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ON THE POSITIVE SOLUTIONS OF SEMILINEAR EQUATIONS $\Delta u + \lambda u - hu^p = 0$ ON THE COMPACT MANIFOLDS

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ABSTRACT. In this paper, we study the existence, nonexistence, and uniqueness of positive solutions of semilinear equations $\Delta u + \lambda u - hu^p = 0$ on compact Riemannian manifolds as well as on bounded smooth domains in R^n with homogeneous Dirichlet or Neumann boundary conditions.

1. INTRODUCTION

In this paper, we study the existence of positive solutions of the semilinear equation

$$\Delta u + \lambda u - hu^p = 0$$

on compact Riemannian manifolds as well as on bounded smooth domains in R^n with homogeneous Dirichlet or Neumann boundary conditions.

Analysis on Riemannian manifolds is a field currently undergoing great development. Analysis proves to be a very powerful tool for solving geometric problems (see e.g. [1]). A basic problem in Riemannian geometry is to determine what curvatures a given manifold can possess.

In this paper we shall limit our discussions to a compact connected smooth n-dimensional manifold without boundary, $n \geq 3$. (Throughout, M will always denote an n-dimensional compact connected Riemannian manifold.) Since we consider several Riemannian metrics on the same manifold M, we denote by (M, g) the Riemannian manifold with metric g. In the tangent space T_P at a point P on M, the Riemannian metric g defines an inner product $g(X, Y)$ of two vectors X and Y on T_P, and the angle θ between X and Y is given by

$$\cos\theta = \frac{g(X, Y)}{\sqrt{g(X, X)}\sqrt{g(Y, Y)}}.$$

Let there be given two metrics g and g^* on M. If the angles between two vectors with respect to g and g^* are always equal to each other at each point of the manifold, we say that g is pointwise conformal to g^*. A necessary and sufficient condition for g to be pointwise conformal to g^* of M is that there exists a function $\rho > 0$ on M such that $g^* = \rho g$ (see e.g. [2]).

Now let (M, g) be a Riemannian manifold of dimension ≥ 3 with scalar curvature k and let K be a given function on M. One may ask the question:

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Can we find a new metric g^* on M such that K is the scalar curvature of g^* and g^* is pointwise conformal to g (i.e., $g^* = u^{\frac{4}{n-2}} g$ for some $u > 0$ on M)? This is equivalent to the problem of finding positive solutions of the equation (see e.g. [4, Chapter 6; 11])

$$\frac{4(n-1)}{n-2}\Delta u - ku + K u^{\frac{n+2}{n-2}} = 0,$$

where Δ is the Laplace-Beltrami operator (simply say Laplacian) in the g metric.

Yamabe [5] attempted to show that any Riemannian structure on a compact manifold of dimension ≥ 3 could be pointwise conformally deformed to one with constant scalar curvature. It was found by Trudinger [6] that Yamabe's paper contained an error. Trudinger was able to correct Yamabe's proof in the case when the total scalar curvature (i.e., the integral of the scalar curvature) is nonpositive. In this case the constant scalar curvature is negative. A couple of years later, Eliasson [7] and Aubin [8] showed that every compact manifold of dimension ≥ 3 possesses a metric whose total scalar curvature is negative. This, together with Trudinger's results, shows that every compact manifold of dimension ≥ 3 admits a Riemannian metric with constant negative scalar curvature.

Kazdan and Warner in [3] studied the first eigenvalue $\lambda_1(g)$ of the operator L with corresponding eigenfunction φ,

$$L\varphi \equiv -\left(\frac{4(n-1)}{n-2}\Delta\varphi - k\varphi\right) = \lambda_1(g)\varphi \quad \text{on } M,$$

and obtained that if $\lambda_1(g) < 0$ then one can always pointwise conformally deform g to a metric of constant negative scalar curvature. In this paper we shall only consider the case where the given metric already has a constant negative scalar curvature $k < 0$.

We now free our problem from geometry and consider instead a general nonlinear equation

$$(1.1) \qquad \begin{cases} \Delta u + \lambda u - h u^p = 0 & \text{on } M, \\ u > 0 & \text{on } M, \end{cases}$$

where $\lambda > 0$, $p > 1$ are constants and $h(x) \geq 0$ is a C^1-function on M.

In [3] Kazdan and Warner observed that if $h > 0$ in (1.1) then there exists a solution of (1.1) for any constant $\lambda > 0$, and posed the question of whether one can prove the same result for the case $h \geq 0$. It turns out that the problem is more subtle than one might expect. The purpose of this paper is to give a complete answer to this question.

Let $M_+ = \{x \in M | h(x) > 0\}$ and $M_0 = M \backslash \overline{M}_+$.

Our main result may be stated as follows.

Theorem 1. *Assume that $h \geq 0$ $(\not\equiv 0)$ is a smooth function on M.*

(i) *If $M_0 = \varnothing$, then for every $\lambda > 0$ there exists a unique solution $u(\lambda)$ of problem* (1.1).

(ii) *If $M_0 \neq \varnothing$, then there is a positive $\bar{\lambda} \in (0, \infty)$ such that for any $\lambda < \bar{\lambda}$ there exists a unique solution $u(\lambda)$ of* (1.1), *and for $\lambda \geq \bar{\lambda}$ there is no solution of* (1.1). *Moreover*

$$\lim_{\lambda \to \bar{\lambda}} \|u(\lambda)\|_{L^2(M)} = \infty.$$

Furthermore, for the open subset $M_0 \subset M$ one can define the first eigenvalue $\lambda_1(M_0)$ of the Laplacian operator on M_0 with zero Dirichlet boundary condition in a natural way (see §2, Definition 2). Let $\lambda_1(M_0) > 0$ be the first eigenvalue and $\varphi > 0$ be the corresponding unit eigenfunction, i.e.,

$$\begin{aligned} \Delta\varphi + \lambda_1\varphi &= 0 \quad \text{in } M_0, \\ \varphi &= 0 \quad \text{on } \partial M_0. \end{aligned}$$

Then $\bar{\lambda} = \lambda_1$.

The conclusion in (ii) of Theorem 1 is independent of the norm and the shape of h, only depends on the support of h, and is also independent of the power $p > 1$.

Returning to the original geometric problem, our result implies that

(i) if $K < 0$ on M, then there exists a conformal metric g^* such that K is the scalar curvature of the manifold (M, g^*);

(ii) if the zero set of K (i.e., $\{x \in M | K(x) = 0\}$) is not too "large," then the same conclusion as in (i) is also true. More precisely, if $-k$ is smaller than the first eigenvalue of the Dirichlet problem on $M_0 = M \backslash \overline{M}_+$, where $M_+ = \{x \in M | K(x) > 0\}$, then there is a conformal metric g^* such that K is the scalar curvature of the manifold (M, g^*);

(iii) if the zero set of K is too "large," then k is not pointwise conformal to K. More precisely, if $-k$ is greater than or equal to the first eigenvalue of the Dirichlet problem on M_0, then there is no conformal metric g^* such that K is a scalar curvature on (M, g^*).

Therefore negative constant scalar curvatures are not always pointwise conformal to nonpositive scalar curvatures; it depends on the measure and the shape of the zero set of K but is independent of the norm and the shape of K.

We also have similar results for the Neumann problem and Dirichlet problem of equation (1.1) in a bounded domain $\Omega \subset R^n$. The result for the Dirichlet problem is the following.

Let Ω be a bounded smooth domain in R^n. Consider the following problem

$$\begin{cases} \Delta u + \lambda u - hu^p = 0 & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

where $\lambda > 0$, $p > 1$ are constants, and $h(x)$ is a function in Ω.

Letting λ_0 be the first eigenvalue of the Dirichlet problem in Ω and

$$\Omega_+ = \{x \in \Omega | h(x) > 0\}, \qquad \Omega_0 = \Omega \backslash \overline{\Omega}_+,$$

we have

Theorem 2. *Assume that $h \geq 0$ $(\not\equiv 0)$ is a smooth function in Ω.*

(i) *If $\Omega_0 = \varnothing$, then for every $\lambda > \lambda_0$ there exists a unique solution $u(\lambda)$ of problem* (1.2).

(ii) *If $\Omega_0 \neq \varnothing$, then there is a positive $\bar{\lambda} \in (\lambda_0, \infty)$ such that for any $\lambda \in (\lambda_0, \bar{\lambda})$ there exists a unique solution of* (1.2), *and for $\lambda \geq \bar{\lambda}$ there is no solution of* (1.2). *Moreover*

$$\lim_{\lambda \to \bar{\lambda}} \|u(\lambda)\|_{L^2(\Omega)} = \infty.$$

Furthermore, suppose $\lambda_1 > 0$ *is the first eigenvalue of the Laplacian in* M_0 *with zero Dirichlet boundary condition and* $\varphi > 0$ *is the unit corresponding eigenfunction, i.e.,*

$$\begin{aligned} \Delta\varphi + \lambda_1\varphi = 0 \quad &\textit{in } \Omega_0, \\ \varphi = 0 \quad &\textit{on } \partial\Omega_0. \end{aligned}$$

Then $\bar{\lambda} = \lambda_1$.

For Neumann problems we have the following result.

Let Ω be a bounded smooth domain in R^n. Consider the problem

$$\tag{1.3} \begin{cases} \Delta u + \lambda u - hu^p = 0 & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\lambda > 0$, $p > 1$ are constants, ν is the unit outer normal vector on $\partial\Omega$, and $h(x)$ is a function in Ω.

Letting

$$\Omega_+ = \{x \in \Omega | h(x) > 0\}, \qquad \Omega_0 = \Omega \backslash \overline{\Omega}_+,$$

we have

Theorem 3. *Assume that* $h \geq 0$ $(\not\equiv 0)$ *is a smooth function in* Ω.

(i) *If* $\Omega_0 = \varnothing$, *then for every* $\lambda > 0$ *there exists a unique solution* $u(\lambda)$ *of problem* (1.3).

(ii) *If* $\Omega_0 \neq \varnothing$, *then there is a positive* $\bar{\lambda} \in (0, \infty)$ *such that for any* $\lambda \in (0, \bar{\lambda})$ *there exists a unique solution of* (1.3), *and for* $\lambda \geq \bar{\lambda}$ *there is no solution of* (1.3). *Moreover*

$$\lim_{\lambda \to \bar{\lambda}} \|u(\lambda)\|_{L^2(\Omega)} = \infty.$$

Furthermore, suppose $\lambda_1 > 0$ *is the first eigenvalue of the Laplacian on* Ω_0 *with zero Dirichlet boundary condition, and* $\varphi > 0$ *is the corresponding unit eigenfunction i.e.,*

$$\begin{aligned} \Delta\varphi + \lambda_1\varphi = 0 \quad &\textit{in } \Omega_0, \\ \varphi = 0 \quad &\textit{on } \partial\Omega_0. \end{aligned}$$

Then $\bar{\lambda} = \lambda_1$.

