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On the Polynomial of the Dunwoody (1, 1)-knots

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Abstract. There is a special connection between the Alexander polynomial of $(1,1)$-knot and the certain polynomial associated to the Dunwoody 3-manifold ([3], [10] and [13]). We study the polynomial(called the Dunwoody polynomial) for the $(1,1)$-knot obtained by the certain cyclically presented group of the Dunwoody 3-manifold. We prove that the Dunwoody polynomial of $(1,1)$-knot in $\mathbb{S}^3$ is to be the Alexander polynomial under the certain condition. Then we find an invariant for the certain class of torus knots and all 2-bridge knots by means of the Dunwoody polynomial.

1. Introduction

We begin with the fact that every closed 3-manifold has a spine called the Heegaard diagram, from which one can obtain the presentation for the group; however, not all group presentations arise from the spines of 3-manifolds. Therefore, determining which cyclic presentations of groups correspond to spines of closed 3-manifolds is an open problem.

In 1968, L. Neuwirth introduced an algorithm for the construction of a connected closed orientable 3-manifold from 2-complex, which corresponds to a group presentation([17]). In 1994, M. J. Dunwoody introduced the 6-tuples yielding a family of genus n Heegaard diagrams of closed orientable 3-manifolds called the Dunwoody 3-manifold ([10]). In 2000, author in [19] first proposed that the branched set in the quotient space of the Dunwoody 3-manifold is a $(1,1)$-knot in $\mathbb{S}^3$. In other words, at least one cyclic symmetry on the Dunwoody 3-manifold induces a $(1,1)$-knot. In 2004, it was shown that some classes of knots represented by the Dunwoody 3-manifolds contain all $(1,1)$-knots in $\mathbb{S}^3$ ([5]).

Conversely, for a given $(1,1)$-knot K, it is an interesting problem to determine a type of the Dunwoody 3-manifold representing K even if it is not unique. Until

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now these problems for all 2-bridge knots and some torus knots were solved in [11], [14] and [19]. For example, the explicit type for the torus knot $T(p,q)$ satisfying $q \equiv \pm 1 \bmod p$ has been obtained in [1] and [5], and for the torus knot $T(p,q)$ satisfying $q \equiv \pm 2 \bmod p$, the type has been obtained in [13]. Furthermore, in [6], it has been obtained for all torus knots with bridge number at most three. However to determining types of Dunwoody 3-manifolds for all torus knots is still unknown.

The Dunwoody 3-manifold plays an important role in determining which cyclically presented group corresponds to a 3-manifold. Indeed, in order to study a 3-manifold with some particular group as the fundamental group, the Dunwoody 3-manifold has a Heegaard diagram from which one can obtain a presentation for the group. Thus, to find the Dunwoody 3-manifold, one must seek a cyclically presented group associated with a 3-manifold. Furthermore, as in [12], the branched covering space of the spatial Θ-curve containing $(1,1)$-knots as the constituent knots is related to the Dunwoody 3-manifold. Therefore the concept of the Dunwoody 3-manifold is important in knot, branched covering and graph theories.

In section 2, we introduce a set of 4-tuples representing all $(1,1)$-knots , which is determined by two permutations, and so 3-manifolds related to a set of 4-tuples are containing the Dunwoody 3-manifolds. In particular, we prove that the strongly-cyclic branched covering space of the Dunwoody $(1,1)$-knot represented by the certain 4-tuples is homeomorphic to the Dunwoody 3-manifold. Moreover we show some conditions of the Dunwoody $(1,1)$-knot representing a torus knot; and we also discuss about the type of the Dunwoody 3-manifold representing the torus knot.

In Section 3, we show that the fundamental group of the Dunwoody 3-manifold admits a cyclic presentation, which is independent of results in [4] and [16]; and we define the Dunwoody polynomial from the cyclic presentation. As the main result, we show that the Dunwoody polynomial for the Dunwoody $(1,1)$-knot in $\mathbb{S}^3$ is the Alexander polynomial under some condition. For the results in [3] and [4], they are shown the connection between the Dunwoody polynomial and the projection of Alexander polynomial into $\mathbb{Z}[t]/(t^n-1)$ for some $n > 1$. Moreover we show that a certain numerical number from the Dunwoody polynomial is an invariant for some torus knots and all 2-bridge knots of $(1,1)$-knots. This result gives an answer to a question in [10]. In this note, all lens spaces will be assumed to include $\mathbb{S}^3$ but not $\mathbb{S}^1 \times \mathbb{S}^2$. The basic facts about lens spaces are covered in [18].

2. On the Dunwoody $(1,1)$-knots

Let (V_1, V_2) be a Heegaard splitting with genus n of a closed orientable 3-manifold M. A properly embedded disc D in the handlebody V_2 is called a meridian disc of V_2 if cutting V_2 along D yields a handlebody of genus $n-1$. A collection of n mutually disjoint meridian discs $\{D_i\}$ of V_2 is called a complete system of meridian discs of V_2 if cutting V_2 along $\cup_i D_i$ gives a 3-ball. Let c_i denote the 1-sphere ∂D_i which lies in the closed orientable surface $\partial V_1 = \partial V_2$ of genus n. Then the system is said to be a Heegaard diagram of M denoted by $(V_1; c_1, c_2, \cdots, c_n)$. Let M be a lens space and K be a knot in M. Then the pair (M, K) admits a $(1,1)$-decomposition if there exists a Heegaard splitting of genus one $(V_1, K_1) \cup_\phi (V_2, K_2)$ of (M, K) such that $(V_1; c_1)$ is a Heegaard diagram of M, and $K_1 \subset V_1$ and $K_2 \subset V_2$ are properly embedded trivial arcs, where ϕ is an attaching homeomorphism.

We now introduce the Dunwoody $(1,1)$-decomposition of (M, K) determined by two permutations and 4-tuples (a, b, c, r), where $a > 0$, $b \geq 0$, $c \geq 0$, $r \in \mathbb{Z}_d$, and $d = 2a+b+c$. Let m^+ and m^- be two circles with each other different orientations, and let $X^+ = \{1, 2, \cdots, d\}$ and $X^- = \{-1, -2, \cdots, -d\}$ be sets of d vertices in m^+ and m^-, respectively. We now define two permutations α and β as below, where all numbers are under $\mod d$.

$$\alpha(j) = \begin{cases} d-j+1 & \text{if} \quad 1 \leq j \leq a \\ -j-c & \text{if} \quad a+1 \leq j \leq a+b \\ -j+b & \text{if} \quad a+b+1 \leq j \leq a+b+c \\ -d-j-1 & \text{if} \quad -a \leq j \leq -1 \end{cases}$$

and

$$\beta(j) = \begin{cases} -j+r & \text{if } r < j \\ -j+r-d & \text{otherwise} \end{cases} .$$

The cycle expressions of α and β are the following.

$$\alpha = (1, d)(2, d-1)(3, d-2) \cdots (a, d-a+1)$$
$$(a+1, -(a+c+1)) \cdots (a+b, -(a+c+b))$$
$$(a+b+1, -(a+1)) \cdots (a+b+c, -(a+c))$$
$$(-1, -d)(-2, -(d-1)) \cdots (-a, -(d-a+1)) \ (*)$$

and

$$\beta = (1, -(1-r))(2, -(2-r)) \cdots (j, -(j-r))$$
$$\cdots (d, -(d-r)). \quad (**)$$

We note that each 2-cycle in α consists of the end points of a curve connecting m^+ and m^- or themselves as the rule of Figure 1; and each 2-cycles in β generates a meridian disk m, by gluing the corresponding points in m^+ and m^- via β. For example, $(r, -(r-r))$ means that the number r of m^+ is identified with the number $-(r-r) = -0 = -d$ in m^-. Thus $\beta\alpha$ determines the genus one solid torus V_1 and the disjoint simple closed curves on ∂V_1.

