# On the Pósa-Seymour Conjecture

János Komlós,[1] Gábor N. Sárközy, [2]
and Endre Szemerédi[3]

[1] *DEPT. OF MATHEMATICS, RUTGERS UNIVERSITY, NEW BRUNSWICK, NJ 08903*

[2] *DEPT. OF COMPUTER SCIENCE, WORCESTER POLYTECHNIC INSTITUTE, WORCESTER, MA 01609*

[3] *DEPT. OF COMPUTER SCIENCE, RUTGERS UNIVERSITY, NEW BRUNSWICK, NJ 08903*

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**Abstract:** Paul Seymour conjectured that any graph $G$ of order $n$ and minimum degree at least $\frac{k}{k+1}n$ contains the $k^{th}$ power of a Hamilton cycle. We prove the following approximate version. For any $\epsilon > 0$ and positive integer $k$, there is an $n_0$ such that, if $G$ has order $n \geq n_0$ and minimum degree at least $(\frac{k}{k+1} + \epsilon)n$, then $G$ contains the $k^{th}$ power of a Hamilton cycle. © 1998 John Wiley & Sons, Inc. J Graph Theory 29: 167–176, 1998

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## 1. INTRODUCTION

### 1.1. Notations and Definitions

For basic graph concepts see the monograph of Bollobás [1].

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$+$ will sometimes be used for disjoint union of sets. $V(G)$ and $E(G)$ denote the vertex-set and the edge-set of the graph $G$. $(A, B, E)$ denotes a bipartite graph $G = (V, E)$, where $V = A + B$, and $E \subset A \times B$. For a graph $G$ and a subset $U$ of its vertices, $G|_U$ is the restriction to $U$ of $G$. $N(v)$ is the set of neighbors of $v \in V$. Hence, $|N(v)| = \deg(v) = \deg_G(v)$, the degree of $v$. $\delta(G)$ stands for the minimum, and $\Delta(G)$ for the maximum degree in $G$. For $A \subset V(G)$, we write $N(A) = \cap_{v \in A} N(v)$, the set of common neighbors. $N(x, y, z, \cdots)$ is shorthand for $N(\{x, y, z, \cdots\})$. When $A, B$ are disjoint subsets of $V(G)$, we denote by $e(A, B)$ the number of edges of $G$ with one endpoint in $A$ and the other in $B$. In particular, we write $\deg(v, U) = e(\{v\}, U)$ for the number of edges from $v$ to $U$. For nonempty $A$ and $B$,

$$d(A, B) = \frac{e(A, B)}{|A||B|}$$

is the **density** of the graph between $A$ and $B$.

**Definition 1.** *The bipartite graph $G = (A, B, E)$ is $\epsilon$-**regular** if*

$$X \subset A, Y \subset B, |X| > \epsilon|A|, |Y| > \epsilon|B| \text{ imply } |d(X, Y) - d(A, B)| < \epsilon,$$

*otherwise it is $\epsilon$-irregular*.
We will often say simply that "the pair $(A, B)$ is $\epsilon$-regular" with the graph $G$ implicit.

**Definition 2.** *$(A, B)$ is $(\epsilon, \delta)$-**super-regular** if it is $\epsilon$-regular and*

$$\deg(a) > \delta|B| \text{ for all } a \in A, \deg(b) > \delta|A| \text{ for all } b \in B.$$

## 1.2. Powers of Cycles

The $k^{th}$ **power** of a graph $G$ is the graph obtained from $G$ by joining every pair of vertices with distance at most $k$ in $G$. We will write $C^k$ and $P^k$ for the $k^{th}$ power of a cycle and a path.

Let $G$ be a graph on $n \geq 3$ vertices. A classical result of Dirac [2] (see also [1]) asserts that if $\delta(G) \geq n/2$, then $G$ contains a Hamilton cycle. As a natural generalization of Dirac's theorem, Pósa conjectured the following in 1962.

**Conjecture 1 (Pósa).** *Let $G$ be a graph on $n$ vertices. If $\delta(G) \geq \frac{2}{3}n$, then $G$ contains the square of a Hamilton cycle*.

Later, in 1974, Seymour [13] generalized this conjecture.

**Conjecture 2 (Seymour).** *Let $G$ be a graph on $n$ vertices. If $\delta(G) \geq \frac{k}{k+1}n$, then $G$ contains the $k^{th}$ power of a Hamilton cycle.*

Seymour indicated the difficulty of the conjecture by observing that the truth of this conjecture would imply the notoriously difficult Hajnal–Szemerédi theorem [10] (see below).