2. Preliminaries

The following theorems will be needed in the proof of Theorem 1. The first one, Strong Maximum Principle, is a manifold version of the regular strong maximum principle in R^n domain.

Strong Maximum Principle. *Let* (M^n, g) *be a smooth compact and connected manifold without boundary,* Δ *the Laplacian on* M, *and* $u \in C^2(M)$ *satisfying*

$$\begin{aligned} \Delta u + cu \leq 0 \quad &\textit{on } M, \\ u \geq 0 \ (\not\equiv 0) \quad &\textit{on } M, \end{aligned}$$

where c *is a bounded function on* M. *Then* $u > 0$ *on* M.

Proof (this is a modification of the proof of Theorem 3.5 in [10, p. 35]). Let $M_+ = \{x \in M | u(x) > 0\}$. If $M_+ = M$, then we are done. So we assume

$$M_0 = \{x \in M | u(x) = 0\}, \qquad M_0 \neq \varnothing.$$

By the definition of the Riemannian manifold (M^n, g), for all $x \in M$ there is a neighborhood U_x of x and a diffeomorphism

$$\phi_x \colon U_x \to V \subset R^n, \qquad V \text{ is a open set in } R^n.$$

Since M is compact, we have

$$M = \bigcup_{x \in M} U_x = \bigcup_{i=1}^{m} U_{x_i}$$

and

$$M = M_+ \cup M_0 = \bigcup_{i=1}^{m} U_{x_i}.$$

Let

$$U_+ = \bigcup_{U_{x_i} \subset M_+} U_{x_i}, \qquad U_0 = \bigcup_{U_{x_i} \subset M_0} U_{x_i}.$$

Then $U_+ \subset M_+$, $U_0 \subset M_0$. We claim that $U_+ \cup U_0 \underset{\neq}{\subset} M$, except for $u \equiv 0$.

Suppose $M = U_+ \cup U_0$; we claim $U_0 = M_0$. In fact, $\forall x \in M_0$, we have $x \notin U_+$ (by definition of U_+), so x must belong to U_0. Hence $M_0 \subset U_0$ and $U_0 = M_0$, and therefore M_0 is both an open and closed set. By the connectedness of M, we have $M_0 = M$ and therefore $u \equiv 0$ on M.

This means that if $u \not\equiv 0$, then there is a U_x such that

$$U_x \cap M_+ \neq \varnothing, \qquad U_x \cap M_0 \neq \varnothing$$

and a corresponding $\phi_x \colon U_x \to V \subset R^n$ which is a diffeomorphism of U_x onto an open set $V \subset R^n$. In the coordinate neighborhood V, the Laplacian is

$$\Delta = \frac{1}{\sqrt{|g|}} \sum_{j,k} \frac{\partial}{\partial x_j} \left(g^{jk} \sqrt{|g|} \frac{\partial}{\partial x_k} \right),$$

where $|g| = \det(g_{ij})$, $(g^{jk}) = \operatorname{inverse}(g_{ij})$, and $g_{ij} = g(\phi^{-1}(x_i), \phi^{-1}(x_j))$. Since g is a symmetric, positive definite, bilinear form and M is compact, it follows that Δ is uniformly elliptic. Also $u(\phi^{-1})$ is a C^2-function in V satisfying

$$\Delta u(\phi^{-1}) + c u(\phi^{-1}) \leq 0 \quad \text{in } V,$$
$$u(\phi^{-1}) \geq 0 \ (\not\equiv 0) \quad \text{in } V,$$

and the set

$$V_+ = \{ x \in V \mid u(\phi^{-1})(x) > 0 \}$$

satisfying $V_+ \underset{\neq}{\subset} V$. Let

$$V_0 = \{ x \in V \mid u(\phi^{-1})(x) = 0 \}.$$

Choose $x_0 \in V_+$ such that

$$\operatorname{dist}(x_0, V_0) < \operatorname{dist}(x_0, \partial V)$$

and consider the largest ball $B \subset V_+$ centered at x_0. Then there is a point $y \in \partial B \cap V_0$ such that

$$u(\phi^{-1})(y) = 0, \qquad u(\phi^{-1}) > 0 \quad \text{in } B.$$

The Hopf boundary lemma implies $\mathrm{D}u(\phi^{-1})(y) \neq 0$, which contradicts the fact that y is an interior minimum in V. Hence $u > 0$ on M. □

Definition 1. A function $v \in C^2(M)$ is said to be a super-solution (sub-solution) of the problem

$$\Delta u + f(x, u) = 0 \quad \text{on } M, \tag{2.1}$$

where $f(x, \xi) \in M \times R \to R$ is a smooth function on M, if v satisfies the inequality

$$\Delta v + f(x, v) \leq (\geq)\ 0 \quad \text{on } M. \tag{2.2}$$

Proposition 1. *Let (M^n, g) be a C^∞-compact Riemannian manifold without boundary, $f(x, \xi) \in M \times R \to R$ be a C^1-function, and $u_1, u_2 \in C^2(M)$ be super-solutions of*

$$\Delta u_i + f(x, u_i) = 0 \quad \text{on } M, \quad i = 1, 2. \tag{2.3}$$

Let

$$u(x) = \min(u_1(x), u_2(x)), \quad x \in M.$$

Then u is a super-solution of (2.3) *in the following weak sense*:

$$\int_M \nabla u \nabla \phi - \int_M f(x, u)\phi \geq 0 \qquad \forall \phi \in C^\infty(M), \phi \geq 0.$$

Proof. Let

$$M_1 = \{x \in M \mid u_1(x) < u_2(x)\},$$
$$M_2 = \{x \in M \mid u_1(x) \geq u_2(x)\}.$$

First, we assume ∂M_1 is a piecewise C^1-boundary.

For all $\phi > 0$, $\phi \in C^\infty(M)$,

$$\begin{aligned} \int_M \nabla u \nabla \phi - \int_M f(x, u)\phi = \int_{M_1} \nabla u_1 \nabla \phi - \int_{M_1} f(x, u_1)\phi \\ + \int_{M_2} \nabla u_2 \nabla \phi - \int_{M_2} f(x, u_2)\phi. \end{aligned} \tag{2.4}$$

Using the divergence theorem, we have

$$\int_{M_i} \nabla u_i \nabla \phi = \int_{\partial M_i} \frac{\partial u_i}{\partial \nu}\phi - \int_{M_i} \Delta u_i \phi, \qquad i = 1, 2. \tag{2.5}$$

Combining (2.4) with (2.5), we have

$$\begin{aligned} &\int_M \nabla u \nabla \phi - \int_M f(x, u)\phi \\ &\quad = -\int_{M_1} [\Delta u_1 + f(x, u_1)]\phi - \int_{M_2} [\Delta u_2 + f(x, u_2)]\phi \\ &\qquad + \int_{\partial M_1} \frac{\partial u_1}{\partial \nu}\phi + \int_{\partial M_2} \frac{\partial u_2}{\partial \nu}\phi \\ &\quad = -\int_{M_1} [\Delta u_1 + f(x, u_1)]\phi - \int_{M_2} [\Delta u_2 + f(x, u_2)]\phi + \int_{\partial M_1} \frac{\partial (u_1 - u_2)}{\partial \nu}\phi \\ &\quad = I_1 + I_2 + I_3. \end{aligned}$$

From the definition of super-solution, we have $I_1 \geq 0$, $I_2 \geq 0$. To show $I_3 \geq 0$, we note

$$u_1 - u_2 < 0 \quad \text{in } M_1 \qquad \text{and} \qquad u_1 - u_2 = 0 \quad \text{on } \partial M_1 .$$

It follows that

$$\left.\frac{\partial(u_1 - u_2)}{\partial \nu}\right|_{\partial M_1} \geq 0,$$

and therefore $I_3 \geq 0$. Hence

$$\int_M \nabla u \nabla \phi - \int_M f(x, u)\phi \geq 0.$$

Next, assume ∂M_1 is not a C^1-boundary. Suppose $u_1 - u_2 \in C^n(M)$ By Sard's theorem there is a sequence $\varepsilon_n > 0$ such that $\lim_{n\to\infty} \varepsilon_n = 0$ and the boundary of $\{u_1 - u_2 < \varepsilon_n\}$ belongs to the C^1-class.

Denote

$$\begin{aligned} M_{1,\varepsilon_n} &= \{x \in M \mid u_1 - u_2 < \varepsilon_n\}, \\ M_{2,\varepsilon_n} &= \{x \in M \mid u_1 - u_2 \geq \varepsilon_n\}, \\ u_{\varepsilon_n}(x) &= \min\{u_1(x),\ u_2(x) + \varepsilon_n\}. \end{aligned}$$

For all $\phi > 0$, $\phi \in C^\infty(M)$,

$$\begin{aligned} &\int_M \nabla u_{\varepsilon_n} \nabla \phi - \int_M f(x, u_{\varepsilon_n})\phi \\ &\quad = \int_{M_{1,\varepsilon_n}} \nabla u_1 \nabla \phi - \int_{M_{1,\varepsilon_n}} f(x, u_1)\phi + \int_{M_{2,\varepsilon_n}} \nabla(u_2 + \varepsilon_n)\nabla \phi \\ &\quad\quad - \int_{M_{2,\varepsilon_n}} f(x, u_2 + \varepsilon_n)\phi \\ &\quad = -\int_{M_{1,\varepsilon_n}} [\Delta u_1 + f(x, u_1)]\phi - \int_{M_{2,\varepsilon_n}} [\Delta u_2 + f(x, u_2 + \varepsilon_n)]\phi \\ &\quad\quad + \int_{\partial M_{1,\varepsilon_n}} \frac{\partial u_1}{\partial \nu}\phi + \int_{\partial M_{2,\varepsilon_n}} \frac{\partial u_2 + \varepsilon_n}{\partial \nu}\phi \\ &\quad = -\int_{M_{1,\varepsilon_n}} [\Delta u_1 + f(x, u_1)]\phi - \int_{M_{2,\varepsilon_n}} [\Delta u_2 + f(x, u_2)]\phi \\ &\quad\quad + \int_{\partial M_{1,\varepsilon_n}} \frac{\partial(u_1 - u_2 - \varepsilon_n)}{\partial \nu}\phi + \int_{M_{2,\varepsilon_n}} [f(x, u_2) - f(x, u_2 + \varepsilon_n)]\phi \\ &\quad = I_1 + I_2 + I_3 + I_4. \end{aligned}$$

Since

$$u_1 - u_2 - \varepsilon_n < 0 \quad \text{in } M_{1,\varepsilon_n} \qquad \text{and} \qquad u_1 - u_2 - \varepsilon_n = 0 \quad \text{on } \partial M_{1,\varepsilon_n},$$

it follows that

$$\left.\frac{\partial(u_1 - u_2 - \varepsilon_n)}{\partial \nu}\right|_{\partial M_{1,\varepsilon_n}} \geq 0$$

and

$$\int_{\partial M_{1,\varepsilon_n}} \frac{\partial u_1 - u_2 - \varepsilon_n}{\partial \nu}\phi \geq 0.$$

By using the definition of super-solutions u_1 and u_2, we have $I_1 \geq 0$ and $I_2 \geq 0$.