Theorem 2.1. *If L is the number of the disjoint simple closed curves on ∂V_1, then there are two permutations α and β such that $|\alpha| - |\beta| + 2L = |\beta\alpha|$, where $|\cdot|$ means the number of disjoint cycles in a permutation.*

Proof. Let $d = 2a+b+c$. Let $X^+ = \{1, 2, \cdots, d\}$ and $X^- = \{-d, -d+1, \cdots, -1\}$ be sets of d points in m^+ and m^-, respectively. Then the permutation α is consisting of

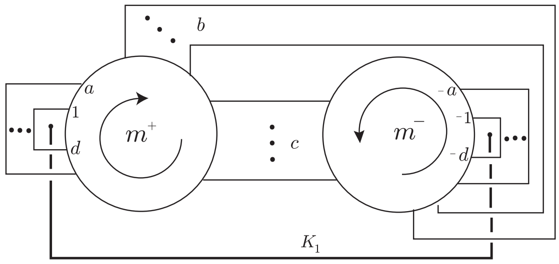

Figure 1: The Dunwoody (1, 1)-decomposition $D(a, b, c, r)$

d 2-cycles by two end points of line segments connecting m^+ and m^- or themselves on ∂V_1, and the permutation β is consisting of d 2-cycles connecting m^+ and m^- on ∂V_1. We now define an equivalence relation on $X = X^+ \cup X^-$ by

$$x \sim y \text{ if } y = (\beta\alpha)^i(x) \text{ or } y = \alpha(\beta\alpha)^i(x) \text{ for some } i.$$

Then we call the equivalence classes of X under the relation the orbits of $\beta\alpha$. Let l be a simple closed curve on ∂V_1 and x be a point on m^+ meeting l. Then l is determined by the repeated applications of α and β as follows;

$$x, \alpha(x), \beta\alpha(x), \alpha\beta\alpha(x), \ldots, \alpha\beta\cdots\alpha(x),$$

which forms an orbit of $\beta\alpha$. Each orbit of $\beta\alpha$ determines a simple closed curve in ∂V_1. Let $Y_1, \ldots, Y_L$ be orbits of $\beta\alpha$. If $x \in Y_i$ and d is the smallest positive integer such that $(\alpha\beta)^d(x) = x$, then on Y_i, $\beta\alpha$ is expressed as a product $\beta_i\alpha_i$ of two disjoint permutations α_i and β_i of the same length:

$$\alpha_i = (x, \beta\alpha(x), (\beta\alpha)^2(x), \ldots, (\beta\alpha)^{d-1}(x))$$

and

$$\beta_i = (\alpha(x), \alpha\beta\alpha(x), \ldots, (\alpha\beta)^{d-1}\alpha(x)).$$

Furthermore the $\beta_i\alpha_i$ are pairwise disjoint and

$$\beta\alpha = (\beta_L\alpha_L)\cdots(\beta_2\alpha_2)(\beta_1\alpha_1).$$

Moreover

$$|\beta\alpha| = |\beta_L\alpha_L| + \cdots + |\beta_1\alpha_1| = 2L. \qquad \square$$

Since two consecutive cycles in $\beta\alpha$ determine a simple closed curve (which is isotopic to $c_1 = \partial D_1$) on ∂V_1, we assume that l is the simple closed curve determined by α and β on ∂V_1 whenever $L = 1$. Let K_1 be a trivial arc in V_1 such that $K_1 \cap \partial V_1 = \partial K_1$, which is situated inside the bigons bounded by 2-cycles $(1, d)$ and $(-1, -d)$ as shown in Figure 1. Then a set of 4-tuples of integers

$$\mathcal{D} = \{(a, b, c, r) | a > 0, b \geq 0, c \geq 0,$$

$$d = 2a + b + c, r \in \mathbb{Z}_d, |\alpha\beta| = 2\}$$

admits a $(1,1)$-decomposition of (M, K) called *the Dunwoody* $(1,1)$*-decomposition.* For each (a, b, c, r) in $\mathcal{D}$, we denote the Dunwoody $(1,1)$-decomposition of (M, K) by $D(a, b, c, r)$. (See Figure 1.) Moreover, we denote a $(1,1)$-knot K represented by $D(a, b, c, r)$ by $K(a, b, c, r)$, and call it *the Dunwoody* $(1,1)$*-knot.* We note that every $(1,1)$-knot can be represented by the Dunwoody $(1,1)$-knot and vice versa([5]). The representation of a $(1,1)$-knot by Dunwoody $(1,1)$-decomposition is not unique. For example, both $K(1, 3, 4, 7)$ and $K(2, 1, 4, 4)$ represent the pretzel knot $P(-2, 3, 7)$ which is a $(1,1)$-knot as was mentioned in [19]. The subset of $\mathcal{D}$ representing all 2-bridge knots was determined by [11], [14] and [19]. However, the subset of $\mathcal{D}$ representing all torus knots is not yet determined completely. In [1], [5], [13] and [6] we have Dunwoody $(1,1)$-decompositions representing the certain class of torus knots.

We now construct a family of 3-manifolds which are the n-fold strongly-cyclic coverings branched over Dunwoody $(1,1)$-knots. Let M be a lens space and K be a Dunwoody $(1,1)$-knot in M. Then the n-fold cyclic covering of M branched over K is completely defined by an epimorphism $C : H_1(M - K) \to \mathbb{Z}_n$, where $\mathbb{Z}_n$ is the cyclic group of order n. Let r_1 be a generator of ∂V_1, which is the boundary of the meridian disk meeting with K_1 at one point and let r_2 be a generator of ∂V_1, which is the longitude curve meeting with r_1 at one point. Then every curve of ∂V_1 determined by two permutations α and β is generated by r_1 and r_2. In other words, the orbit l of $\beta\alpha$ is generated by r_1 and r_2. We define $l_i (1 \le i \le 6)$ from the oriented curve l on $D(a, b, c, r)$ as follows.

- l_1 is the number of left directed arrows from m^+ or m^- to m^+ or m^- in a edges respectively.
- l_2 is the number of right directed arrows from m^+ or m^- to m^+ or m^- in a edges respectively.
- l_3 is the number of arrows directed from m^+ to m^- in b edges.
- l_4 is the number of arrows directed from m^- to m^+ in b edges.
- l_5 is the number of arrows directed from m^+ to m^- in c edges.
- l_6 is the number of arrows directed from m^- to m^+ in c edges.

From now on for $D(a, b, c, r)$ we let $p = (l_3 + l_5) - (l_4 + l_6)$, $q = (l_1 + l_3) - (l_2 + l_4)$ and $d = 2a + b + c$. If $p = \pm 1$ or $p = 0$, then M is $\mathbb{S}^3$ or $\mathbb{S}^1 \times \mathbb{S}^2$([11] and [12]), respectively. Thus $p \neq 0$ if M is not $\mathbb{S}^1 \times \mathbb{S}^2$. We have $\pi(M) = \langle x | x^{\pm p} \rangle = \mathbb{Z}_{|p|}$ and $H_1(M - K) = \langle r_1, r_2 | pr_2 + qr_1 \rangle = \mathbb{Z} \oplus \mathbb{Z}_{gcd(p,q)}$. By definition, the n-fold cyclic covering f of M branched over K is called strongly-cyclic if the branching index of K is n. That is, the fiber $f^{-1}(x)$ of each point $x \in K$ contains a single point. Therefore the homology class of a meridean loop r_1 around K is mapped by C in a generator of $\mathbb{Z}_n$, say $C(r_1) = 1$, and so there exists an n-fold strongly-cyclic covering space $\overline{M}$ of M branched over K if and only if there is $s = C(r_2) \in \mathbb{Z}_n$ such that $ps + q \equiv 0 \bmod n$. We call the diagram in Figure 2 a Heegaard diagram of $\overline{M}$ and denote it by $D_n(a, b, c, r, s)$. If the Dunwoody $(1, 1)$-knot K is in $\mathbb{S}^3$, the

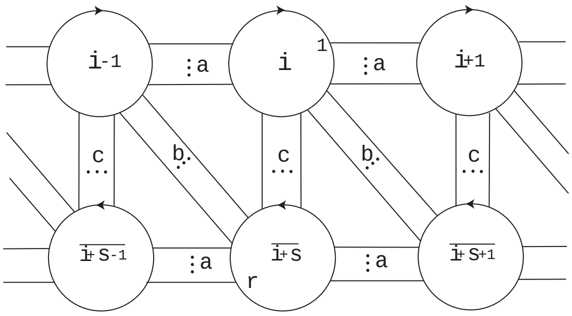

Figure 2: A Heegaard diagram $D_n(a, b, c, r, s)$

strongly-cyclic branched covering is the same that cyclic branched covering. Indeed the n-fold cyclic branched covering of K in $\mathbb{S}^3$ always exists and is unique up to equivalence for $n > 1$ because $H_1(\mathbb{S}^3 - K) = \mathbb{Z}$, the homology class r_1 is mapped by C in a generator of $\mathbb{Z}_n$ and $s = C(r_2) = -q$.

We have proved the following.

