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82-5-111
高工研図書室

Ref.TH.3269-CERN

ON THE POSITIVITY OF THE EFFECTIVE ACTION IN A THEORY OF RANDOM SURFACES

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A B S T R A C T

It is shown that the functional $S[\eta] = \frac{1}{24\pi}\int(\frac{1}{2}|\nabla\eta|^2+2\eta)d\mu_0$, defined on C^∞ functions on the two-dimensional sphere, satisfies the inequality $S[\eta] \geq 0$ if η is subject to the constraint $\int(e^\eta-1)d\mu_0 = 0$. The minimum $S[\eta] = 0$ is attained at the solutions of the Euler-Lagrange equations. The proof is based on a sharper version of Moser-Trudinger's inequality (due to Aubin) which holds under the additional constraint $\int e^\eta \vec{x}\, d\mu_0 = 0$; this condition can always be satisfied by exploiting the invariance of $S[\eta]$ under the conformal transformations of S^2. The result is relevant for a recently proposed formulation of a theory of random surfaces.

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Ref.TH.3269-CERN
29 March 1982

1. - INTRODUCTION

Let $ds^2 = e^\eta ds_0^2$ denote a Riemannian metric on the two-dimensional sphere S^2, conformal to the standard metric $ds_0^2 = d\theta^2 + \sin^2\theta\, d\phi^2$. The points of S^2 will be parametrized, as usual, by a unit vector $\vec{x}$, by polar co-ordinates (θ,ϕ) or by a complex variable ξ, related to $\vec{x}$ by stereographic projection, i.e., $\xi = \cot\frac{\theta}{2}\, e^{i\phi} = (x_1+ix_2/1-x_3)$. The conformal factor e^η is assumed to be C^∞. Let $\Delta = e^{-\eta}\Delta_0$ be the Laplace-Beltrami operator associated to ds^2 and let $0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_n \leq \dots \to \infty$ be the spectrum of $-\Delta$ (Δ_0 and $\{\lambda_n^0\}$ will denote the corresponding objects belonging to ds_0^2).

It was shown in Ref. [1] that the limit

$$\frac{\text{“}\det \Delta\text{”}}{\det \Delta_0} \equiv \lim_{n\to\infty} \prod_{k=1}^{n} \frac{\lambda_k}{\lambda_k^0} = e^{-S[\eta]} \tag{1}$$

exists provided that e^η is normalized, i.e.,

$$\int (e^\eta - 1)\, d\mu_0 = 0 \tag{2}$$

where $d\mu_0 = \sin\theta\, d\theta \wedge d\phi$. A closed expression for $S[\eta]$ was obtained, namely

$$S[\eta] = \frac{1}{24\pi}\int_{S^2} \left\{ \frac{1}{2} |\nabla_0 \eta|^2 + 2\eta \right\} d\mu_0 \tag{3}$$

where ∇_0 is the covariant gradient with respect to ds_0^2, i.e.

$$|\nabla_0 \eta|^2 = \left(\frac{\partial\eta}{\partial\theta}\right)^2 + (\sin\theta)^{-2}\left(\frac{\partial\eta}{\partial\phi}\right)^2 \tag{4}$$

The Euler-Lagrange equations for $S[\eta]$ under the constraint Eq. (2) have the simple geometrical meaning that the metric $e^\eta ds_0^2$ has constant curvature. It follows that the general solution, giving all the stationary points of $S[\eta]$ is the following :

$$\eta = \eta_g^{(0)}(\xi) = 2 \ln \frac{1+|\xi|^2}{|\alpha\xi+\beta|^2+|\gamma\xi+\delta|^2} = -2\ln(\cosh\tau + \operatorname{sh}\tau\, \vec{n}\cdot\vec{x}) \tag{5}$$

where $g = \begin{pmatrix}\alpha & \beta \\ \gamma & \delta\end{pmatrix} \in SL(2,\mathbb{C})$, $\vec{n}$ is a unit vector and $\tau \in (0,+\infty)$. $S[\eta]$ vanishes at $\eta_g^{(o)}$ and its expansion around any of these stationary points has a positive semi-definite quadratic part, hence Eq. (5) gives indeed the local minima of $S[\eta]$. Since $S[\eta]$ is interpreted as the classical action of the field $\eta(\xi)$, it is important to know whether $\eta_g^{(o)}$ are merely local minima (metastable states) or whether they are indeed the absolute minima of $S[\eta]$. The problem is less trivial than it might appear at first sight, actually its solution requires some tools from non-linear analysis which are far from trivial.