The problem has received significant attention lately. In the direction of Conjecture 1, first Jacobson (unpublished) showed that if $\delta(G) \geq \frac{5}{6}n$, then the conclusion of the conjecture holds. Faudree, Gould, Jacobson, and Schelp [8] confirmed the conclusion with $\delta(G) \geq (\frac{3}{4}+\epsilon)n + C(\epsilon)$. Later the same authors improved this to $\delta(G) \geq \frac{3}{4}n$. By using a result in [9], Häggkvist (unpublished) gave a very simple proof for the case $\delta(G) \geq \frac{3}{4}n+1$ and $n \equiv 0 \pmod 4$. Fan and Häggkvist [3] lowered the bound to $\delta(G) \geq \frac{5}{7}n$. Fan and Kierstead [4] improved this further to $\delta(G) \geq \frac{17n+9}{24}$, and Faudree, Gould, and Jacobson [7] to $\delta(G) \geq \frac{7}{10}n$. Finally, Fan and Kierstead [5] improved the condition to the almost optimal $\delta(G) \geq (\frac{2}{3}+\epsilon)n+C(\epsilon)$. (They also announced that the same holds with $\epsilon = C = 0$, if one wants only the square of a Hamilton *path*.)

However, for the general Conjecture 2, the only result available is in the above-mentioned article of Faudree et al. [8], which states that for any $\epsilon > 0$ and positive integer $k$ there is a $C(\epsilon, k)$ such that, if an $n$-graph $G$ satisfies

$$\delta(G) \geq \left(\frac{2k-1}{2k} + \epsilon\right) n + C(\epsilon, k),$$

then $G$ contains the $k^{th}$ power of a Hamilton cycle.

Here we prove the following approximate version of the Seymour conjecture.

**Theorem 1.** *For any $p > 0$ and positive integer $k$ there is an $n_0 = n_0(p,k)$ such that, if $G$ has order $n \geq n_0$ and minimal degree*

$$\delta(G) \geq \left(\frac{k}{k+1} + p\right) n, \tag{1}$$

*then $G$ contains the $k^{th}$ power of a Hamilton cycle.*

## 2. MAIN TOOLS

In the proof, the Regularity Lemma of the third author plays a central role. Here we will use the following variation of the lemma.

**Lemma 1 (Regularity Lemma—Degree form).** *For every $\epsilon > 0$ there is an $M = M(\epsilon)$ such that if $G = (V, E)$ is any graph and $d \in [0, 1]$ is any real number, then there is a partition of the vertex-set $V$ into $l + 1$ sets (so-called clusters) $V_0, V_1, \ldots, V_l$, and there is a subgraph $G' = (V, E')$ with the following properties:*

- $l \leq M$,
- $|V_0| \leq \epsilon |V|$,
- *all clusters $V_i, i \geq 1$, are of the same size $L \leq \lceil \epsilon |V| \rceil$.*
- $\deg_{G'}(v) > \deg_G(v) - (d+\epsilon)|V|$ *for all $v \in V$,*
- $G'|_{V_i} = \emptyset$ *($V_i$ are independent in $G'$),*
- *all pairs $G'|_{V_i \times V_j}, 1 \leq i < j \leq l$, are $\epsilon$-regular, each with a density 0 or exceeding $d$.*

This form can easily be obtained by applying the original Regularity Lemma (with a smaller value of $\epsilon$), adding to the exceptional set $V_0$ all clusters incident to many irregular pairs, and then deleting all edges between any other clusters where the edges either do not form a regular pair, or they do but with a density at most $d$.

Our other main tool is a coloring theorem of Hajnal and Szemerédi, which says that every graph with $n$ vertices and maximum degree $\Delta(G) \leq k$ is $(k+1)$-colorable with all color classes of size $\lfloor n/(k+1) \rfloor$ or $\lceil n/(k+1) \rceil$. We have already pointed out the close connection between Seymour's problem and the Hajnal–Szemerédi theorem, namely, the truth of Conjecture 2 would imply the latter theorem. We use the theorem in the following complementary form.

**Lemma 2 (Hajnal, Szemerédi [10]).** *Let $G$ be a graph on $n = sk$ vertices. If $\delta(G) \geq \frac{k-1}{k} n$ then $G$ contains $s$ vertex-disjoint cliques of order $k$.*

## 3. OUTLINE OF THE PROOF

The proof borrows many elements from [11].