And since

$$\left|\int_{\partial M_{2,\varepsilon_n}} [f(x, u_2) - f(x, u_2+\varepsilon_n)]\phi\right| \leq \int_{\partial M_{2,\varepsilon_n}} |f_\xi(x, u_2+\theta)|\varepsilon_n\phi$$
$$\leq \|f_\xi\|_{C(M\times(-\|u_2\|_{C(M)}-\varepsilon_n, \|u_2\|_{C(M)}+\varepsilon_n))}\|\phi\|_{C(M)}\varepsilon_n,$$

we have

$$\int_M \nabla u_{\varepsilon_n}\nabla\phi - \int_M f(x, u_{\varepsilon_n})\phi \geq -\|f_\xi\|_{C(M\times(-\|u_2\|_{C(M)}-\varepsilon_n, \|u_2\|_{C(M)}+\varepsilon_n))}\|\phi\|_{C(M)}\varepsilon_n.$$

Letting $\varepsilon_n \to 0$, we have $\lim_{n\to\infty} u_{\varepsilon_n}(x) = u(x)$ and

$$\int_M \nabla u\nabla\phi - \int_M f(x, u)\phi \geq 0 \quad \forall\phi > 0, \phi \in C^\infty(M).$$

Finally, if $u_1 - u_2 \notin C^n(M)$ and since $u_1, u_2 \in C^2(M)$, then u_1 and u_2 can be approximated by $u_{1\varepsilon}$ and $u_{2\varepsilon}$ respectively such that $u_{1\varepsilon}, u_{2\varepsilon} \in C^n$ and for $\varepsilon > 0$

$$\|u_i - u_i\varepsilon\|_{C^2(M)} \leq \varepsilon \qquad \text{for } i = 1, 2,$$

and

$$\Delta u_{i\varepsilon} + f(x, u_{i\varepsilon}) \leq \varepsilon \quad \text{on } M, \quad i = 1, 2.$$

Let $u_\varepsilon = \min(u_{1\varepsilon}, u_{2\varepsilon})$. Then it follows from the above argument that

$$\int_M \nabla u_\varepsilon\nabla\phi - \int_M f(x, u_\varepsilon)\phi \geq C\varepsilon \quad \forall\phi > 0, \phi \in C^\infty(M).$$

Letting $\varepsilon \to 0$, we have

$$\int_M \nabla u\nabla\phi - \int_M f(x, u)\phi \geq 0 \quad \forall\phi > 0, \phi \in C^\infty(M). \quad \square$$

Proposition 2 (Sub-super-solution method). *Let $\overline{u}$ ($\underline{u}$) be a super-solution (sub-solution) of the equation*

$$\Delta + f(x, u) = 0 \quad \text{on } M, \tag{2.6}$$

where $f(x, u) \in C^1(M \times \mathscr{R})$, and satisfy

$$\underline{u} < \overline{u} \quad \text{on } M.$$

Then there exists a solution u of equation (2.6) *satisfying*

$$\underline{u} < u < \overline{u} \quad \text{on } M.$$

Proof. This is a well-known result (see e.g. [3]).

Next, let us define the first eigenvalue of the Laplacian operator Δ on M_0. We can decompose M_0 into at most countably infinitely many connected components and express M_0 as

$$M_0 = \bigcup_{n=1}^{\infty} M_i \qquad \text{and} \qquad M_i \cap M_j = \varnothing \quad \text{for } i \neq j.$$

For each $M_i \in M_0$, $\overline{M}_i$ is a compact and connected subset of M with $\partial M_i \neq \varnothing$. The first eigenvalue $\lambda_1(M_i)$ of the Laplacian operator Δ on M_i with zero Dirichlet boundary condition is defined as

$$\lambda_1(M_i) = \inf_{\substack{\varphi \in H_0^1(M_i) \\ \|\varphi\|_{L^2(M_i)}=1}} \int_M |\nabla \varphi|^2. \tag{2.7}$$

Definition 2. The first eigenvalue of the Laplacian operator Δ on M_0 with zero Dirichlet boundary condition is

$$\lambda_1(M_0) = \inf_{1 \leq i < \infty} \lambda_1(M_i). \tag{2.8}$$

It is not difficult to show that there is an M_i, $1 \leq i < \infty$, such that

$$\lambda_1(M_0) = \lambda_1(M_i). \tag{2.9}$$

In fact, if M_0 has only finite, say N, components M_i, where $1 \leq i \leq N$, then (2.9) is true. If M_0 has infinite components M_i, $1 \leq i < \infty$, and since M is compact, we have

$$\lim_{i \to \infty} \mathrm{Vol}(M_i) = \lim_{i \to \infty} \int_{M_i} dV = 0.$$

Therefore

$$\lim_{i \to \infty} \lambda_1(M_i) = \infty,$$

and we have

$$\lambda_1(M_0) = \inf_{1 \leq i \leq N} \lambda_1(M_i)$$

for sufficiently large N.

Definition 3. The inner boundary $\partial'\Omega$ of a subset $\Omega \subset M$ consists of the points on $\partial\Omega$ which are not on the boundary of any component of $M \backslash \Omega$.

From the smoothness of h and the definition of M_0 in the previous section, we have that

$$\partial' M_0 = \varnothing. \tag{2.10}$$

3. Main proof

Lemma 1. *Assume $h(x) \geq 0$. Then, for any $\lambda > 0$, there exists at most one positive solution $u(\lambda)$ of (1.1).*

Proof. Suppose, for some $\lambda > 0$, there exist two positive solutions u_1 and u_2 of (1.1) with $u_1 \neq u_2$. We may assume

$$u_1 \geq u_2 \quad \text{on } M. \tag{$*$}$$

If $u_1 \ngeq u_2$ and $u_2 \ngeq u_1$, then we set

$$\overline{u}(x) = \min\{u_1(x), u_2(x)\}, \qquad x \in M.$$

It is easy to see that $\overline{u} > 0$ on M and from Proposition 1 in the previous section we know that $\overline{u}$ is a super-solution of (1.1). It is also easy to check that

$$u_c(x) = \text{const} < \min\left\{\left(\frac{\lambda}{H}\right)^{\frac{1}{p-1}}, \min_{x \in M} \overline{u}(x)\right\},$$

where $H = \|h\|_{L^\infty(M)}$, is a sub-solution of (1.1). By the sub-super-solution method there is a solution v of (1.1) satisfying

$$u_c \le v \le \overline{u} \quad \text{on } M.$$

So we may choose v to replace u_2 such that the new pair of solutions satisfy $(*)$. Moreover, we can assume

$$u_1 > u_2 \quad \text{on } M.$$

In fact, if $u_1 \ge u_2$ and $u_1(x_0) = u_2(x_0)$ for some $x_0 \in M$, then set

$$w(x) = u_1(x) - u_2(x).$$

It follows that

$$\begin{aligned} \Delta w + \lambda w + f(x)w &= 0 \quad \text{on } M, \\ w &\ge 0 \quad \text{on } M, \\ w(x_0) &= 0, \quad x_0 \in M, \end{aligned}$$

where $f(x) = -h(u_1^p - u_2^p)/(u_1 - u_2)$ is a continuous function on M. Since M is compact, f is bounded on M and therefore

$$\Delta w + f(x)w \le 0 \quad \text{on } M.$$

By using the Strong Maximum Principle, we have $w > 0$ on M, hence $u_1 > u_2$ on M. Since u_1 and u_2 are solutions of (1.1),

$$\Delta u_1 + \lambda u_1 - hu_1^p = 0 \quad \text{on } M, \tag{3.1}$$

$$\Delta u_2 + \lambda u_2 - hu_2^p = 0 \quad \text{on } M. \tag{3.2}$$

Multiplying both sides of (3.1) by u_2 and integrating by parts over M, we have

$$-\int_M \nabla u_1 \nabla u_2 + \lambda \int_M u_1 u_2 - \int_M hu_1^p u_2 = 0. \tag{3.3}$$

Similarly, we have

$$-\int_M \nabla u_1 \nabla u_2 + \lambda \int_M u_1 u_2 - \int_M hu_2^p u_1 = 0. \tag{3.4}$$

Subtracting (3.4) from (3.3), we obtain

$$\int_M hu_1 u_2({u_1}^{p-1} - {u_2}^{p-1}) = 0.$$

But $h \ge 0$ $(\not\equiv 0)$ and $u_1 > u_2 > 0$, so the left-hand side of the above equation must be positive. This contradiction means $u_1 \equiv u_2$. □

In order to prove the existence of a positive solution of (1.1) we need the bifurcation theorem below.

Let X and Y be Banach spaces and let $F : R \times X \to Y$ be continuously differentiable. Suppose $F(\lambda, 0) = 0$ for $\lambda \in \Lambda$, where $\Lambda \subset R$ is an open interval containing λ_0, and that every neighborhood of $(\lambda_0, 0)$ contains a zero of $F(\lambda, x)$ which does not lie on the curve $\hbar = \{(\lambda, 0); \lambda \in \Lambda\}$. Then $(\lambda_0, 0)$ is said to be a bifurcation point of $F(\lambda, x)$ with respect to $\hbar$.