Theorem 2.2. *Let $D(a, b, c, r)$ be the Dunwoody $(1, 1)$-decomposition of (M, K) and $n > 1$. Then $\overline{M}$ is homeomorphic to a 3-manifold if and only if there is an integer s such that $ps + q \equiv 0$ mod n.*

We notice that $D_n(a, b, c, r, s)$ satisfies the conditions for the Heegaard diagram of the Dunwoody 3-manifold considered in [10]. Thus we have the following.

Corollary 2.3. *Let M be a lens space and K be a Dunwoody $(1, 1)$-knot in M. Then the n-fold strongly-cyclic covering space $\overline{M}$ of M branched over K is homeomorphic to the Dunwoody 3-manifold.*

Corollary 2.4. *[12] $D(a, b, c, r)$ is the $(1, 1)$-decomposition of $(\mathbb{S}^3, K)$ if and only if $|p| = 1$.*

From the result of Corollary , if $d = 2a + b + c$ is even, then $D(a, b, c, r)$ cannot be a $(1, 1)$-decomposition of $(\mathbb{S}^3, K)$ because d has the same parity of p. (See [15] for detail.)

Generally, for the following set

$$\mathcal{S} = \{(a, b, c, r) | a > 0, b \geq 0, c \geq 0,$$

$$d = 2a + b + c, r \in \mathbb{Z}_d, |\alpha\beta| = 2L\},$$

we suppose that $L \geq 2$ is the number of simple closed curves determined by α and β on ∂V_1. Given an $(a, b, c, r) \in \mathcal{S}$, it is possible to represent a link in lens spaces containing $\mathbb{S}^3$ and $\mathbb{S}^1 \times \mathbb{S}^2$. Thus the orientable 3-manifold $\overline{M}$ in existing is a generalization of the Dunwoody 3-manifold introduced in [10], called *the generalized Dunwoody* 3*-manifold*. (See [15],[13] or [7] for some examples.)

We let $L = 1$. That is, for each $(a, b, c, r) \in \mathcal{D}$, $K(a, b, c, r)$ is the Dunwoody $(1, 1)$-knot in a lens space or $\mathbb{S}^3$. We now consider the Dunwoody $(1, 1)$-knot representing the torus knot. The torus knot is a knot embedded in the standard torus T in $\mathbb{S}^3$. Regarding T as the boundary of tubular neighborhood of trivial knot in $\mathbb{S}^3$, we take a meridian-longitude system (m_1, m_2) of trivial knot on T. The torus knot is said to be *of type* (k_1, k_2), denoted by $T(k_1, k_2)$, if it is homologous to $k_1m_1 + k_2m_2$ in T for some coprime integers k_1 and k_2.

The Dunwoody $(1, 1)$-knots representing $T(k_1, k_2)$ with $k_2 \equiv \pm 1 \bmod k_1$ have been obtained in [1] and [5]. Moreover, the Dunwoody $(1, 1)$-knots representing the torus knots with bridge number at most three have been obtained in [6]. Furthermore the Dunwoody $(1, 1)$-knots representing $T(k_1, k_2)$ with $k_2 \equiv \pm 2 \bmod k_1$ have been obtained in [13] with explicit formulae: (i) $T(k_1, k_2)$ with $k_2 \equiv 2 \bmod k_1$ is represented by $K(a, b, c, r)$, where

$$(2.1) \qquad \begin{cases} a = \frac{k_1 - 1}{2} \\ b = 1 \\ c = \frac{(k_1+1)(k_1-1)(k_2-2)}{2k_1} \\ r = \frac{-k_1 + (k_1)^2 k_2 - k_2 + 2 - (k_1)^3}{2k_1} \end{cases},$$

and (ii) $T(k_1, k_2)$ with $k_2 \equiv -2 \bmod k_1$ is represented by $K(a, b, c, r)$ where

$$(2.2) \qquad \begin{cases} a = \frac{k_1 - 1}{2} \\ b = 1 \\ c = \frac{(k_1)^2 k_2 - 2(k_1)^2 - k_2 - 2}{2k_1} \\ r = \frac{1}{2}(k_1)^2 - \frac{3}{2} \end{cases}.$$

In the following theorem the conditions for $K(a, b, c, r)$ to represent a torus knot will be given where $|X_1 \cap X_2|$ means the number of intersecting points between two sets X_1 and X_2.

Theorem 2.5. *Let $D(a, b, c, r)$ be the Dunwoody $(1, 1)$-decomposition of $(\mathbb{S}^3, K)$. Suppose that m is the meridian disk determined by β and l is the simple closed curve defined by $\beta\alpha$ such that $|K_1 \cap K_2| = 2$, $|K_1 \cap l| = k_1$, and $|K_2 \cap m| = k_2$, for some coprime integers k_1 and k_2. Then $K(a, b, c, r)$ is $T(k_1, k_2)$, where $k_1 = 2a + b$ and $k_2 \leq c + 2$.*

Proof. There exists a Heegaard splitting of genus one $(V_1, K_1) \cup_\phi (V_2, K_2)$ of $(\mathbb{S}^3, K)$, where V_1 and V_2 are solid tori, $K_1 \subset V_1$ and $K_2 \subset V_2$ are properly embedded trivial arcs, and $\phi : (\partial V_2, \partial K_2) \to (\partial V_1, \partial K_1)$ is an attaching homeomorphism. Since $|K_1 \cap K_2| = 2$, K_1 and K_2 do not meet each other except the bigons determined by the 2-cycles $(1, d)$ and $(-1, -d)$. Thus the meridian-longitude system (m, l) satisfies $|K_1 \cap l| = 2a + b$ and $|K_2 \cap m| \le c + 2$. Let $k_1 = 2a + b$ and $k_2 = |K_2 \cap m|$ be integers satisfying $\gcd(k_1, k_2) = 1$. Then $K = K_1 \cup_\phi K_2$ is homologous to $k_1 m + k_2 l$ in V_1. Therefore $K(a, b, c, r)$ is $T(k_1, k_2)$. □

The inequality $k_2 \le c + 2$ in Theorem 2.5 is the generalization of Theorem 4.2(iii) in [8]. That is, $K(1, 0, 2k - 1, 2)$ is equivalent to $T(2k + 1, 2)$. We also note that the Dunwoody $(1, 1)$ -decomposition $D(a, b, c, r)$ representing $T(k_1, k_2)$ with $k_2 \equiv \pm 2 \bmod k_1$ satisfies the conditions in Theorem 2.5.

Example 1. Let $D(1, 2, 3, 3)$ be a $(1, 1)$-decomposition of $(\mathbb{S}^3, K) = (V_1, K_1) \cup_\phi (V_2, K_2)$ and $|(l_3 + l_5) - (l_4 + l_6)| = 1$. (See Figure 3.) Then $|K_1 \cap l| = 4$, $|K_2 \cap m| = 5$, and $|K_1 \cap K_2| = 2$, which are marked by circled numbers, numbers and dots respectively in Figure 3. Thus $D(1, 2, 3, 3)$ satisfies the conditions of Theorem 2.5 and so $K(1, 2, 3, 3)$ is $T(4, 5)$.

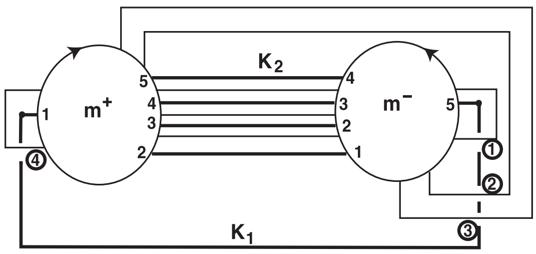

Figure 3: A $(1, 1)$-decomposition $D(1, 2, 3, 3)$ of $T(4, 5)$

3. The Alexander polynomial vs the Dunwoody polynomial

In this section, we show that (i) the certain polynomial of the Dunwoody $(1, 1)$-knot in $\mathbb{S}^3$ is the Alexander polynomial, and that (ii) if $K(a, b, c, r)$ is the Dunwoody $(1, 1)$-knot representing 2-bridge knot or some torus knots, then the number $d = 2a + b + c$ is an invariant for $K(a, b, c, r)$. For the Dunwoody polynomial, (i) gives an answer to a question in [10].