We will assume throughout the article that $n$ is sufficiently large. We apply the Regularity Lemma (Lemma 1) for $G$, with $d = p/3$ and $\epsilon$ small enough, compared to $p$, to get a partition of $V = \cup_{0 \leq i \leq l} V_i$ and a subgraph $G'$ as described in the lemma. We will assume that the number of clusters is divisible by $k+1$:

$$l = s(k+1)$$

(by adding a few clusters to the exceptional set $V_0$, if necessary). We define the following so-called **reduced graph R:** The vertices of $\mathbf{R}$ are the clusters $V_i, i \geq 1$, in the partition and there is an edge between two clusters if they form an $\epsilon$-regular pair in $G'$ with density exceeding $d$. Since $G'$ still satisfies condition (1) with $p$ replaced by $p/2$, an easy calculation shows that each cluster $V_i, i \geq 1$, has a degree (in $\mathbf{R}$) at least

$$\left( \frac{k}{k+1} + d \right) l. \tag{2}$$

Let us apply Lemma 3 for $\mathbf{R}$ to get a covering of $V(\mathbf{R})$ by vertex disjoint cliques of order $k+1$. Denote these cliques by $K_1, K_2, \ldots, K_s$.

In each clique $K_i$ we take an arbitrary ordering of the $k+1$ clusters, and we denote the clusters in this order by $V_1^i, V_2^i, \ldots, V_{k+1}^i$. We think of this sequence as a cycle of length $k+1$, where we have all the possible chords.

The rough idea of the proof is the following: We find the $k^{th}$ power of a path in $K_1$ by going around the cycle as many times as we can. Then we connect this path to $K_2$ with the use of a few extra vertices, then find the $k^{th}$ power of a path in $K_2$, etc. However, for technical reasons we will start with constructing the connecting paths between the subsequent cliques (for the last one $K_s$ the next one is $K_1$). This will be the first part of the proof. In the second part, we will take care of the exceptional vertices and make some adjustments to achieve that the distribution of the vertices inside each clique is perfect, i.e., there are the same number of vertices in each cluster of the clique. Finally, in the last part of the proof, we string the vertices inside each clique into the $k^{th}$ power of a path.

### 3.1. Connecting the Cliques

We are repeatedly going to use the following fact, which is a consequence of (2).

**Fact 1.** *Let $V_1, V_2, \ldots, V_{k+1}$ be $k+1$ arbitrary clusters in $\mathbf{R}$. Then*

$$|N_{\mathbf{R}}(V_1, V_2, \ldots, V_{k+1})| \geq (k+1)dl.$$

In other words, every set of $k+1$ clusters has a large common neighborhood set in $\mathbf{R}$.

We construct the connecting path between $K_1$ and $K_2$; the remaining $s-1$ connecting paths are constructed in exactly the same way. First, we determine the sequence of clusters from which the connecting path will use vertices. This sequence will be the square of a path in $\mathbf{R}$ (however, it will not be a simple path). Our goal is to define a sequence of cliques of size $k+1$ in $\mathbf{R}$

$$K^0, K^1, \ldots, K^t \tag{3}$$

with the following properties:

- $K^0 = K_1, K^t = K_2$,
- $|K^{i+1} \cap K^i| = k$ for every $0 \leq i \leq t-1$,
- $t = O(k^2)$.

For this purpose, if $K$ and $K'$ are two cliques of size $k+1$, for every cluster $C$ in $\mathbf{R} \backslash (K \cup K')$, we determine a label $\ell_{K,K'}(C) = (a,b), 0 \leq a \leq k+1, 0 \leq b \leq k+1$ in the following way:

$$a = \deg_{\mathbf{R}}(C, K) \text{ and } b = \deg_{\mathbf{R}}(C, K').$$

We are going to use the following fact.