Bifurcation theorem [9]. *Let X, Y be Banach spaces, let V be a neighborhood of 0 in X, and let $F : (-1, 1) \times V \to Y$ have the properties:*

(1) $F(t, 0) = 0$ *for* $|t| < 1$;
(2) *The partial derivatives F_t, F_x, and F_{tx} exist and are continuous;*
(3) $N(F_x(0, 0))$ *and* $Y/R(F_x(0, 0))$ *are one dimensional;*
(4) $F_{tx}(0, 0)x_0 \notin R(F_x(0, 0))$ *where* $N(F_x(0, 0)) = \operatorname{span}\{x_0\}$.

If Z is any complement of $N(F_x(0, 0))$ in X, then there is a neighborhood U of $(0, 0)$ in $R \times X$, an interval $(-a, a)$, and continuous functions $\varphi: (-a, a) \to R$ and $\psi: (-a, a) \to Z$ such that $\varphi(0) = 0$, $\psi(0) = 0$, and

$$F^{-1}(0) \cap U = \{(\varphi(\alpha), \alpha x_0 + \alpha\psi(\alpha)) \mid |\alpha| < a\} \cup \{(t, 0) \mid (t, 0) \in U\}.$$

Lemma 2. *Assume $h(x) \geq 0$ ($\not\equiv 0$). Then for any fixed $p > 1$ there exists a bifurcation solution curve $(\lambda, u(\lambda))$ of problem (1.1) starting from $(0, 0)$ which is positive.*

Proof. In order to apply the Bifurcation Theorem, let

$$X = \{u \in C^{2,\alpha}(M) \mid \|u\|_X = \|u\|_{C^{2,\alpha}(M)}\},$$

$$Y = \{u \in C^{0,\alpha}(M) \mid \|u\|_Y = \|u\|_{C^{0,\alpha}(M)}\},$$

where $0 < \alpha < 1$ is a fixed constant. Obviously X, Y, are Banach spaces. Let

$$F(\lambda, u) = \Delta u + \lambda u - hu^p.$$

We have

$$F_\lambda(\lambda, u) = u, \quad F_u(\lambda, u)v = \Delta v + \lambda v - phu^{p-1}v, \quad F_{\lambda u}(\lambda, u)v = v,$$

and for $l = 0$, $u = 0$, we have

$$F_\lambda(0, 0) = 0, \quad F_u(0, 0)v = \Delta v, \quad F_{\lambda u}(0, 0)v = v.$$

Δ has 0 as its first eigenvalue with a constant as a corresponding eigenfunction. Obviously $N(F_u(0, 0)) = N(\Delta \text{ operator})$ and $Y/R(F_u(0, 0))$ are one dimensional with

$$N(F_u(0, 0)) = \operatorname{span}\{1\},$$

$$R(F_u(0, 0)) = \left\{ f \in Y \mid \int_M f = 0 \right\},$$

$$F_{\lambda u}(0, 0)1 \notin R(F_u(0, 0)).$$

So by the Bifurcation Theorem, there is a bifurcation curve $(\lambda(s), u(s))$ starting from $(0, 0)$ with

$$\lambda = \lambda(s), \quad u(s) = s1 + s\psi(s) \qquad \text{for } s \text{ near } 0,$$

where $\lambda : (-a, a) \to R$ and $\psi : (-a, a) \to C^2(M)$, such that

$$\lambda(0) = 0, \qquad \psi(0) = 0,$$

$$F^{-1}(0) = \{(\lambda(s), u(s)), \ |s| < a\} \cup \{(\lambda, 0), \lambda \in (-a, a)\}$$

in a neighborhood of $(0, 0)$ in $R \times X$. Replacing u and λ in equation (1.1) by the above expressions, we have

$$\text{(3.5)} \qquad \begin{aligned} \Delta u + \lambda u - hu^p &= s\Delta\psi(s) + \lambda(s)(s + s\psi(s)) - h(s + s\psi(s))^p \\ &= 0 \quad \text{on } M, \end{aligned}$$

$$u(x, s) = s(1+\psi(s)) > 0 \quad \text{on } M,$$

for $s > 0$ small enough. Integrating both sides of (3.5) over M, we have

$$\lambda(s)s(1+o(1)) - s^p \int_M h(1+o(1))^p = 0,$$

$$\lambda(s) = s^{p-1} \int_M h + o(s^{p-1}) > 0 \quad \text{for } s > 0 \text{ small.}$$

Hence from the point $(0, 0) \in R \times C^2(M)$ there is a bifurcation curve $(\lambda, u(\lambda))$ such that

$$\Delta u(\lambda) + \lambda u(\lambda) - hu(\lambda)^p = 0 \quad \text{on } M$$

and

$$\lambda > 0, \quad u(\lambda) = \frac{\lambda^{\frac{1}{p-1}}}{(\int_M h)^{\frac{1}{p-1}}} + o(\lambda^{\frac{1}{p-1}}) \quad \text{for } \lambda \text{ near } 0. \quad \square$$

Remark 2. Elementary arguments show that there is a maximum number $\bar{\lambda} \in (0, \infty]$ such that there exists a continuous function $u : [0, \bar{\lambda}) \to C^2(M)$ satisfying

$$F(\lambda, u(\lambda)) = 0 \quad \text{on } M$$

and

$$F_u(\lambda, u(\lambda)) \text{ is nonsingular (invertible) for } \lambda < \bar{\lambda}.$$

This means we can continue to extend the above bifurcation $(\lambda, u(\lambda))$ to all $l < \bar{l}$. To study the properties of the bifurcation curve $(\lambda, u(\lambda))$, we claim

Lemma 3. *For all $\lambda \in (0, \bar{\lambda})$, $u(\lambda)$ is differentiable with respect to λ, and is monotone increasing, i.e., $u'(\lambda) > 0$ on M, where $u'(\lambda)$ is the derivative of $u(\lambda)$ with respect to λ.*

Proof.

Step 1. We claim $u(\lambda)$ is nondecreasing for $\lambda \in (0, \bar{\lambda})$.

In fact, suppose the claim is not true. Then there are two pairs of solutions $(\lambda_1, u(\lambda_1))$ and $(\lambda_2, u(\lambda_2))$ such that $\lambda_1 < \lambda_2$ and, for some $x \in M$, $u(\lambda_1)(x) > u(\lambda_2)(x)$. Since

$$\Delta u(\lambda_2) + \lambda_2 u(\lambda_2) - hu^p(\lambda_2) = 0 \quad \text{on } M,$$

we have

$$\Delta u(\lambda_2) + \lambda_1 u(\lambda_2) - hu^p(\lambda_2) = -(\lambda_2 - \lambda_1)u(\lambda_2) < 0 \quad \text{on } M.$$

Therefore $u(\lambda_2)$ is a super-solution of (1.1) at $\lambda = \lambda_1$. It follows from Proposition 1 that

$$u(x) = \min(u(\lambda_1)(x), u(\lambda_2)(x)) \quad \text{on } M$$

is a super-solution of (1.1) and

$$0 < u \le u(\lambda_1) \quad \text{on } M.$$

We also know that $u_c =$ constant small enough is a subsolution of (1.1) with $u_c \le u$. Using sub-super-solution methods, there is a second positive solution $u_2(\lambda_1)$ of (1.1) at $\lambda = \lambda_1$. This contradicts the uniqueness in Lemma 1. Hence the claim is true.

Step 2. First, we claim that for $\lambda \in (0, \bar{\lambda})$, $F_u(\lambda, u(l))$ is invertible.

Let (μ, v) be the first eigenvalue and eigenfunction of $F_u(\lambda, u(\lambda))$, which satisfy

$$\begin{aligned} \Delta v + \lambda v - phu^{p-1}(\lambda)v &= -\mu v \quad \text{on } M, \\ v &> 0 \quad \text{on } M. \end{aligned} \tag{3.6}$$

Recall that

$$\Delta u(\lambda) + \lambda u(\lambda) - hu^p(\lambda) = 0 \quad \text{on } M. \tag{1.1}$$

Multiplying both sides of (3.6) by $u(\lambda)$ and integrating by parts, we have

$$\int_M \nabla u \nabla v + \lambda \int_M uv - p\int_M hu^p v = -\mu \int_M uv. \tag{3.7}$$

Multiplying both sides of (1.1) by v and integrating by parts, we have

$$\int_M \nabla u \nabla v + \lambda \int_M uv - \int_M hu^p v = 0. \tag{3.8}$$

Subtracting (3.8) from (3.7), we have

$$(p-1)\int_M hu^p v = \mu \int_M uv.$$

Since $u, v > 0$, we have $\mu > 0$. Therefore $F_u(\lambda, u(\lambda))$ is invertible. It follows from the Implicit Function Theorem that $u(\lambda)$ is differentiable with respect to λ. Combining this with the fact that u is nondecreasing we have

$$u'(\lambda) \geq 0 \quad \text{on } M \text{ for } \lambda \in (0, \bar{\lambda}).$$

Differentiating (1.1) with respect to λ, we have

$$\begin{aligned} \Delta u'(\lambda) + \lambda u'(\lambda) - phu^{p-1}(\lambda)u'(\lambda) + u(\lambda) &= 0 \quad \text{on } M, \\ u'(\lambda) &\geq 0 \quad \text{on } M. \end{aligned}$$

By using the Strong Maximum Principle we have $u'(\lambda) > 0$ on M for all $\lambda \in (0, \bar{\lambda})$. □

Remark 3. We claim that there exists a function $h^* \in C^1(M)$ satisfying

(1) supp $h^* = M_+$;
(2) $0 < h^*(x) \leq h(x)\ \forall x \in M_+$;
(3) $\sup_{x\in M_+} |\nabla h^*/h^{*\,1-\varepsilon}| < C(\varepsilon) = C/\varepsilon^2$ for $\forall \varepsilon > 0$.

For example, let $d(x)$ denote the distance between x and ∂M_+, i.e.,

$$d(x) = \operatorname{dist}(x, \partial M_+) \quad \text{for } x \in M_+.$$

Choose $\delta > 0$ small enough and define h^* as follows

$$h^*(x) = \begin{cases} e^{-1/h(x)} & \text{for } x \in M_+ \text{ and } d(x) < \delta, \\ h(x) & \text{for } x \in M_+ \text{ and } d(x) > 2\delta, \\ 0 & \text{for } x \in M \backslash M_+. \end{cases}$$

It can be shown that h^* satisfies our assumption.