Theorem 3.1. *The n-fold strongly-cyclic branched covering of the Dunwoody $(1, 1)$-knot in a lens space admits a cyclically presented fundamental group.*

Proof. When we consider the Dunwoody $(1, 1)$-knot $K(a, b, c, r)$ in a lens space, $\beta\alpha$ has two cycles of length d such that $(\beta\alpha)^d(x) = x$ for each x on $D(a, b, c, r)$ by Theorem 2.1. Thus the n-fold strongly-cyclic branched covering of $K(a, b, c, r)$ is homeomorphic to the Dunwoody 3-manifold $D_n(a, b, c, r, s)$. Since $ps + q \equiv 0 \bmod n$, the path corresponding to this cycle connects the endpoint labelled 1 in the hole labelled 0 to the endpoint labelled 1 in the hole labelled $\overline{ps+q}$ under mod n. That is, the condition $ps+q \equiv 0$ mod n ensures that the path corresponding to the cycles is a simple closed curve with an orientation. Since $D_n(a,b,c,r,s)$ has n simple closed curves, each path starting at the endpoint labelled 1 in the hole labelled i corresponding to the cycles of $\beta\alpha$ will be connected to the endpoint labelled 1 in the hole labelled $\bar{i}$ under mod n. With notations in [13], $w(C_i)$(resp. $w(\bar{C}_i)$) is a cyclic presentation obtained by reading off simple closed curves around the hole labelled i(resp. $\bar{i}$). Thus the identification of C_i and $\bar{C}_i$ by r on $D_n(a,b,c,r,s)$ induces $w(C_i) \approx_r w(\bar{C}_i)$. If $i=0$, then

$$u\eta^s(c)\eta^{s-1}(b)\eta^{-1}(u^{-1}) \approx_r abc\eta^{-1}(a^{-1}),$$

from which we have a cyclic presentation for the fundamental group. □

For the specific example, let the Dunwoody $(1,1)$-knot $K(a,b,c,r)$ represent $T(p,q)$ such that p is odd and $q \equiv \pm 2$ mod p. Then the n-fold cyclic covering of $\mathbb{S}^3$ branched over $K(a,b,c,r)$ satisfies $u\eta^s(c)\ \eta^{s-1}(b)\ \eta^{-1}(u^{-1}) \approx_r abc\ \eta^{-1}(a^{-1})$ (for reference see [13]), where the parameter s is equal to $-s$ in [13].

From [3], [9] and [20], we recall the definition of the Alexander polynomial of a knot in compact connected 3-manifold. We also note that every finitely generated abelian group G is a direct sum of a torsion-free part $F(G)$ and a torsion-part $T(G)$. For the group G, we denote its integral group ring by $\mathbb{Z}[G]$. In particular, the first homology group $H_1(N)$ of a compact connected 3-manifold N has a decomposition

$$H_1(N) \cong F(H_1(N)) \oplus T(H_1(N)).$$

The projection $J: H_1(N) \to H_1(N)/T(H_1(N))$ induces the ring homomorphism $J': \mathbb{Z}[H_1(N)] \to \mathbb{Z}[H_1(N)/T(H_1(N))]$. If k is the first Betti number of N and $t_1, \cdots, t_k$ are generators of $H_1(N)/T(H_1(N))$, then we have

$$\mathbb{Z}[H_1(N)/T(H_1(N))] \cong \mathbb{Z}[t_1, t_1^{-1}, \cdots, t_k, t_k^{-1}].$$

Let $h: \pi_1(N,*) \to H_1(N)$ be the Hurewitz homomorphism, where $*$ is a fixed point in N. Denote $E_1(N) \subset \mathbb{Z}[H_1(N)]$ and $E_1'(N)$ with the first elementary ideal of $\pi_1(N,*)$ and the smallest principal ideal of $\mathbb{Z}[H_1(N)/T(H_1(N))]$ containing $J'(E_1(N))$, respectively. The generator $\triangle_N$ of $E_1'(N)$ is well-defined up to multiplication by units of $\mathbb{Z}[t_1, t_1^{-1}, \cdots, t_k, t_k^{-1}]$ and is said to be the Alexander polynomial of N. Let K be a knot in a compact connected 3-manifold M. Then the Alexander polynomial $\triangle_N$ of $N = M - K$ is the Alexander polynomial of K and it will be denoted by $\triangle_K$ instead of $\triangle_N$ for a knot K.

Let R be a unital commutative ring and let

$$G_n \cong < x_0, x_1, \cdots, x_{n-1} | r_0, \cdots, r_{n-1} >$$

be a finitely-presented R-module, where each relation r_i is a linear combination of the generators x_j: $r_i = a_{i0}x_0 + \cdots + a_{i(n-1)}x_{n-1}$. In other words, G_n is generated as an R module by the elements $x_0, \cdots, x_{n-1}$, and $r_0 = 0, \cdots, r_{n-1} = 0$ are relations among the x_i's. Then we can define a presentation matrix to be an $n \times n$ matrix with entries a_{ij} for $0 \leq i \leq n-1$, $0 \leq j \leq n-1$. An Alexander matrix is a presentation matrix for $H_1(\tilde{X})$ as a $\mathbb{Z}[t, t^{-1}]$ module, where $\tilde{X}$ is the infinite cyclic cover of the knot complement N. The ideal generated by the Alexander matrix is the Alexander ideal of the knot, so the Alexander ideal is principal([18], P.207). Any generator of this principal ideal is the Alexander polynomial $\triangle_K$ for a knot K. In fact, it was discovered by Alexander [2] in the 1920s, early in the history of topology, using the homology of the infinite cyclic cover of a knot complement.

In this section, let M be a lens space and denoted by $L(p, q')$, where p and q' are relatively prime. Considering the Dunwoody $(1,1)$-knot K in M, we have $F(H_1(N)) \cong \mathbb{Z}$ and $T(H_1(N)) \cong \mathbb{Z}_{gcd(p,q)}$. Thus $\triangle_K(t) \in \mathbb{Z}[t, t^{-1}]$ is the Alexander polynomial of K. In particular, for K in $\mathbb{S}^3$, $\triangle_K(t) \in \mathbb{Z}[t, t^{-1}]$ has the properties (i) $\triangle_K(t) = \triangle_K(t^{-1})$ and (ii) $\triangle_K(1) = \pm 1$. Vice verse, every polynomial in $\mathbb{Z}[t, t^{-1}]$ satisfying (i) and (ii) is the Alexander polynomial of a knot in $\mathbb{S}^3$ ([9]). However, by Theorem B in [20], for the Alexander polynomial of K in M, it is true for the condition (i), but the condition (ii) is no longer true. See also [21].

Now we introduce the Dunwoody polynomial of the Dunwoody $(1,1)$-knot $K = K(a, b, c, r)$, and study the connections between the Dunwoody polynomial and the Alexander polynomial for K. Let $n > 1$. Then the n-fold strongly-cyclic branched covering $D_n(a, b, c, r, s)$ of K in M admits a cyclically presented fundamental group by Corollary 2.3 and Theorem 3.1. Due to the cyclic symmetry of $D_n(a, b, c, r, s)$, the fundamental group has the cyclic presentation induced by a single word $w(x_0, x_1, \cdots, x_{n-1})$ as following:

$$G_n(w(x_0, x_1, \cdots, x_{n-1})) = \langle x_0, x_1, \cdots, x_{n-1} |$$
$$\theta^j(w(x_0, x_1, \cdots, x_{n-1})), 0 \leq j \leq n-1 \rangle$$

where θ is the automorphism on the free group $F_n = \langle x_0, \cdots, x_{n-1} \rangle$ of rank n defined by $\theta(x_i) = x_{i+1}$ and all indices are taken under mod n. Since θ is an automorphism of order n, the relations

$$\{\theta^j(w(x_0, x_1, \cdots, x_{n-1})) | 0 \leq j \leq n-1\}$$

are independent of j with $0 \leq j \leq n-1$, that is, for any $0 \leq j \leq n-1$,

$$G_n(w(x_0, x_1, \cdots, x_{n-1})) \cong$$
$$G_n(\theta^j(w(x_0, x_1, \cdots, x_{n-1}))).$$

The relations $\theta^j(w(x_0, x_1, \cdots, x_{n-1}))$, $0 \leq j \leq n-1$, are determined by n disjoint simple closed curves on $D_n(a, b, c, r, s)$. For a relation $w(x_0, x_1, \cdots, x_{n-1}) \in$

$\{\theta^j(w(x_0, x_1, \cdots, x_{n-1}))|0 \le j \le n-1\}$, the relation $w(x_0, x_1, \cdots, x_{n-1})$ is said to be *principal* if all indices in $w(x_0, x_1, \cdots, x_{n-1})$ are independent of n. For the cyclic presentation $G_n(w(x_0, x_1, \cdots, x_{n-1}))$ by $w(x_0, x_1, \cdots, x_{n-1})$, we obtain the abelianized word $\sum_{i=0}^{k} a_i\bar{x}_i$, $(a_i \in \mathbb{Z})$, of w and a polynomial $f_w^n(t) = \sum_{i=0}^{k} a_i t^i \in \mathbb{Z}[t, t^{-1}]$ obtained by substituting t^i into $\bar{x}_i$ is called *the Dunwoody polynomial* determined by $D_n(a, b, c, r, s)$. Moreover, by the multiplications of $\pm t^j$ $(j \in \mathbb{Z})$, we can normalize $f_w^n(t) \in \mathbb{Z}[t, t^{-1}]$ in order to have the polynomial with a positive constant term and positive exponents in $\mathbb{Z}[t]$. Let $f_w^n(t) \in \mathbb{Z}[t]$ and $n > 1$. Then $f_{\theta^j(w)}^n(t)(0 \le j \le n-1)$ are different polynomials in $\mathbb{Z}[t]$, but they are the same in the quotient ring $\mathbb{Z}[t]/(t^n - 1)$, up to units. From now on, we will consider the Dunwoody polynomial $f_{w_k}^n(t)$ as an element of $\mathbb{Z}[t]/(t^n - 1)$. For some n, we say that $D_n(a, b, c, r, s)$ admits the principal cyclic presentation $G_n(w)$ if w is principal.