**Fact 2.** *The number of clusters $C$ with $\ell_{K,K'}(C) = (k+1, k)$ or $(k, k+1)$ is at least $(k+1)dl$.*

We construct the sequence in (3) in two steps. First, we construct two sequences of cliques of size $k+1 : A_1, A_2, \ldots, A_{t_1}$ and $B_1, B_2, \ldots, B_{t_2}$ with the following properties:

**(a)** $A_1 = K_1, B_1 = K_2$,
**(b)** $|A_{i+1} \cap A_i| = k, |B_{j+1} \cap B_j| = k$ for every $0 \leq i \leq t_1 - 1, 0 \leq j \leq t_2 - 1$,
**(c)** either

$$\deg_{\mathbf{R}}(C, A_{t_1}) \geq k \text{ for every } C \in B_{t_2}, \tag{4}$$

or

$$\deg_{\mathbf{R}}(C', B_{t_2}) \geq k \text{ for every } C' \in A_{t_1}. \tag{5}$$

We construct these two sequences in the following way. $A_1 = K_1, B_1 = K_2$ and assume that $A_1, \ldots, A_i$ and $B_1, \ldots, B_j$ are already constructed, but (c) does not hold for $A_i$ and $B_j$. Using Fact 2 we can find a cluster $C$ with $\ell_{A_i,B_j}(C) = (k, k+1)$ or $(k+1, k)$, say $(k, k+1)$. In $B_j$ there must be a cluster $C'$ with $\deg_{\mathbf{R}}(C', A_i) \leq k-1$. To get $B_{j+1}$, we remove $C'$ from $B_j$ and we add $C$. We repeat this procedure until (c) holds. From the fact that one application of this procedure strictly increases the number of edges in $\mathbf{R}$ between the two cliques, it follows that, in at most $(k+1)^2 = O(k^2)$ steps, we can achieve (c). Thus, $t_1 + t_2 = O(k^2)$. In the second step of the construction of the clique sequence in (3), we construct a clique sequence

$$D_0, D_1, \ldots, D_{k+1},$$

which forms a gradual transition between $A_{t_1}$ and $B_{t_2}$. More precisely, first we assume that in (c) (4) holds. Then we will have $D_i \subset A_{t_1} \cup B_{t_2}$ and $|D_i \cap B_{t_2}| = i$. The construction is the following. $D_0$ is just $A_{t_1}$. To get $D_1$, we add $V_1^2$ to $D_0$ and we remove the cluster from $D_0 = A_{t_1}$, which is not adjacent to $V_1^2$ (if there is any). If all the clusters in $D_0$ are adjacent to $V_1^2$, then we remove an arbitrary cluster from $D_0$. In general, to get $D_{i+1}$ from $D_i, 0 \leq i \leq k$, we add $V_{i+1}^2$ to $D_i$ and remove the cluster from $D_i \cap D_0$, which is not adjacent to $V_{i+1}^2$ (if there is any). If all the clusters in $D_i \cap D_0$ are adjacent to $V_{i+1}^2$, then we remove an arbitrary cluster from $D_i \cap D_0$. Second, if in (c) (5) holds, then we do the same procedure backwards, we construct the gradual transition backwards from $B_{t_2}$ (starting with $D_{k+1}$) to $A_{t_1}$ (ending at $D_0$). (As a referee noticed, here, we used a somewhat abusive notation when added $V_1^2, V_2^2, \cdots$ to the sets $D_0, D_1, \ldots$, since here clusters from $B_{t_2}$ were used, and they are not necessarily numbered the same way as the corresponding ones in $B_1 = K_2$.)

The desired clique sequence in (3) is obtained in the following way:

$$A_1, A_2, \ldots, A_{t_1}, D_1, D_2, \ldots, D_k, B_{t_2}, \ldots, B_1.$$

For this sequence we use the notation in (3), so $t = t_1 + t_2 + k = O(k^2)$.

We get the sequence of clusters from which the connecting path will use vertices in the following way. We start by going around $K^0 = K_1$, so by $V_1^1, V_2^1, \ldots, V_{k+1}^1$, then we start a second cycle and we stop at the last cluster before the cluster in $K^0 \setminus K^1$. The next cluster is the cluster in $K^1 \setminus K^0$, then we go around $K^1$ once and, in the second cycle, we stop at the last cluster before the cluster in $K^1 \setminus K^2$. The next cluster is the cluster in $K^2 \setminus K^1$, etc. We continue in this fashion, and we get a sequence of clusters (note that this sequence contains repetitions):

$$C_1, C_2, \ldots, C_{t'},$$

where $C_i = V_i^1$ for $1 \leq i \leq k+1$, the last $k+1$ clusters are the clusters $V_1^2, V_2^2, \ldots, V_{k+1}^2$ in some permutation and $t' = O(k^2)$.