In fact, when $d(x) < \delta$, we have that $\nabla h^* = e^{-\frac{1}{h(x)}} \nabla h / h^2(x)$ and

$$\left| \frac{\nabla h^*}{{h^*}^{1-\varepsilon}} \right| = |\nabla h| \left| \frac{e^{-\frac{\varepsilon}{h}}}{h^2} \right| \le C \left(\frac{e^{-\frac{\varepsilon}{h}}}{h^2} \right).$$

Moreover, it is easy to verify that the maximum of $e^{-\frac{\varepsilon}{h}}/h^2$ for $d(x)$ near zero is bounded by C/ε^2 In fact,

$$\left(\frac{e^{-\frac{\varepsilon}{h}}}{h^2} \right)' = \frac{e^{-\frac{\varepsilon}{h}}\varepsilon - 2he^{-\frac{\varepsilon}{h}}}{h^4},$$

when $h(x) = \varepsilon/2$ it gets its maximum, and therefore

$$\left| \frac{\nabla h^*}{{h^*}^{1-\varepsilon}} \right| < C(\varepsilon) = \frac{C}{\varepsilon^2}.$$

Lemma 4. *For any $\varepsilon > 0$, if $u(\lambda)$ is the positive solution of (1.1) with $\lambda < \overline{\lambda}$, then $h^* u^{p-1-\varepsilon} \in L^\infty(M_+)$ and*

$$\sup_{x \in M_+} h^* u^{p-1-\varepsilon} \le C(\varepsilon, \lambda).$$

If λ is finite, so is $C(\varepsilon, \lambda)$.

Proof. Again let (μ, v) satisfy (3.6). From Lemma 3 we know that $\mu(\lambda) > 0$ for all $\lambda \in (0, \overline{\lambda})$. By the variational properties of the first eigenvalue of $F_u(\lambda, u(\lambda))$, we have that for all $\varphi \in H^1(M)$

$$\int_M |\nabla \varphi|^2 - \lambda \int_M \varphi^2 + p \int_M h u^{p-1} \varphi^2 \ge \mu \int_M \varphi^2. \tag{3.9}$$

Choosing $\varphi = (h^*)^s u^k$ in (3.9), where $s > 0$ and $k > p$ will be determined later, we have

$$\nabla \varphi = s(h^*)^{s-1} u^k \nabla(h^*) + k(h^*)^s u^{k-1} \nabla u,$$
$$|\nabla \varphi|^2 = s^2 (h^*)^{2s-2} u^{2k} |\nabla(h^*)|^2 + k^2 (h^*)^{2s} u^{2k-2} |\nabla u|^2 + 2sk h^{2s-1} u^{2k-1} \nabla h \nabla u.$$

Replacing the terms in (3.9) by the above expressions, we have

(3.10)
$$\begin{aligned} s^2 \int_M (h^*)^{2s-2} u^{2k} |\nabla(h^*)|^2 + 2sk \int_M (h^*)^{2s-1} u^{2k-1} \nabla h^* \nabla u \\ + k^2 \int_M (h^*)^{2s} u^{2k-2} |\nabla u|^2 - \lambda \int_M (h^*)^{2s} u^{2k} + p \int_M h (h^*)^{2s} u^{p-1+2k} \\ \ge \mu \int_M (h^*)^{2s} u^{2k}. \end{aligned}$$

Recall

$$\Delta u + \lambda u + h u^p = 0 \quad \text{on } M. \tag{1.1}$$

Choosing $\psi = k u^{2k-1} (h^*)^{2s}$, multiplying both sides of (1.1) by ψ, and integrating by parts over M, we have

$$\begin{aligned} 2sk \int_M u^{2k-1} (h^*)^{2s-1} \nabla(h^*) \nabla u + k(2k-1) \int_M u^{2k-2} (h^*)^{2s} |\nabla u|^2 \\ - \lambda k \int_M (h^*)^{2s} u^{2k} + k \int_M h (h^*)^{2s} u^{p+2K-1} = 0. \end{aligned} \tag{3.11}$$

Subtracting (3.11) from (3.10), we have

$$s^2 \int_M (h^*)^{2s-2} u^{2k} |\nabla (h^*)|^2 - k(k-1) \int_M (h^*)^{2s} u^{2k-2} |\nabla u|^2 + \lambda(k-1) \int_M (h^*)^{2s} u^{2k} + (p-k) \int_M h(h^*)^{2s} u^{p+2k-1} \geq \mu \int_M (h^*)^{2s} u^{2k}.$$

Since $k > p$, we have

$$(3.12) \qquad \begin{aligned} &(k-p) \int_M (h^*)^{1+2s} u^{p+2k-1} + k(k-1) \int_M (h^*)^{2s} u^{2k-2} |\nabla u|^2 \\ &\leq s^2 \int_M (h^*)^{2s-2} u^{2k} |\nabla (h^*)|^2 + \lambda(k-1) \int_M (h^*)^{2s} u^{2k}. \end{aligned}$$

By using (3) in Remark 2, we have

$$(3.13) \qquad \begin{aligned} \int_M (h^*)^{2s-2} u^{2k} |\nabla (h^*)|^2 &= \int_{M_+} (h^*)^{2(s-\varepsilon)} u^{2k} \left| \frac{\nabla (h^*)}{(h^*)^{1-\varepsilon}} \right|^2 \\ &\leq \frac{C}{\varepsilon^4} \int_{M_+} (h^*)^{2(s-\varepsilon)} u^{2k} \\ &\leq \frac{C}{\varepsilon^4} \left[\int_{M_+} ((h^*)^{2(s-\varepsilon)} u^{2k})^{\frac{2k+p-1}{2k}} \right]^{\frac{2k}{2k+p-1}} |M_+|^{\frac{p-1}{2k+p-1}} \end{aligned}$$

$$(3.14) \qquad \left(\begin{aligned} &\text{Setting } \frac{(s-\varepsilon)(2k+p-1)}{k} = 1 + 2s, \\ &\text{we have} \quad s = \frac{(1+2\varepsilon)k}{p-1} + \varepsilon, \quad s - \varepsilon = \frac{1+2\varepsilon}{p-1} k. \end{aligned} \right)$$

$$= \frac{C}{\varepsilon^4} \left[\int_{M_+} (h^*)^{1+2s} u^{2k+p-1} \right]^{\frac{2k}{2k+p-1}} |M_+|^{\frac{p-1}{2k+p-1}}.$$

By using Young's inequality, we have

$$(3.15) \qquad \begin{aligned} s^2 &\int_{M_+} (h^*)^{2s-2} u^{2k} |\nabla (h^*)|^2 \\ &\leq \frac{Cs^2}{\varepsilon^4} \left[\int_{M_+} (h^*)^{1+2s} u^{2k+p-1} \right]^{\frac{2k}{2k+p-1}} |M_+|^{\frac{p-1}{2k+p-1}} \\ &\leq \int_{M_+} (h^*)^{1+2s} u^{2k+p-1} + \left(\frac{Cs^2}{\varepsilon^4} \right)^{\frac{2k+p-1}{p-1}} |M_+|, \end{aligned}$$

$$(3.16) \qquad \begin{aligned} l(k-1) &\int_{M_+} (h^*)^{2s} u^{2k} \\ &\leq \lambda(k-1) \sup_{M_+} |(h^*)^{2\varepsilon}| \int_{M_+} (h^*)^{2(s-\varepsilon)} u^{2k} \\ &\leq \int_{M_+} (h^*)^{1+2s} u^{2k+p-1} + \left(\lambda(k-1) \sup_{M_+} |(h^*)^{2\varepsilon}| \right)^{\frac{2k+p-1}{p-1}} |M_+|. \end{aligned}$$

Combining (3.12), (3.14), (3.15), and (3.16), we have

$$\int_{M_+} (h^*)^{1+2s} u^{p+2k-1} \leq \left(\frac{Cs^2}{\varepsilon^4} + Ck\right)^{\frac{2k+p-1}{p-1}} |M_+|,$$

and

$$\int_{M_+} (h^*)^{2s} u^{2k-2} |\nabla u|^2 \leq \left(\frac{Cs^2}{\varepsilon^4} + Ck\right)^{\frac{2k+p-1}{p-1}} |M_+|,$$

where $s = (1+2\varepsilon)k/(p-1) + \varepsilon$ and $k > p+3$ will be determined later.

By using (3.14) and the fact $s = O(k)$, we have

$$\int_{M_+} \left((h^*)^{\frac{1+2\varepsilon}{p-1}} u\right)^{p+2k-1} \leq \left(\frac{Cs^2}{\varepsilon^4} + Ck\right)^{\frac{p+2k-1}{p-1}} |M_+|, \tag{3.17}$$

and

$$\int_{M_+} (h^*)^{\frac{2k(1+\varepsilon)}{p-1}+\varepsilon} u^{2k-2} |\nabla u|^2 \leq \left(\frac{Cs^2}{\varepsilon^4} + Ck\right)^{\frac{p+2k-1}{p-1}} |M_+|, \tag{3.18}$$

where $C = C(\sup_{M_+} |h^*|,\ \sup_{M_+} |\nabla h^*|,\ \lambda,\ p)$ and $0 < \varepsilon \ll 1$.

Now set $\omega = (h^*)^{\frac{s}{k}} u$, where $s/k = (1+2\varepsilon)/(p-1) + \varepsilon/k$. We have

$$\begin{aligned}
\int_{M_+} |\nabla \omega^k|^2 &= s^2 \int_{M_+} (h^*)^{2s-2} u^{2k} |\nabla h^*|^2 + k^2 \int_{M_+} (h^*)^{2s} u^{2k-2} |\nabla u|^2 \\
&\quad + 2sk \int_{M_+} (h^*)^{2s-1} u^{2k-1} \nabla(h^*) \nabla u \\
&\leq 2s^2 \int_{M_+} (h^*)^{2s-2} u^{2k} |\nabla h^*|^2 + 2k^2 \int_{M_+} (h^*)^{2s} u^{2k-2} |\nabla u|^2 \\
&\leq 2s^2 \int_{M_+} (h^*)^{2s-2} u^{2k} |\nabla h^*|^2 \\
&\quad + \frac{2k^2}{k(k-1)} \left[s^2 \int_{M_+} (h^*)^{2s-2} u^{2k} |\nabla h^*|^2 + \lambda(k-1) \int_{M_+} (h^*)^{2s} u^{2k} \right],
\end{aligned}$$

where (3.12) is used.