Example 2. For $n > 4$, $\pi(D_n(3, 1, 2, 2, -1))$ has a cyclic presentation induced by

$$w(x_0, x_1, x_2, x_3, x_4) = x_1x_2x_3x_4^{-1}x_3x_2^{-1}x_2^{-1}x_1x_0^{-1}.$$

Thus $\pi(D_n(3, 1, 2, 2, -1))$ is presented by

$$\langle x_0, x_1, \cdots, x_{n-1}|\theta^j(x_1x_2x_3x_4^{-1}x_3x_2^{-1}x_2^{-1}x_1x_0^{-1}), \quad 0 \le j \le n-1\rangle.$$

Thus $f_w^n(t) = 1 - 2t + t^2 - 2t^3 + t^4$ is the Dunwoody polynomial determined by $D_n(3, 1, 2, 2, -1)$. Since

$$w(x_0, x_1, x_2, x_3, x_4) = x_1x_2x_3x_4^{-1}x_3x_2^{-1}x_2^{-1}x_1x_0^{-1}$$

is independent of $n(> 4)$, it is a principal relation. Indeed that the Dunwoody $(1, 1)$-knot $K(3, 1, 2, 2)$ represents the knot 9_{42} in the knot table classified by Rolfsen. It is interesting to note that the Dunwoody polynomial $f_w^n(t)$ for $n > 4$ is the Alexander polynomial of 9_{42}.

For the Dunwoody $(1, 1)$-knot $K = K(a, b, c, r)$ in $\mathbb{S}^3$, the following shows that $f_w^n(t)$ is to be the Alexander polynomial of K.

- If $w_k = w(x_0, x_1, \cdots, x_k)$ is the principal relation of the Dunwoody 3-manifold represented by $D_n(a, b, c, r, s)$ for some n, then $f_w^n(t)$ is the Alexander polynomial of K in $\mathbb{S}^3$.

In particular, the Dunwoody $(1, 1)$-knot $K = K(a, b, c, r)$ defined in (2.1) and (2.2) for $K = T(k, hk \pm 2)$ with $h, k > 0$ induces the Dunwoody 3-manifold represented by $D_n(a, b, c, r, s)$ for some s. Using the principal relation, say w, of $D_n(a, b, c, r, s)$, we can obtain the Dunwoody polynomial $f_w^n(t)$ such that $f_w^n(t) \doteq \triangle_K(t)$. For example, let K be a torus knot $T(5, 7)$ satisfying $7 \equiv 2 \bmod 5$. On $D_n(2, 1, 12, 13, -5)$ with $n > 29$, we can obtain a relation w as the following

$$w = x_0^{-1}x_1^1x_6^1x_{11}^1x_{16}^1x_{17}^{-1}x_{12}^{-1}x_7^{-1}x_8^1x_{13}^1x_{18}^1x_{23}^1x_{24}^{-1}$$

$$x_{19}^{-1}x_{14}^{-1}x_{10}^{-1}x_5^{-1}.$$

Then the relation w is principal in $D_n(2,1,12,13,-5)$ for condition $n > 29$. Thus it was shown in [13] that $f_w^n(t)$ is the Alexander polynomial of K.

For each knot K in $\mathbb{S}^3$, we denote the projection of $\triangle_K(t)$ into $\mathbb{Z}[t]/(t^n-1)$ with $\triangle_K^n(t)$. The following corollary shows the connections between the projection of Alexander polynomial into $\mathbb{Z}[t]/(t^n-1)$ and the Dunwoody polynomial for the Dunwoody $(1,1)$-knot in $\mathbb{S}^3$.

Corollary 3.2([4]). *Let $D(a,b,c,r)$ be the Dunwoody $(1,1)$-decomposition of $(\mathbb{S}^3,K)$ and $n>1$. Then $f_w^n(t) \doteq \triangle_K^n(t)$ in $\mathbb{Z}[t]/(t^n-1)$, where $\doteq$ means equal up to units.*

The following corollary is required the degree of the Alexander polynomial in order to obtain the Dunwoody polynomial.

Corollary 3.3([4]). *Let K be the Dunwoody $(1,1)$-knot representing $T(k,hk\pm 1)$ with $h,k>0$ and $n>1$. Suppose that $f_w^n(t)$ is the Dunwoody polynomial associated to the cyclic presentation obtained by applying Theorem 7 in [4]. Then $f_w^n(t) \doteq \triangle_K(t)$ in $\mathbb{Z}[t]/(t^n-1)$ if $n > deg(\triangle_K(t))$, where $\doteq$ means equal up to units.*

For example, no more the result of Corollary 3.3 is true for $T(5,7)$ and $deg(\triangle_K(t)) = 24$. In fact, for $n=25$, the relation w is not principal because x_{23} and x_{24}^{-1} in w are equal to x_{-2} and x_{-1}^{-1} under $D_{25}(2,1,12,13,-5)$, respectively, that is, w is not independent of 25. Thus w is equal to the relation

$$x_0^{-1}x_1^1x_6^1x_{11}^1x_{16}^1x_{17}^{-1}x_{12}^{-1}x_7^{-1}x_8^1x_{13}^1x_{18}^1x_{-2}^1x_{-1}^{-1}x_{19}^{-1}x_{14}^{-1}x_{10}^{-1}x_5^{-1}$$

on $D_{25}(2,1,12,13,-5)$. In case of Corollary 3.3, we have to know the degree of $\triangle_K(t)$ in order to show that $f_w^n(t)$ is to be the Alexander polynomial of K. However, without the condition for the degree of $\triangle_K(t)$, we can show that $f_w^n(t)$ is to be the Alexander polynomial of the torus knot K(generally, the Dunwoody $(1,1)$-knot in $\mathbb{S}^3$) from properties itself.

As the main result of this section, we show that the Dunwoody polynomial is to be the Alexander polynomial. In other words we give the condition for n in order that $D_n(a,b,c,r,s)$ admits always the principal cyclic presentation.

Given p and q defined on $D(a,b,c,r)$ which is the Dunwoody $(1,1)$-decomposition determined by two permutations α and β such that $|\beta\alpha| = 2$, we recall that $D_n(a,b,c,r,s)$ satisfies $ps+q \equiv 0 \bmod n$ for some $n>1$ and $s \in \mathbb{Z}$. First of all, we define *a cyclic sequence* from $D_n(a,b,c,r,s)$ as follows. We set

$$A^+ = \{1,2,\cdots,a\},$$

$$B^+ = \{a+1,a+2,\cdots,a+b\},$$

$$C^+ = \{a+b+1,a+b+2,\cdots,a+b+c\},$$

$$E^+ = \{a+b+c+1, a+b+c+2, \cdots, a+b+c+a = d\},$$
$$A^- = \{-1, -2, \cdots, -a\},$$
$$C^- = \{-a-1, -a-2, \cdots, -a-c\},$$
$$B^- = \{-a-c-1, -a-c-2, \cdots, -a-c-b\}, \quad \text{and}$$
$$E^- = \{-a-c-b-1, -a-c-b-2, \cdots, -a-c-b-a = -d\}.$$