However, for technical reasons we would like to end the sequence with $V_1^2, V_2^2, \ldots, V_{k+1}^2$ in this order. For this purpose, it is obviously sufficient to show that if $V_1, V_2, \ldots, V_{k+1}$ is an arbitrary permutation of $K_2$, then we can change the order to

$$V_1, \dots, V_{i-1}, V_j, V_{i+1}, \dots, V_{j-1}, V_i, V_{j+1}, \dots, V_{k+1}$$

for any $1 \le i < j \le k+1$.

We do the following. Using Fact 1, we find a cluster $C \in N_{\mathbf{R}}(V_1, V_2, \dots, V_{k+1})$. The sequence of clusters is then the following:

$$V_1, \dots, V_{k+1}, V_1, \dots, V_{i-1}, C, V_{i+1}, \dots, V_{k+1}, V_1, \dots, V_{i-1}, C, V_{i+1}, \dots, V_{j-1},$$

$$V_i, V_{j+1}, \dots, V_{k+1}, V_1, \dots, V_{i-1}, C, V_{i+1}, \dots, V_{j-1}, V_i, V_{j+1}, \dots, V_{k+1},$$

$$V_1, \dots, V_{i-1}, V_j, V_{i+1}, \dots, V_{j-1}, V_i, V_{j+1}, \dots, V_{k+1},$$

as desired.

Thus, we may assume that we have a sequence of clusters

$$C_1, C_2, \dots, C_{t''},$$

which form the square of a path in $\mathbf{R}$, and where $C_i = V_i^1$ for $1 \le i \le k+1$ and $C_{t''-i} = V^2_{(k+1)-i}$ for $1 \le i \le k+1$ with $t'' = O(k^2)$. We also define

$$C_0 = V^1_{k+1}, C_{-1} = V^1_k, C_{-2} = V^1_{k-1}, \dots, C_{-k+1} = V^1_2,$$

and similarly,

$$C_{t''+1} = V_1^2, C_{t''+2} = V_2^2, \dots, C_{t''+k} = V_k^2.$$

Now we choose a vertex $p_i$ from each cluster $C_i, 1 \le i \le t''$, such that $p_i$ is connected to all $p_j$ with $1 \le |j - i| \le k$. They will also have the following additional properties for all $i, 1 \le i \le k$:

$$|N(p_1, p_2, \dots, p_i) \cap C_{i-k}| > (d-\epsilon)^i L$$

$$|N(p_{t''}, p_{t''-1}, \dots, p_{t''+1-i}) \cap C_{t''+1+k-i}| > (d-\epsilon)^i L, \tag{6}$$

which ensure that they can later be extended to the $k$th power of a Hamilton cycle of $G'$. We will select them one-by-one. We maintain $t'' + 2k$ sets $H_{i,j}$ from which the points will be selected. We start with $H_{0,j} = C_j, 1 - k \le j \le t'' + k$.

Then, when selecting the point $p_i$ from $H_{i-1,i}, 1 \le i \le t''$, we choose one with the following property:

$$\deg(p_i, H_{i-1,j}) > (d-\epsilon)|H_{i-1,j}| \text{ for all } j \ne i, |j-i| \le k.$$

This holds for all but at most $2k\epsilon|C_i|$ vertices in $H_{i-1,i}$, so we can choose such a $p_i \in H_{i-1,i}$. (Here we used that $(d-\epsilon)^k > \epsilon$.)

Then we update the sets $H$ as follows:

$$H_{i,j} = \begin{cases} H_{i-1,j} \cap N(p_i) & \text{if } 1 \le |j-i| \le k \\ H_{i-1,j} \setminus \{p_i\} & \text{otherwise.} \end{cases}$$

Note that we did not choose any points from the sets $H_{t'',j}$ for $j < 1$ and $j > t''$; this selection will be done later.

### 3.2. Adjustments and the Handling of the Exceptional Vertices

We already have an exceptional set $V_0$ of vertices in $G'$. We add some more vertices to $V_0$. From a cluster $V_j^i$ in a clique $K_i$ we remove all vertices $v$ for which there exists a $j'$ with $1 \leq j' \leq k+1, j' \neq j$ such that

$$\deg(v, V_j^i) \leq d|V_{j'}^i|.$$

$\epsilon$-regularity guarantees that at most $k\epsilon|V_j^i| \leq k\epsilon L$ such vertices exist in each cluster $V_j^i$.

We may have a small discrepancy among the number of remaining vertices in each clique $K_i$ (we removed some for the connecting paths, and some in the last step). By removing extra vertices from certain clusters (and putting them into the exceptional set $V_0$) we achieve that each cluster has exactly $L'$ vertices. (We will still use the notation $V_0$ for the enlarged exceptional set.) We still have $|V_0| \leq 2k\epsilon n$.