For k big enough, we have

$$\frac{k^2}{k(k-1)} < 2, \qquad k - 1 < k^2,$$

and

$$\int_{M_+} (h^*)^{2s} u^{2k} \leq \sup_{M_+} |(h^*)^{2\varepsilon}| \int_{M_+} (h^*)^{2(s-\varepsilon)} u^{2k},$$

$$\int_{M_+} (h^*)^{2s-2} u^{2k} |\nabla(h^*)|^2 = \int_{M_+} (h^*)^{2(s-\varepsilon)} u^{2k} \left| \frac{\nabla(h^*)}{(h^*)^{1-\varepsilon}} \right|^2 \leq \frac{C}{\varepsilon^4} \int_{M_+} (h^*)^{2(s-\varepsilon)} u^{2k}.$$

Combining the above inequalities, we have

$$\int_{M_+} |\nabla \omega^k|^2 \leq C\left(\frac{s^2}{\varepsilon^4} + \lambda k\right) \int_{M_+} (h^*)^{2(s-\varepsilon)} u^{2k} \leq C(1+\lambda) \frac{k^2}{\varepsilon^4} \int_{M_+} (h^*)^{2(s-\varepsilon)} u^{2k}$$

since $s = (1+2\varepsilon)k/(p-1) + \varepsilon$, $0 < \varepsilon \ll 1$.

By using the Sobolev inequality, we have

$$\left(\int_{M_+}((h^*)^s u^k)^{\frac{2n}{n-2}}\right)^{\frac{n-2}{n}} \le C\int_{M_+}|\nabla\omega^k|^2 \le C(1+\lambda)\frac{k^2}{\varepsilon^4}\int_{M_+}(h^*)^{2(s-\varepsilon)}u^{2k},$$

i.e.,

$$\|(h^*)^{s/k}u\|_{L^{2kn/(n-2)}(M_+)} \le C^{\frac{1}{2k}}(1+\lambda)^{\frac{1}{2k}}\left(\frac{k^2}{\varepsilon^4}\right)^{\frac{1}{2k}}\|(h^*)^{\frac{s-\varepsilon}{k}}u\|_{L^{2k}(M_+)},$$

where $s/k = (1+2\varepsilon)/(p-1)+\varepsilon/k$. Now set

$$\chi = \frac{n}{n-2}, \qquad k = \chi^m, \tag{3.19}$$

where m is a sufficiently large integer. Then we have

$$\|(h^*)^{\frac{1+2\varepsilon}{p-1}+\frac{\varepsilon}{k}}u\|_{L^{2\chi^{m+1}}(M_+)} \le C^{\frac{1}{2\chi^m}}\left(\frac{\chi^{2m}}{\varepsilon^4}\right)^{\frac{1}{2\chi^{2m}}}\|(h^*)^{\frac{1+2\varepsilon}{p-1}}u\|_{L^{2\chi^m}(M_+)}. \tag{3.20}$$

Choose $\varepsilon = \varepsilon_0 > 0$ small enough and set

$$\frac{1+2\varepsilon_0}{p-1}+\frac{\varepsilon_0}{\chi^m} = \frac{1+2(1+\frac{p-1}{2\chi^M})\varepsilon_0}{p-1} = \frac{1+2\varepsilon_1}{p-1},$$

where $\varepsilon_1 = (1+(p-1)/2\chi^m)\varepsilon_0$. Then (3.20) becomes

$$\|(h^*)^{\frac{1+2\varepsilon_1}{p-1}}u\|_{L^{2\chi^{m+1}}(M_+)} \le C^{\frac{1}{2}\chi^{-m}}\left(\frac{\chi^{2m}}{\varepsilon_0^4}\right)^{\frac{1}{2}\chi^{-2m}}\|(h^*)^{\frac{1+2\varepsilon_0}{p-1}}u\|_{L^{2\chi^m}(M_+)}.$$

By the same procedure as before, we have

$$\begin{aligned}
&\|(h^*)^{\frac{1+2\varepsilon_1}{p-1}+\frac{\varepsilon_1}{\chi^{m+1}}}u\|_{L^{2\chi^{m+2}}(M_+)} \\
&\le C^{\frac{1}{2}\chi^{-(m+1)}}\left(\frac{\chi^{2(m+1)}}{\varepsilon_1{}^4}\right)^{\frac{1}{2}\chi^{-2(m+1)}}\|(h^*)^{\frac{1+2\varepsilon_1}{p-1}}u\|_{L^{2\chi^{m+1}}(M_+)} \\
&\le C^{\frac{1}{2}\chi^{-(m+1)}}\left(\frac{\chi^{2(m+1)}}{\varepsilon_0{}^4}\right)^{\frac{1}{2}\chi^{-2(m+1)}}\|((h^*))^{\frac{1+2\varepsilon_1}{p-1}}u\|_{L^{2\chi^{m+1}}(M_+)} \\
&\qquad\qquad\left(\text{since } \varepsilon_1 = \left(1+\frac{p-1}{2\chi^m}\right)\varepsilon_0 > \varepsilon_0\right) \\
&\le C^{\frac{1}{2}(\chi^{-(m+1)}+\chi^{-m})}\left(\frac{\chi^{(m+1)\chi^{-2(m+1)}+m\chi^{-2m}}}{\varepsilon_0^{2(\chi^{-2(m+1)}+\chi^{-2m})}}\right)\|(h^*)^{\frac{1+2\varepsilon_0}{p-1}}u\|_{L^{2\chi^m}(M_+)}.
\end{aligned}$$

Setting $\varepsilon_{n+1} = (1+(p-1)/2\chi^{m+n})\varepsilon_n$ and iterating the above inequality, we have

$$\begin{aligned}
\|(h^*)^{\frac{1+2\varepsilon_{n+1}}{p-1}}u\|_{L^{2\chi^{m+n+1}}(M_+)} &= \|(h^*)^{\frac{1+2\varepsilon_n}{p-1}+\frac{\varepsilon_n}{\chi^{m+n}}}u\|_{L^{2\chi^{m+n+1}}(M_+)} \\
&\le C^{\frac{1}{2}\sum_{i=0}^n\chi^{-(m+i)}}\frac{\chi^{\sum_{i=0}^n(m+i)\chi^{-2(m+i)}}}{\varepsilon_0^{2\sum_{i=0}^n\chi^{-2(m+i)}}}\|(h^*)^{\frac{1+2\varepsilon_0}{p-1}}u\|_{L^{2\chi^m}(M_+)}.
\end{aligned}$$

Since $\chi = n/(n-2) > 1$, we have

$$\sum_{i=0}^{\infty} \frac{1}{2\chi^{m+i}} = \frac{n}{4\chi^m} \le \frac{C}{\chi^m}, \qquad \sum_{i=0}^{\infty} \frac{m+i}{2\chi^{m+i}} \le \frac{C}{\chi^{2m}},$$

and

$$\varepsilon_n = \varepsilon_{n-1}\left(1 + \frac{p-1}{2\chi^{m+n-1}}\right) = \prod_{i=0}^{n-1}\left(1 + \frac{p-1}{2\chi^{m+i}}\right)\varepsilon_0.$$

Let

$$\sigma = \prod_{i=0}^{\infty}\left(1 + \frac{p-1}{2\chi^{m+i}}\right).$$

We have that σ is finite and

$$\begin{aligned} \|(h^*)^{\frac{1+2\sigma\varepsilon_0}{p-1}} u\|_{L^{2\chi^{m+n}}(M_+)} &\le C\|(h^*)^{\frac{1+2\varepsilon_n}{p-1}} u\|_{L^{2\chi^{m+n}}(M_+)} \\ &\le C^{\sum_{i=0}^{n-1}\frac{1}{2}\chi^{-(m+i)}} \chi^{\sum_{i=0}^{n-1}(m+i)\chi^{-2(m+i)}} \varepsilon_0^{-2\sum_{i=0}^{n-1}\chi^{-2(m+i)}} \|(h^*)^{\frac{1+2\varepsilon_0}{p-1}} u\|_{L^{2\chi^m}(M_+)} \\ &\le C(\chi, \varepsilon_0, \|h\|_{L^\infty(M_+)}) \|h^{\frac{1+2\varepsilon_0}{p-1}} u\|_{L^{2\chi^m}(M_+)}. \end{aligned}$$

Letting $n \to \infty$, we have

$$\sup_{x\in M_+} |(h^*)^{\frac{1+2\sigma\varepsilon_0}{p-1}} u| \le C(\chi, \varepsilon_0, \|h\|_{C^1(M_+)}) \|(h^*)^{\frac{1+2\varepsilon_0}{p-1}} u\|_{L^{2\chi^m}(M_+)}.$$

Combining the above inequality with (3.17), we have

$$\sup_{x\in M_+} |(h^*)^{\frac{1+2\sigma\varepsilon_0}{p-1}} u| \le C(\lambda, \varepsilon_0, p, \|h\|_{C^1(M_+)})|M_+|.$$

Since $1 < \sigma < C$, for all $\varepsilon > 0$, we can choose ε_0 such that $\varepsilon = 2\sigma\varepsilon_0$ and

$$\sup_{x\in M_+} |(h^*)^{\frac{1+2\varepsilon}{p-1}} u| \le C(\lambda, p, \|h\|_{C^1(M_+)})\varepsilon^{-\mu}|M_+|, \tag{3.21}$$

where $\mu = 2\sum_{i=0}^{\infty} 1/\chi^{2(m+i)} < +\infty$ $\square$.