Then $A^+ \cup B^+ \cup C^+ \cup E^+ = X^+$ and $A^- \cup C^- \cup B^- \cup E^- = X^-$. For each $0 \le i \le n-1$, let C_i be the i-th meridian disk i of the Heegaard diagram $D_n(a,b,c,r,s)$ as in Figure 2, and $\bar{C}_i$ the i-th meridian disk $\bar{i}$ of the Heegaard diagram $D_n(a,b,c,r,s)$. For $0 \le i \le n-1$ and $1 \le j \le d$, a point (i,j) on $D_n(a,b,c,r,s)$ means the number j in i-th meridian disk C_i, and a point $(\bar{i}, -j)$ means the number $-j$ in i-th meridian disk $\bar{C}_i$, So if $(i,j) \in D_n(a,b,c,r,s)$ is a starting point at C_i, then $\theta^{n-1}(i,j) = (i+n-1, j)$ is a starting point at $\theta^{n-1}(C_i)$. We define the rules corresponding to i on $D_n(a,b,c,r,s)$ and α on $D(a,b,c,r)$ by

$$\tag{3.1} \begin{cases} (i,a) \to (i+1, \alpha(a)) & \text{if } a \in A^+ \\ (i,b) \to (\overline{i+s+1}, \alpha(b)) & \text{if } b \in B^+ \\ (i,c) \to (\overline{i+s}, \alpha(c)) & \text{if } c \in C^+ \\ (i,e) \to (i-1, \alpha(e)) & \text{if } e \in E^+ \end{cases}.$$

The rules corresponding to $\bar{i}$ on $D_n(a,b,c,r,s)$ and α on $D(a,b,c,r)$ are defined by

$$\tag{3.2} \begin{cases} (\bar{i}, -a) \to (\overline{i+1}, \alpha(-a)) & \text{if } -a \in A^- \\ (\bar{i}, -c) \to (i-s, \alpha(-c)) & \text{if } -c \in C^- \\ (\bar{i}, -b) \to (i-(s+1), \alpha(-b)) & \text{if } -b \in B^- \\ (\bar{i}, -e) \to (\overline{i-1}, \alpha(-e)) & \text{if } -e \in E^- \end{cases}.$$

Moreover, the identification between i-th meridian disk C_i and i-th meridian disk $\bar{C}_i$ on $D_n(a,b,c,r,s)$ is defined by

$$\tag{3.3} \begin{cases} (i,x) \to (\bar{i}, \beta(x)) & \text{if } x \in X^+ \\ (\bar{i}, -x) \to (i, \beta(-x)) & \text{if } -x \in X^- \end{cases}.$$

By the property of the n-fold strongly-cyclic branched covering space, we have the following.

Lemma 3.4. *Let $(0,1)$ be a starting point on $D_n(a,b,c,r,s)$ and $d = 2a+b+c$. Then we have $(\beta\alpha)^d(0,1) = (ps+q, 1)$.*

Proof. From α and β defined in Theorem 2.1, $\beta\alpha$ with length d determines the simple closed curve in $D(a,b,c,r)$ with the starting point $(0,1)$. The simple closed curve is lifted to n simple closed curves on $D_n(a,b,c,r,s)$ which is determined by (3.1), (3.2) and (3.3). For $0 \le i \le n-1$ and $1 \le j \le d$, if $(i,j) \in D_n(a,b,c,r,s)$ is a starting point of a curve of the n simple closed curves on $D_n(a,b,c,r,s)$, we have $(\beta\alpha)^d(i,j) = (ps+q+i,j)$ because of $ps+q \equiv 0 \bmod n$. In particular, let $(0,1)$ be a starting point on $D_n(a,b,c,r,s)$. Then the proof follows from the above result. □

For $0 \le i \le n-1$ and $1 \le j \le d$, if $(i,j) \in D_n(a,b,c,r,s)$ is a starting point, we have a sequence

$$(i,j) \to (\beta\alpha)(i,j) \to (\beta\alpha)^2(i,j) \cdots \to$$
$$(\beta\alpha)^{d-1}(i,j) \to (\beta\alpha)^d(i,j) = (ps+q+i,j).$$

The sequence from (i,j) to $(ps+q+i,j)$ determined by (3.1), (3.2) and (3.3) is called *a cyclic sequence* of $D_n(a,b,c,r,s)$. We note that the cyclic sequences of $D_n(a,b,c,r,s)$ are independent of the choice of the starting points on itself.

We now suppose that $(0,1)$ is a starting point which is the number 1 in 0-th meridian disk of $D_n(a,b,c,r,s)$. Since $1 \in A^+$ and $\alpha(1) = d$, we have $(0,1) \to (1,d)$ by (3.1). Since $r < d$, $\beta(d) = -d+r$ and so $(1,d) \to (\bar{1}, -d+r)$ by (3.3). Thus $(0,1) \to (\bar{1}, -d+r)$ under $\beta\alpha$, or $\beta\alpha(0,1) = (\bar{1}, -d+r)$. Applying repeated process, we obtain $(\beta\alpha)^d(0,1) = (ps+q,1)$ by Lemma 3.4. We define a relation

$$w = x_0 x^{\pm 1}_{\beta\alpha(0)} x^{\pm 1}_{(\beta\alpha)^2(0)} x^{\pm 1}_{(\beta\alpha)^3(0)} \cdots x^{\pm 1}_{(\beta\alpha)^{d-1}(0)}$$

on $D_n(a,b,c,r,s)$ induced by the cyclic sequence of $D_n(a,b,c,r,s)$, where

$$x^{\pm 1}_{(\beta\alpha)^k(0)} = \begin{cases} x_{(\beta\alpha)^k(0)} & \text{if } (\beta\alpha)^k(0) = i \\ x^{-1}_{(\beta\alpha)^k(0)} & \text{if } (\beta\alpha)^k(0) = \bar{i}. \end{cases}$$

If $(\beta\alpha)^k(0) = \overline{-i}$ and $(\beta\alpha)^k(0) = -i$, then $x^{\pm 1}_{(\beta\alpha)^k(0)} = x^{-1}_{-i}$ and $x^{\pm 1}_{(\beta\alpha)^k(0)} = x_{-i}$, respectively. We can assume that $(\beta\alpha)^k(0) \ge 0$ under mod n for all $1 \le k \le d$. For each $0 \le i \le n-1$, we note that $\theta^i(w)$ has the starting point $(i,1)$ and the final point $(ps+q+i,1)$ with $(\beta\alpha)^d(i,1) = (ps+q+i,1)$. Thus the abelianized word of w induces *the Dunwoody polynomial*

$$f^n_w(t) = t^0 \pm t^{\beta\alpha(0)} \pm t^{(\beta\alpha)^2(0)} \pm \cdots \pm t^{(\beta\alpha)^{d-1}(0)}$$

by substituting $-t^{-i}$ into x^{-1}_{-i}. If $|(\beta\alpha)^k(0)| < n$ for each $1 \le k \le d-1$, then w is the principal relation of the cyclic presentation of $D_n(a,b,c,r,s)$ and we have $deg(f^n_w(t)) = M^+ - M^-$ where $M^+ = \max_{0\le k\le d-1}\{(\beta\alpha)^k(0)\}$ and $M^- = \min_{0\le k\le d-1}\{(\beta\alpha)^k(0)\}$.

Suppose that $D(a,b,c,r)$ is the Dunwoody $(1,1)$-decomposition representing a $(1,1)$-knot K in $\mathbb{S}^3$. Since $p = \pm 1$ and $ps+q = 0$, $s = \mp q$. For each $n > 1$, there exists the Dunwoody 3-manifold represented by $D_n(a,b,c,r,s)$, which is the n-fold (strongly-) cyclic covering of $\mathbb{S}^3$ branched over K. By Lemma 3.4, the cyclic sequence of $D_n(a,b,c,r,s)$ has $(\beta\alpha)^d(0,1) = (0,1)$. Since each $\{C_i, \bar{C}_{i+s}\}$

in $D_n(a,b,c,r,s)$ is connected by c edges, the relation w is independent of n if $n > M^+ - M^- + |s|$. Therefore $f_w^n(t)$ is the Alexander polynomial of the Dunwoody $(1,1)$-knot K in $\mathbb{S}^3$ if $n > M^+ - M^- + |s|$. We remark that there is a natural way to obtain the Dunwoody polynomial $f_w^n(t) \in \mathbb{Z}[t]/(t^n - 1))$ associated to the Dunwoody $(1,1)$-knot K in $\mathbb{S}^3$ because of the uniqueness of s in $\mathbb{Z}_n$ with $ps + q \equiv 0 \bmod n$. Summarizing, we have proved the following.