Next we take care of the vertices in $V_0$. For each vertex $v \in V_0$, we find all $K_i$-s such that

$$\deg(v, V_j^i) \geq d|V_j^i| \text{ for all } j \in \{1, \dots, k+1\}.$$

Equation (1) easily shows that we have at least $kps$ such cliques for each $v \in V_0$. We assign each $v \in V_0$ to one of these cliques in such a way that we do not assign too many vertices to a particular clique. A simple greedy algorithm leads to an assignment in which no clique is assigned more than $\epsilon_1 L'$ vertices provided that $|V_0| \leq \epsilon_1 L' kps$, which holds for $\epsilon_1 = 3\epsilon(k+1)/p$. Now let us take an arbitrary vertex $v$ assigned to $K_i$. We will add $v$ to the connecting path between $K_{i-1}$ and $K_i$ by also using some vertices from $K_i$ in such a way that the extended path is still extendable to the $k^{th}$ power of a Hamilton cycle, and we use the same number of vertices from each cluster in $K_i$ (in fact, exactly three from each cluster):

Let us denote the connecting path between $K_{i-1}$ and $K_i$ by $P_1, P_2, \dots, P_{k'}$. We extend this path in essentially the same way as in the previous section with vertices $P_{k'+1}, P_{k'+2}, \dots, P_{k''}$, where $k'' = k' + 3(k+1) + 1$. We go around the clusters of the clique three times. The only change in the procedure described in the previous section is that the new points $P_j, k'+1 \leq k''$ should have the additional property

$$|V_\ell \cap N(v) \cap \{N(P_j) : k'+1 \leq j \leq k'', P_j \notin V_\ell\}| > d^{3(k+1)+1} L' \text{ for each } V_\ell \in K_i.$$

This guarantees that the new vertex $v$ can be added as $P_{k'+2(k+1)+1}$, and the previous and next $k$ vertices can be chosen from $N(v)$.

Thus, we are left with the following situation: In each clique $K_i$ we have the same number of remaining vertices in each cluster $V_1^i, V_2^i, \dots, V_{k+1}^i$. On the connecting path between $K_{i-1}$ and $K_i$, the last $k$ vertices have many common neighbors in $V_1^i$, the last $k-1$ vertices have many common neighbors in $V_2^i$, etc., and finally the last vertex has many neighbors in $V_k^i$. On the connecting path between $K_i$ and $K_{i+1}$, the first $k$ vertices have many common neighbors in $V_{k+1}^i$, the first $k-1$ vertices many common neighbors in $V_k^i$, etc., and finally the first vertex has many neighbors in $V_2^i$. These properties guarantee that in the next section we can close the $k^{th}$ power of a Hamilton cycle inside each clique.

### 3.3. Building the $k^{th}$ Power of Hamilton Paths within the Cliques

The fact that we can close the $k^{\text{th}}$ power of a Hamilton cycle within each clique is an easy consequence of the following lemma (and the remark after that), which in turn is a special case of the so-called Blow-up Lemma [12].

**Lemma 3.** *For every $\delta > 0$ and positive integer $k$ there is an $\epsilon > 0$ such that the following holds for any positive integer $L$. Let us construct a graph $G$ by replacing each vertex $v_i$ of the complete graph $K$ on $k + 1$ vertices by an $L$-set $V_i$, and replacing the edges of $K$ with $(\epsilon, \delta)$-super-regular pairs (possibly different pairs for different edges). Then $G$ contains $P^k$, the $k^{\text{th}}$ power of a Hamilton path.*

The problem addressed in Lemma 3 is really part of a bigger embedding problem, and, thus, the following remark is needed to guarantee that the first and last constant number of vertices in $P^k$ connect appropriately to the rest of the graph (see the end of the previous subsection).

**Remark.** (See the remark at the end of [12].) *The statement of Lemma* 3 *remains true if we a priori restrict the location of the first and last $m$ points $p$ of $P^k$ to some sets $S_p$ as follows: Each $S_p$ has size at least $\gamma L$, any one $S_p$ is a subset of one cluster $V_i$, the sets $S_p$ belong to the right clusters $V_i$(that is, such a restricted $P^k$ should exist at least within the complete $k$-partite graph with color-classes $V_i$), and the parameter $\epsilon$ is small enough also in terms of the constants $\gamma > 0$, and $m$.*

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