Lemma 5. *Assume that $M_0 \neq \varnothing$. Then $\bar{\lambda} = \lambda_1$ in problem (1.1), where λ_1 is the first eigenvalue of the Dirichlet problem on M_0 with the unit corresponding eigenfunction $\varphi > 0$.*

Proof. From the definition of λ_1 and φ, we have

$$\begin{aligned} \Delta\varphi + \lambda_1\varphi &= 0 \quad \text{on } M_0, \\ \varphi &> 0 \quad \text{on } M_0, \\ \varphi &= 0 \quad \text{on } \partial M_0. \end{aligned} \tag{3.22}$$

Assume $u(\lambda)$ is a positive solution of (1.1) with $\lambda > 0$,

$$\Delta u(\lambda) + \lambda u(\lambda) - hu^p(\lambda) = 0 \quad \text{on } M. \tag{1.1}$$

Denote

$$\varphi^* = \begin{cases} \varphi(x), & x \in M_0, \\ 0, & x \in M\backslash M_0. \end{cases}$$

We know $\varphi^* \in H^1(M)$, $\varphi^* \geq 0$ on M. Multiplying both sides of (1.1) by φ^* and integrating by parts over M, we have

$$-\int_{M_0} \nabla\varphi\nabla u + \lambda \int_{M_0} \varphi u = 0. \tag{3.23}$$

Multiplying both sides of (3.22) by $u(\lambda)$ and integrating by parts over M_0, we have

$$\int_{\partial M_0} u\frac{\partial\varphi}{\partial\nu} - \int_{M_0} \nabla\varphi\nabla u + \lambda_1 \int_{M_0} \varphi u = 0. \tag{3.24}$$

Subtracting (3.24) from (3.23), we have

$$\int_{\partial M_0} u\frac{\partial\varphi}{\partial\nu} + (\lambda_1 - \lambda)\int_{M_0} \varphi u = 0.$$

Since $\frac{\partial\varphi}{\partial\nu} < 0$ on ∂M_0, we have $\lambda_1 > \lambda$. Therefore $\overline{\lambda} \leq \lambda_1$. We claim that

$$\lim_{\lambda\to\overline{\lambda}} \|u(\lambda)\|_{L^2(M)} = +\infty. \tag{3.25}$$

In fact, suppose (3.25) is not true. Then there exist a constant $C < \infty$ such that

$$\int_M u^2(\lambda) \leq C \quad \text{for all } \lambda < \overline{\lambda}. \tag{3.26}$$

From equation (1.1) we have

$$\int_M |\nabla u|^2 + \int_M hu^{p+1} = \lambda \int_M u^2 \leq \lambda C,$$

and

$$k\int_M u^{k-1}|\nabla u|^2 + \int_M hu^{p+k} = \lambda \int_M u^{k+1}, \quad \text{for all } k > 1.$$

By using a similar iteration as in the proof of Lemma 4, we can prove that

$$\|u\|_{L^\infty(M)} \leq C\|u\|_{L^2(M)} \leq C. \tag{3.27}$$

From (3.26) and (3.27) we have

$$u(\overline{\lambda})(x) = \lim_{\lambda\to\overline{\lambda}} u(\lambda)(x)$$

is a positive C^2-solution of (1.1).

It is easy to show that the functional $F(\lambda, u) = \Delta u + \lambda u - hu^p$ is nonsingular at $(\overline{\lambda}, u(\overline{\lambda}))$. Therefore we can extend the bifurcation curve beyond the $\overline{\lambda}$ by using the Implicit Function Theorem. But this contradicts the definition of $\overline{\lambda}$. Hence the claim of (30) is true.

Define

$$\omega(\lambda) = \frac{u(\lambda)}{\|u(\lambda)\|_{L^2(M)}}.$$

Then $\omega(\lambda)$ is a positive solution of

$$\Delta\omega + \lambda\omega - hu^{p-1}\omega = 0 \quad \text{on } M, \quad \text{and} \quad \|\omega\|_{L^2(M)} = 1.$$

Let

$$F(x) = hu^{p-1}\omega(x) = \frac{hu^p}{\|u\|_{L^2(M)}}.$$

It follows that $F(x) \in L^1(M)$ with $\|F\|_{L^1(M)} \le C(\bar{\lambda}, n)$, and since

$$\Delta\omega + \lambda\omega = F(x) \ge 0 \quad \text{on } M,$$

ω is a positive sub-solution of

$$\Delta\omega + \lambda\omega = 0 \quad \text{on } M. \tag{3.28}$$

By using the same argument as applied in the estimate of (3.27), we have

$$\|\omega\|_{L^\infty(M)} \le C(\bar{\lambda}, n), \quad \text{and} \quad \|\omega(\lambda)\|_{H^1(M)} \le C(\bar{\lambda}, n).$$

By the boundedness of $\omega(\lambda)$ in $H^1(M)$, there is a subsequence of $\{\omega(\lambda)\}_{\lambda<\bar{\lambda}}$, say $\{\omega(\lambda_n)\}_{n=1}^{\infty}$, such that

$$\lim_{n\to\infty} \lambda_n = \bar{\lambda}, \quad \text{and} \quad \lim_{n\to\infty} \omega(\lambda_n) = \omega(\bar{\lambda}),$$

where $\omega(\lambda_n)$ is strong convergence in $L^2(M)$ and weak convergence in $H^1(M)$.

In M_0, for any $\lambda < \bar{\lambda}$, $\omega(\lambda)$ is a positive solution of

$$\Delta\omega + \lambda\omega = 0 \quad \text{on } M_0.$$

So $\omega(\bar{\lambda})$ is a weak solution of

$$\Delta\omega + \bar{\lambda}\omega = 0 \quad \text{on } M_0.$$

On the other hand, since $\omega(\lambda)$ is a sub-solution of (3.28), by the semicontinuity of the weak convergence we have that $\omega(\bar{\lambda})$ is a weak sub-solution of (3.28) in the following sense:

$$\int_M \nabla\omega(\bar{\lambda})\nabla\varphi - \bar{\lambda}\int_M \omega(\bar{\lambda})\varphi \le 0 \qquad \forall\varphi \in H^1(M), \ \varphi \ge 0.$$

By the same methods as before, we have

$$\|\omega(\bar{\lambda})\|_{L^\infty(M)} \le C\|\omega(\bar{\lambda})\|_{L^2(M)} \le C(\bar{\lambda}, n).$$

So by the regularity of the linear elliptic equation, we have that $\omega(\bar{\lambda})$ is a strong nonnegative solution of

$$\Delta\omega + \bar{\lambda}\omega = 0 \quad \text{on } M_0. \tag{3.29}$$

From Lemma 4, we know that

$$\lim_{\lambda\to\bar{\lambda}} \omega(\lambda)(x) = 0 \quad \text{a.e. on } M_+.$$

Hence

$$\omega(\bar{\lambda})(x) = 0 \quad \text{a.e. on } M_+. \tag{3.30}$$

We have already obtained that $\omega \in H^1(M)$, where $\omega|_{M_+} = 0$, M_+ is an open set in M. Now we claim that ω actually belongs to $H_0^1(M_0)$. It should be noted that the above problem with a more general setting has been studied in [12, 13] by L. I. Hedberg. He claimed that

Theorem H1 (Theorem 1.1 in [12]). *Let $f \in W_m^q(R^d)$ for some $q > 2 - \frac{1}{d}$ and some positive integer m. Let $K \subset R^d$ be closed, and suppose that $D^\alpha f|_K = 0$ for all α, $0 \le |\alpha| \le m-1$. Then $f \in \overset{\circ}{W}{}_m^q(K^c)$, where K^c denotes the complement of K, i.e. $R^d \backslash K$.*

Theorem H2 (Theorem 11 and Corollary 3 in [13]). *Let E be compact, $2 \le p < \infty$. If the inner boundary $\partial' E = 0$, $\varphi \in W_1^q$, and $\varphi = 0$ q-a.e. on E^c, then $\varphi \in \overset{\circ}{W}{}_1^q(E^o)$.*

Here we give a straightforward proof which is suited particularly to the case where ∂M_0 is C^1, i.e., we assert that

$$\omega(\bar{\lambda})(x) = 0 \quad \text{on } \partial M_0.$$

In fact, from (3.29) we have

$$\omega(\bar{\lambda}) \in C^\infty(M_0) \cap H^1(M) \cap L^\infty(M).$$

Let $M_r \subset\subset M_0$ be a subset in M_0 such that

$$M_r = \{x \in M_0 \mid \operatorname{dist}(x, \partial M_0) > r\}, \tag{3.31}$$

where $0 < r \ll 1$.

Since ∂M_0 is a C^1-boundary, ∂M_r is a piecewise C^1-boundary of M_r, for $r > 0$ small enough. Let $\vec{\mathbf{n}} = (n_1, n_2, \ldots, n_n)$ denote the unit normal vector on ∂M_r. Since ∂M_r is piecewise C^1, the $n_i(x)$, $x \in \partial M_r$, $i = 1, 2, \ldots, n$, are piecewise continuous on ∂M_r and we can modify n_i, $i = 1, 2, \ldots, n$, by a piecewise C^1-vector function $\vec{\psi} = (\psi_1, \psi_2, \ldots, \psi_n)$ on ∂M_r such that

$$\vec{\psi} \cdot \vec{\mathbf{n}} > \tfrac{1}{2} \quad \text{on } \partial M_r. \tag{3.32}$$

Furthermore we can extend the boundary vector function $\vec{\psi}$ to a global piecewise C^1-vector function, denoted by $\vec{\psi}$ again, on M such that

$$\|\vec{\psi}\|_{C^1(M)} \le C.$$

Since $\omega(\bar{\lambda}) \in H^1(M)$, we have

$$\int_M (\nabla\omega) \cdot \vec{\psi} = -\int_M \omega \operatorname{div}(\vec{\psi}) = -\int_{M_0} \omega \operatorname{div}(\vec{\psi}), \tag{3.33}$$

$$\begin{aligned} \int_M (\nabla\omega) \cdot \vec{\psi} &= \int_{M_r} (\nabla\omega) \cdot \vec{\psi} + \int_{M \backslash M_r} (\nabla\omega) \cdot \vec{\psi} \\ &= \int_{M_r} (\nabla\omega) \cdot \vec{\psi} + \int_{M_0 \backslash M_r} (\nabla\omega) \cdot \vec{\psi} \qquad \text{(by (3.30))} \\ &= \int_{\partial M_r} \omega(\vec{\psi} \cdot \vec{\mathbf{n}}) - \int_{M_r} \omega \operatorname{div} \vec{\psi} + \int_{M_0 \backslash M_r} (\nabla\omega) \cdot \vec{\psi}. \end{aligned} \tag{3.34}$$

Combining (3.33) with (3.34), we have

$$\int_{\partial M_r} \omega(\vec{\psi} \cdot \vec{\mathbf{n}}) = -\int_{M_0 \backslash M_r} \omega \operatorname{div} \vec{\psi} - \int_{M_0 \backslash M_r} (\nabla\omega) \cdot \vec{\psi}.$$

Since

$$\left| \int_{M_0 \backslash M_r} \omega \operatorname{div} \vec{\psi} \right| \le C \int_{M_0 \backslash M_r} dV \le Cr$$

and

$$\left|\int_{M_0\backslash M_r} \nabla\omega\cdot\vec{\boldsymbol{\psi}}\right| \le C\int_{M_0\backslash M_r} |\nabla\omega|$$
$$\le \left(\int_{M_0\backslash M_r} |\nabla\omega|^2\right)^{\frac{1}{2}} \left(\int_{M_0\backslash M_r} dV\right)^{\frac{1}{2}} \le Cr^{\frac{1}{2}},$$

we have

$$\int_{\partial M_r} \omega(\vec{\boldsymbol{\psi}}\cdot\vec{\mathbf{n}}) \le Cr^{\frac{1}{2}}.$$

From (3.32), we have

$$\int_{\partial M_r} \omega \le 2\int_{\partial M_r} \omega(\vec{\boldsymbol{\psi}}\cdot\vec{\mathbf{n}}) \le Cr^{\frac{1}{2}}, \tag{3.35}$$

where the constant C is independent of r.

