1} = Q P F_{k,k+1} ,$$

$$F_{k+1,k+1} = M F_{k,k+1} .$$

We continue this process till all the significant short period effects are eliminated.

CONCLUSION

The process of determining S_1, S_2, $\cdots$ in the Brown-Shook method seems to be somewhat simpler than the corresponding process in von Zeipel's method.

If we are pursuing only the elimination of short period terms and the determination of long period effects, but not the determination of the original elements in terms of the final elements, then the method of Brown and Shook can provide very good service. However, the process of determination of the original, osculating, elements in terms of the elements affected by the long period perturbations only is more complicated than von Zeipel's method.

It is so because performing a chain of canonical transformations will require the application of a chain of Lagrange operators to accomplish the return to the original elements. The same chain of different Lagrange operators is to be used in the Brown-Shook method if one wants to form the canonical transformation from the original Hamiltonian to the Hamiltonian without the short period terms. In this respect the Brown-Shook method resembles the classical method of Delaunay. Of course, if we are interested in the first order effects only, then both methods coincide.

In this case the statement of Jeffreys (1961) about the identity of both methods remains valid.

If all the angular canonical elements, of y and η type, are eliminated, then the transformed Hamiltonian will contain only the elements of x and ξ type. These remaining elements become constants and the y, η elements become the linear functions of time. Such a result, for example, can be obtained in the case of an artificial satellite, because in the transformed Hamiltonian the order of the long period terms is higher than the order of the significant secular term.

Unfortunately, with the present day mathematical means, such a goal cannot be accomplished in all problems of celestial mechanics. In such a case, we suggest numerical integration with the large integration step of the canonical equations associated with the transformed Hamiltonian.

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