Theorem 3.5. *Let $D(a,b,c,r)$ be the Dunwoody $(1,1)$-decomposition of $(\mathbb{S}^3, K)$, and α and β be two transpositions defined in Theorem 2.1. Let $w(x_0, \cdots, x_{n-1})$ be a relation induced by the cyclic sequence of $D_n(a,b,c,r,\mp q)$ for $n > 1$, $M^+ = \max_{0\le k\le d-1}\{(\beta\alpha)^k(0)\}$ and $M^- = \min_{0\le k\le d-1}\{(\beta\alpha)^k(0)\}$. Then $f_w^n(t)$ is the Alexander polynomial of K if $n > M^+ - M^- + |s|$.*

We give two canonical examples as follows.

Example 3.

$$\begin{array}{ccccc}
(0, & 1) & \xrightarrow{\alpha} & (1, & 9) \\
 & & \overset{\beta}{\swarrow} & & \\
(\bar{1}, & -4) & \xrightarrow{\alpha} & (8, & 7) \\
 & & \swarrow_{\beta} & & \\
(\bar{8}, & -2) & \xrightarrow{\alpha} & (\bar{9}, & -8) \\
 & & \swarrow_{\beta} & & \\
(9, & 4) & \xrightarrow{\alpha} & (\bar{3}, & -6) \\
 & & \swarrow_{\beta} & & \\
(3, & 2) & \xrightarrow{\alpha} & (4, & 8) \\
 & & \swarrow_{\beta} & & \\
(\bar{4}, & -3) & \xrightarrow{\alpha} & (11, & 6) \\
 & & \swarrow_{\beta} & & \\
(\bar{11}, & -1) & \xrightarrow{\alpha} & (\bar{12}, & -9) \\
 & & \swarrow_{\beta} & & \\
(12, & 5) & \xrightarrow{\alpha} & (\bar{6}, & -7) \\
 & & \swarrow_{\beta} & & \\
(6, & 3) & \xrightarrow{\alpha} & (\bar{0}, & -5) \\
 & & \swarrow_{\beta} & & \\
(0, & 1). & & &
\end{array}$$

Let $D(2,3,2,5)$ be a Dunwoody $(1,1)$-decomposition. Then we obtain $p = 1$ and $q = 7$ from an oriented curve on $D(2,3,2,5)$ defined in section 2. Since $p = 1$, it is representing a Dunwoody (1,1)-knot $K(2,3,2,5)$ in $\mathbb{S}^3$. Since $q = 7$, we have $s = -7$. Therefore, for all $n > 1$, there exists a Dunwoody 3-manifold represented by $D_n(2,3,2,5,-7)$. In order to show a principal relation on $D_n(2,3,2,5,-7)$ we need a cyclic sequence as above. For a Dunwoody $(1,1)$-decomposition $D(2,3,2,5)$, setting $A^+ = \{1,2\}$, $B^+ = \{3,4,5\}$, $C^+ = \{6,7\}$, $E^+ = \{8,9\}$, and $A^- = \{-1,-2\}$, $C^- = \{-3,-4\}$, $B^- = \{-5,-6,-7\}$, and $E^- = \{-8,-9\}$, then we have $A^+ \cup B^+ \cup$

$C^+ \cup E^+ = X^+$ and $A^- \cup C^- \cup B^- \cup E^- = X^-$. Let $(0,1)$ be the starting point on $D_n(2,3,2,5,-7)$. Then we have a cyclic sequence by applying (3.1), (3.2) and (3.3) as above. Thus the relation w induced by the above cyclic sequence is

$$w = x_0 x_1^{-1} x_8^{-1} x_9 x_3 x_4^{-1} x_{11}^{-1} x_{12} x_6$$

and the Dunwoody polynomial $f_w^n(t)$ is

$$f_w^n(t) = 1 - t + t^3 - t^4 + t^6 - t^8 + t^9 - t^{11} + t^{12}.$$

Since $M^+ = 12$ and $M^- = 0$, the relation w on $D_n(2,3,2,5,-7)$ is principal for $n > 19$. In fact, $f_w^n(t)$ is the Alexander polynomial of $K(2,3,2,5)$ representing $T(3,7)$, which can be obtained by considering the principal cyclic presentation of $D_{20}(2,3,2,5,-7)$.

Example 4.

$$\begin{array}{cccccc}
(0, & 1) & \xrightarrow{\alpha} & (1, & 9) \\
 & & \swarrow_{\beta} & & \\
(\bar{1}, & -3) & \xrightarrow{\alpha} & (-2, & 4) \\
 & & \swarrow_{\beta} & & \\
(\overline{-2}, & -7) & \xrightarrow{\alpha} & (-6, & 3) \\
 & & \swarrow_{\beta} & & \\
(\overline{-6}, & -6) & \xrightarrow{\alpha} & (-9, & 7) \\
 & & \swarrow_{\beta} & & \\
(\overline{-9}, & -1) & \xrightarrow{\alpha} & (\overline{-8}, & -9) \\
 & & \swarrow_{\beta} & & \\
(-8, & 6) & \xrightarrow{\alpha} & (\overline{-5}, & -5) \\
 & & \swarrow_{\beta} & & \\
(-5, & 2) & \xrightarrow{\alpha} & (-4, & 8) \\
 & & \swarrow_{\beta} & & \\
(\overline{-4}, & -2) & \xrightarrow{\alpha} & (\overline{-3}, & -8) \\
 & & \swarrow_{\beta} & & \\
(-3, & 5) & \xrightarrow{\alpha} & (\bar{0}, & -4) \\
 & & \swarrow_{\beta} & & \\
(0, & 1). & & &
\end{array}$$

Let $D(2,1,4,6)$ be a Dunwoody $(1,1)$-decomposition with $p=-1$ and $q=3$. Since $ps+q=0$, there exists a Dunwoody 3-manifold represented by $D_n(2,1,4,6,3)$ for all $n>1$. To obtain a principal relation for $D_n(2,1,4,6,3)$, we need a cyclic sequence as above. For $D(2,1,4,6)$, setting $A^+ = \{1,2\}$, $B^+ = \{3\}$, $C^+ = \{4,5,6,7\}$, $E^+ = \{8,9\}$, and $A^- = \{-1,-2\}$, $C^- = \{-3,-4,-5,-6\}$, $B^- = \{-7\}$, and $E^- = \{-8,-9\}$, then we have $A^+ \cup B^+ \cup C^+ \cup E^+ = X^+$ and $A^- \cup C^- \cup B^- \cup E^- = X^-$. By applying (3.1), (3.2) and (3.3), we have a cyclic sequence as above. Thus we have a relation

$$w = x_1 x_{-2} x_{-6} x_{-9} x_{-8}^{-1} x_{-5}^{-1} x_{-4} x_{-3}^{-1} x_0^{-1}.$$

Hence the Dunwoody polynomial is

$$f_w^n(t) = t^{-9}(1 - t + t^3 - t^4 + t^5 - t^6 + t^7 - t^9 + t^{10}).$$

Since $M^+ = 1$ and $M^- = -9$, the relation w on $D_n(2,1,4,6,3)$ is principal for $n > 13$, so $f_w^n(t)$ is the Alexander polynomial of $K(2,1,4,6)$ representing the pretzel knot $P(-2,3,7)$, which can be obtained by considering the principal cyclic presentation of $D_{14}(2,1,4,6,3)$.

The next corollaries are immediate consequences of the previous considerations.

Corollary 3.6. *Let $D(a,b,c,r)$ be the Dunwoody $(1,1)$-decomposition of $(\mathbb{S}^3, K)$. Suppose that α and β are two permutations defined by Theorem 2.1. Let $w(x_0, \cdots, x_k)$ be a relation induced by the cyclic sequence of $D_n(a,b,c,r,\mp q)$ for each $n > 1$, $M^+ = \max_{0\le k\le d-1}\{(\beta\alpha)^k(0)\}$ and $M^- = \min_{0\le k\le d-1}\{(\beta\alpha)^k(0)\}$, where $d = 2a + b + c$. Then $f_w^n(t)$ is the Alexander polynomial of K if n is the smallest positive integer n_0 such that $n_0 > M^+ - M^- + |s|$.*

Let $T(i,j)$ be the torus knot such that $2 \le i \le j$ and $j = \bar{j} + ik$ for some $k \in \mathbb{Z}$. Let $\bar{j} = \pm 1$, then the following $(\star)$ is the families of the Dunwoody 3-manifolds and their branched sets $T(i,j)$, where $i \ge 3$ and $k \ge 1$. Note that these are different with the families introduced in [4]. As applications of Theorem 3.5, for the Dunwoody $(1,1)$-knots representing $T(k_1,k_2)$ satisfying $k_2 \equiv \pm 1 \mod k_1$ as

$$T(i, ki+1) \;\leftrightarrow\; D_n(1, i-2, (i-1)+(k-1)(2i-2), (i-1)+(k-1)(2i-2), i)$$

$$T(i, (k+1)i-1) \;\leftrightarrow\; D_n(1, i-2, (3i-5)+(k-1)(2i-2), 3i-4, -i) \qquad (\star)$$

and $k_2 \equiv \pm 2 \bmod k_1$ as (2.1) or (2.2), we show their Alexander polynomial and certain invariant. In our case Corollary 3.3 can be modified as follows.