Define $M_r^* = M_0\backslash M_r$. From (3.35) we have

$$\int_{M_r^*} \omega(\bar{\lambda}) \le Cr^{\frac{3}{2}}$$

and therefore

$$\left(\int_{M_r^*} \omega^2(\bar{\lambda})\right)^{\frac{1}{2}} \le C\left(\int_{M_r^*} \omega(\bar{\lambda})\right)^{\frac{1}{2}} \le Cr^{\frac{3}{4}}.$$

Now we are going to construct a sequence of $\omega_i \in W_0^{1,q}(M_0)$ for some $q > 1$, such that

$$\omega_i \to \omega(\bar{\lambda}) \quad \text{in } W_0^{1,q}(M_0).$$

Let $S_r = \{x \in M \mid \operatorname{dist}(x, \partial M_0) < r\}$ be a strip containing ∂M_0, and

$$\begin{aligned} \partial S_r^1 &= \{x \in M_+ | \operatorname{dist}(x, \partial M_0) = r\}, \\ \partial S_r^2 &= \{x \in M_0 | \operatorname{dist}(x, \partial M_0) = r\}, \\ S_r^- &= M_0\backslash\overline{M}_r. \end{aligned}$$

We claim that, for any $r > 0$ small enough, there is a "shrinking" diffeomorphism

$$\Psi : S_r \to S_r^- \tag{3.36}$$

with $\Psi(\partial S_r^1) = \partial M_0$, $\Psi(\partial S_r^2) = \partial S_r^2$, and

$$\sup_{x\in S_r} |\nabla\Psi| < C, \qquad \sup_{x\in S_r} |\nabla\Psi^{-1}| < C,$$

where C is a constant independent of r. Assume the above claim is true and let

$$\omega_r = \begin{cases} \omega(\Psi^{-1}(x)) & \text{if } x \in S_r, \\ \omega(x) & \text{if } x \in M_r. \end{cases}$$

It follows from (3.30) that $\omega_r \in H_0^1(M_0)$ for all $r > 0$ small enough, and

$$\|\omega_r - \omega\|_{L^2(M_0)} = \|\omega_r - \omega\|_{L^2(M_0\backslash M_r)} \le \|\omega_r\|_{L^2(M_0\backslash M_r)} + \|\omega\|_{L^2(M_0\backslash M_r)}$$
$$\le 2\|\omega\|_{L^2(M_0\backslash M_r)} \le Cr^{3/2},$$

and

$$\begin{aligned}\|\nabla(\omega_r - \omega)\|_{L^{2n/(n+1)}(M_0)} &= \|\nabla(\omega_r - \omega)\|_{L^{2n/(n+1)}(M_0 \setminus M_r)} \\ &\le \|\nabla \omega_r\|_{L^{2n/(n+1)}(M_0\setminus M_r)} + \|\nabla\omega\|_{L^{2n/(n+1)}(M_0\setminus M_r)} \\ &\le 2\sup|\nabla\Psi^{-1}|\, \|\nabla\omega\|_{L^{2n/(n+1)}(M_0\setminus M_r)} \le Cr^{1/n}.\end{aligned}$$

Therefore $\omega_r \to \omega$ strongly in $W_0^{1,\frac{2n}{n+1}}(M_0)$ and $\omega \in W_0^{1,\frac{2n}{n+1}}(M_0)$. Now applying the L^p estimates for the strong solutions of the second order elliptic equation (see e.g. [10, Theorem 9.14]), we have

$$\omega \in W^{2,\frac{2n}{n+1}}(M_0) \cap W_0^{1,\frac{2n}{n+1}}(M_0),$$

and $\|\omega\|_{W^{2,2n/(n+1)}}M_0 \le C$. Using the Sobolev imbedding inequality, we have $\|\nabla\omega\|_{L^{2n/(n-1)}(M_0)} \le C$, and therefore

$$(*) \qquad \|\nabla\omega\|_{L^2(M_0\setminus M_r)} \le Cr^{1/n}.$$

It follows from $(*)$ and the above argument that $\omega_r \to \omega$ strongly in $W_0^{1,2}(M_0)$ and $\omega \in H_0^1(M_0)$.

Next we need to show that the claim (3.36) is true.

In fact, for any $x_0 \in \partial M_0$, since ∂M_0 is of class C^1, there is an open set $U_{x_0} \subset M$ and a diffeomorphism

$$\phi_{x_0}\colon U_{x_0} \to V_{\phi(x_0)} \subset \mathscr{R}^n, \qquad V_{\phi(x_0)} \text{ is an open set in } \mathscr{R}^n.$$

Moreover, ϕ_{x_0} straightens the boundary $\partial M_0 \cap U_{x_0}$ in the following way: Let $B_{x_0} = U_{x_0} \cap S_r$; then

(1) $\phi_{x_0}(B_{x_0} \cap M_0) \subset \mathscr{R}^n_+$;
(2) $\phi(\partial M_0 \cap U_{x_0}) \subset \partial\mathscr{R}^n_+$;
(3) $\phi \in C^1(U_{x_0})$, $\phi^{-1} \in C^1(D)$, where $D = \phi(U_{x_0})$;
(4) $\phi(\partial S_r^1) \subset \mathscr{R}^n_-$ and $\phi(\partial S_r^2) \subset \mathscr{R}^n_+$.

Considering a finite covering of ∂M_0 by $U_i = U_{x_i}$, $i = 1, 2, \dots, m$, without loss of generality we may assume $\phi_{x_i}(B_{x_i})$ is a coordinate cube $\{x \in \mathscr{R}^n | -2 < x_i < 2,\ i = 1, \dots, n\}$ for $i = 1, \dots, m$, and $S_r \subset \bigcup_{i=1}^m B_i$.

For simplicity we may assume that ϕ_{x_i} also straightens the boundaries $\partial S_r^1 \cap U_{x_0}$ and $\partial S_r^2 \cap U_{x_0}$ such that $\phi(\partial S_r^1 \cap U_{x_0}) \subset \{x \in \mathscr{R}^n |\ 0 < x_i < 1,\ i = 1, 2, \dots, n-1;\ x_n = -1\}$ and $\phi(\partial S_r^2 \cap U_{x_0}) \subset \{x \in \mathscr{R}^n | 0 < x_i < 1,\ i = 1, 2, \dots, n-1;\ x_n = 1\}$ respectively. Since $r > 0$ small, ∂S_r^i also belongs to the C^1 class for $i = 1, 2$.

Let $\{\eta_i\}$, $i = 1, 2, \dots, m$, be a partition of unity subordinate to the covering $\{U_i\}_{i=1}^m$ satisfying

(1) $\eta_i \in C_0^1(U_i)$ for $i = 1, 2, \dots, m$;
(2) $\eta_i > 0$, $\sum \eta_i = 1$ in S_r,

and set

$$\Phi_r^j = \sum_{i=1}^m \eta_i \phi_i^j(x) \quad \text{for } j = 1, \dots, n.$$

It is easy to check that

$$\Phi_r = (\Phi_r^1, \Phi_r^2, \dots, \Phi_r^n): S_r \to \{x \in \mathscr{R}^n \mid -1 < x_n < 1\}$$

is a differomorphism.

Let

$$\rho_r = \left(x_1, x_2, \dots, \frac{x_n + r}{2}\right), \qquad \Psi_r = \Phi_r^{-1} \circ \rho_r \circ \Phi_r .$$

Then $\Psi_r : S_r \to S_r^-$ is a diffeomorphism with $\Psi_r(\partial S_r^1) = \partial M_0$ and $\Psi_r(\partial S_r^2) = \partial S_r^2$.

Therefore $\omega \in H_0^1(M_0)$. Hence $\omega(\bar\lambda)$ is the unit positive eigenfunction of the Dirichlet problem on M_0 and hence $\bar\lambda = \lambda_1$. □

Proof of Theorem 1. Conclusion (ii) of Theorem 1 follows from Lemma 5.

To prove (i), we need the following fact: Suppose $\{\Omega_n\}$, $n = 1, 2, \dots,$ is a sequence of normal connected open sets on M such that

$$\Omega_1 \supset \Omega_2 \supset \Omega_3 \supset \cdots \supset \Omega_n \supset \cdots$$

and Vol $\Omega_n \to 0$, as $n \to \infty$. Then the first eigenvalue $\lambda_1(n)$ of the Dirichlet problem on Ω_n satisfies

$$\lim_{n\to\infty} \lambda_1(n) = \infty. \tag{3.37}$$

Suppose the $\bar\lambda$ in (i) is finite. Then from the above geometric fact we can find a small connected open set $\Omega \subset M_+$ with smooth boundary such that the first eigenvalue λ_1 of the Dirichlet problem on Ω satisfies $\lambda_1 > \bar\lambda$. It is easy to construct a function h^* satisfying

$$0 \le h^* \le h \quad \text{and} \quad M_0(h^*) = \Omega. \tag{3.38}$$

Then for $\lambda \in (\bar\lambda, \lambda_1(h^*))$ there is a finite positive solution $u(\lambda)$ of

$$\Delta u + \lambda u - h^* u^p = 0 \quad \text{on } M.$$

From (3.38) we know that $u(\lambda)$ is a super-solution of (1.1), which means there is a positive solution of (1.1) for $\lambda > \bar\lambda$ but which contradicts the definition of $\bar\lambda$. Hence the conclusion of (i) is true. □

Proof of Theorem 2. Since the first eigenvalue λ_0 of the Dirichlet problem in a bounded smooth domain $\Omega \in \mathscr{R}^n$ is strictly positive and the corresponding eigenfunction v_1 keeps the same sign on Ω, we may choose it to be strictly positive in Ω.

The proof exactly follows that of Theorem 1, except we replace the first eigenvalue 0 of the Δ operator on the compact Riemannian manifold by the first eigenvalue $\lambda_0 > 0$ of the Dirichlet problem in Ω, and in the proof of Lemma 3, we Replace the small positive constant (as a sub-solution of problem (1.1)) by cv_1, where c is a small positive constant, as a sub-solution of problem (1.2).

Therefore we omit the details of the proof. □

Proof of Theorem 3. Since problems (1.3) and (1.1) have the same eigenvalue 0 and constant eigenfunction for the Δ, the proof of Theorem 3 follows exactly from the proof of Theorem 1. □

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