Corollary 3.7. *Let $K = K(a,b,c,r)$ be the Dunwoody $(1,1)$-knot as in $(\star)$ and $n > 1$. Then $f_w^n(t) \doteq \triangle_K(t)$ if $n > M^+ - M^- + |s|$, where $\doteq$ means equal up to units.*

For the Dunwoody $(1,1)$-knots satisfying (2.1) or (2.2), we have the following.

Corollay 3.8. *Let $K = K(a,b,c,r)$ be the Dunwoody $(1,1)$-knot representing $T(k_1,k_2)$ with $k_2 \equiv \pm 2 \bmod k_1$ as in (2.1) or (2.2). Then $f_w^n(t) \doteq \triangle_K(t)$ if $n > M^+ - M^- + |s|$, where $\doteq$ means equal up to units.*

We suppose that $K(a,b,c,r)$ is the Dunwoody $(1,1)$-knot representing $T(k_1,k_2)$ satisfying $k_2 \equiv \pm 1$ or $\pm 2 \bmod k_1$ as $(\star)$ or (2.1) and (2.2). Then the following shows that $d = 2a + b + c$ is an invariant for $K(a,b,c,r)$.

Theorem 3.9. *Let $T(k_1, k_2)$ be the torus knot with $k_2 \equiv \pm 2$ mod k_1 as in (2.1) or (2.2). Then d is an invariant of $T(k_1, k_2)$, where*

$$d = \begin{cases} k_1 + \frac{(k_1^2-1)(k_2-2)}{2k_1} & \text{if} \quad k_2 \equiv 2 \bmod k_1 \\ k_1 + \frac{k_1^2(k_2-2)-(k_2+2)}{2k_1} & \text{if} \quad k_2 \equiv -2 \bmod k_1 \end{cases}.$$

Proof. We suppose that $T(k_1, k_2)$ be the torus knot with $k_2 \equiv \pm 2$ mod k_1. Then the Dunwoody 3-manifold represented by $D_n(a, b, c, r, s)$ satisfies (2.1) or (2.2). Let $n > M^+ - M^- + |s|$. On $D_n(a, b, c, r, s)$, we have a principal relation w from a cyclic sequence by applying (3.1), (3.2) and (3.3). Thus the Dunwoody polynomial $f_w(t)$ of degree $M^+ - M^-$ is the Alexander polynomial of $T(k_1, k_2)$, and $M^+ - M^- = (k_1 - 1)(k_2 - 1)$. Since the length of w is d, the number of terms of $\triangle(k_1, k_2)$ is $d = 2a + b + c$. Therefore d is an invariant of $T(k_1, k_2)$. □

For $(\star)$, the similar argument can be applied as the following.

Corollary 3.10. *Let $T(i, j)$ be the torus knot with $3 \le i \le j$ and $j = ki \pm 1$ for some $k \ge 1$ in $\mathbb{Z}$. Then d is an invariant of $T(i, j)$, where*

$$d = \begin{cases} (2i-1) + (k-1)(2i-2) & \text{if} \ \ j = ki + 1 \\ (4i-5) + (k-1)(2i-2) & \text{if} \ \ j = ki - 1 \end{cases}.$$

We recall that if $\triangle_K^n(t) \in \mathbb{Z}[t]/(t^n - 1)$ is the projection of the Alexander polynomial of $K = K(a, b, c, r)$, then there is a connection between $f_w^n(t)$ and $\triangle_K^n(t)$, which follows from the result of Theorem 4.1 in [3].

Corollary 3.11. *Let $K = K(a, b, c, r)$ be a $(1,1)$-knot in the lens space $L(p, q')$ and $H_1(L(p, q') - K) = \mathbb{Z} \oplus \mathbb{Z}_{\bar{d}}$, where $\bar{d} = gcd(p, q)$. Then for each $n > 1$ such that $gcd(n, p) = 1$, we have*

$$f_w^n(t^{p/\bar{d}}) = \triangle_K^n(t) \cdot \frac{(t^{p/\bar{d}} - 1)}{(t-1)}$$

up to units of $\mathbb{Z}[t]/(t^n - 1)$.

In Corollary 3.11, the cyclotomic polynomial

$$\frac{(t^{p/\bar{d}} - 1)}{(t-1)} = 1 + t + t^2 + \cdots + t^{p/\bar{d}-1}$$

is *irreducible polynomial* if $p/\bar{d}$ is prime. Let $n > 1$ and $gcd(p, n) = 1$. Then the following example explains one way to obtain the Alexander polynomial of $K(a, b, c, r)$ in $L(p, q')$ from the Dunwoody polynomial on $D_n(a, b, c, r, s)$ with $ps + q \equiv 0$ mod n.

Example 5. Let $D(1, 5, 0, 6)$ be a $(1,1)$-decomposition with $p = 5$ and $q = 7$ and

$K = K(1,5,0,6)$ a $(1,1)$-knot in the lens space $L(5,1)$. Then there is a unique $s \in \mathbb{Z}_{12}$ such that $5s+7 \equiv 0 \bmod 12$. On $D_{12}(1,5,0,6,1)$ we have a principal relation

$$w = x_0 x_1^{-1} x_2 x_4 x_6 x_8 x_{10}$$

induced by a cyclic sequence as follow:

$$\begin{array}{ccccc}
(0, & 1) & \xrightarrow{\alpha} & (1, & 7) \\
 & & \swarrow_{\beta} & & \\
(\bar{1}, & -1) & \xrightarrow{\alpha} & (\bar{2}, & -7) \\
 & & \swarrow_{\beta} & & \\
(2, & 6) & \xrightarrow{\alpha} & (\bar{4}, & -6) \\
 & & \swarrow_{\beta} & & \\
(4, & 5) & \xrightarrow{\alpha} & (\bar{6}, & -5) \\
 & & \swarrow_{\beta} & & \\
(6, & 4) & \xrightarrow{\alpha} & (\bar{8}, & -4) \\
 & & \swarrow_{\beta} & & \\
(8, & 3) & \xrightarrow{\alpha} & (\bar{10}, & -3) \\
 & & \swarrow_{\beta} & & \\
(10, & 2) & \xrightarrow{\alpha} & (\bar{12}, & -2) \\
 & & \swarrow_{\beta} & & \\
(12, & 1). & & &
\end{array}$$

Thus we obtain the Dunwoody polynomial

$$\begin{aligned}
f_w^{12}(t) &\doteq 1 + t^2 + t^4 + t^6 + t^8 + t^{10} - t^{-1} \\
&\doteq 1 + t^2 + t^4 + t^6 + t^8 + t^{10} - t^{11}.
\end{aligned}$$

By Corollary 3.11, putting $t^{p/\bar{d}} = t^5$, we have

$$\begin{aligned}
f_w^{12}(t^5) &\doteq 1 + t^{10} + t^{20} + t^{30} + t^{40} + t^{50} - t^{55} \\
&\doteq t^{-10}(1 + t^4 - t^5 + t^6 + t^2 + t^8 + t^{10}) \\
&= (1 - t + t^2 - t^3 + t^4 - t^5 + t^6) \\
&\quad \cdot (1 + t + t^2 + t^3 + t^4) \\
&= (1 - t + t^2 - t^3 + t^4 - t^5 + t^6)\frac{(t^5 - 1)}{(t - 1)}.
\end{aligned}$$

and so $\triangle_K^{12} = 1 - t + t^2 - t^3 + t^4 - t^5 + t^6$ is the Alexander polynomial of the $(1,1)$-knot $K(1,5,0,6)$, where the multiplication for t^{10} requires condition $n > 10$.

Indeed we have the same result from $D_{22}(1,5,0,6,3)$. However the Dunwoody polynomial induced by $D_7(1,5,0,6,0)$ does not give the Alexander polynomial of the $(1,1)$-knot $K(1,5,0,6)$ because of $7 < 10$. In other words, the Dunwoody polynomial induced by $D_7(1,5,0,6,0)$ is not the Alexander polynomial of the $(1,1)$-knot $K(1,5,0,6)$. From this example, we may have the possibility that the Alexander polynomial of the Dunwoody $(1, 1)$-knot in a lens space can be obtained from the Dunwoody polynomial.

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