The answer turns out to be very simple, however, as given by the following theorem :

Theorem : $S[\eta]$ is positive semi-definite under the constraint $\int(e^{\eta}-1)d\mu_0 = 0$ and $S[\eta] = 0$ implies $\eta = \eta_g^{(o)}$ for some $g \in SL(2,\mathbb{C})$.

2. - PROOF OF THE MAIN THEOREM

The proof of the theorem makes essential use of an "exponential" Sobolev inequality due to Aubin, combined with the invariance of $S[\eta]$ under conformal transformations.

Let us dispose of the constraint [Eq. (2)] by introducing

$$\eta = \psi - \ln \int e^{\psi} \frac{d\mu_0}{4\pi} \tag{6}$$

(ψ is defined up to an additive constant, which we may fix by requiring $\int \psi d\mu_0 = 0$, but this will not be necessary). The unconstrained functional is now

$$S[\eta] = \frac{1}{3}\int \left\{ \frac{1}{4} |\nabla_0 \psi|^2 + \psi \right\} \frac{d\mu_0}{4\pi} - \frac{1}{3} \ln \int e^{\psi} \frac{d\mu_0}{4\pi} \tag{7}$$

which was introduced long ago in a purely geometrical context [2]. It was shown by Moser [3] that $S[\eta]$ is bounded from below by some absolute constant. A sharper version of the inequality may hold, however, under additional constraints on ψ such as a parity condition [4] $\psi(x) = \psi(-x)$. More generally, Aubin [5] proved that if ψ satisfies

$$\int e^{\psi} \vec{x} \, d\mu_0 = 0 \tag{8}$$

then

$$\int e^{\psi} \frac{d\mu_0}{4\pi} \leq C(\varepsilon) \exp\left\{ \left(\tfrac{1}{8}+\varepsilon\right) \int |\nabla_0 \psi|^2 \frac{d\mu_0}{4\pi} + \int \psi \frac{d\mu_0}{4\pi} \right\} \tag{9}$$

for any $\varepsilon > 0$ and some constant $C(\varepsilon)$. Since the coefficient in the exponential is now $\frac{1}{8} + \varepsilon < \frac{1}{4}$ it follows that

$$3\, S[\eta] \geq \left(\tfrac{1}{8}-\varepsilon\right) \int |\nabla_0 \eta|^2 \frac{d\mu_0}{4\pi} - \ln C(\varepsilon) \tag{10}$$

Under these circumstances it is known that <u>the infimum of S is actually attained at the solutions of Euler-Lagrange equation</u> (see Aubin [5] for details on this point and Berger [6] for the general theory).

At this point, provided η satisfies the additional contraint (8), one has the sharp inequality

$$\begin{cases} S[\eta] \geq 0 \\ S[\eta] = 0 \Rightarrow \eta = 0 \end{cases} \tag{11}$$

In fact the Euler-Lagrange equation under the constraints (2) and (8) is

$$-\Delta_0 \eta + 2 = \lambda\, e^{\eta} + \vec{\mu}\cdot\vec{x}\, e^{\eta} \tag{12}$$

By integrating over S^2 one finds $\lambda = 2$. It is also known (Kazdan and Warner [7]) that the equation

$$\Delta_0 \eta = 2 - (2 + \vec{\mu}\cdot\vec{x})\, e^{\eta} \tag{13}$$

<u>does not admit any solution except for $\vec{\mu} \equiv 0$</u>, in which case we are led back to the general solution Eq. (5). Only $\eta = 0$ satisfies the constraint (8).

Now we come to the crucial observation that allows us to apply Aubin's result in general :

<u>Lemma</u> : The functional $S[\eta]$ is invariant under the transformations

$$\eta \to (T_g \eta)(\xi) = \eta(g^{-1}\xi) + \chi(g^{-1}, \xi) \tag{14}$$

where

$$g\xi = \frac{\alpha\xi+\beta}{\gamma\xi+\delta} \quad , \quad g = \begin{pmatrix}\alpha & \beta \\ \gamma & \delta\end{pmatrix} \in SL(2,\mathbb{C}) \tag{15}$$

$$\chi(g,\xi) = 2\ln\frac{1+|\xi|^2}{|\alpha\xi+\beta|^2+|\gamma\xi+\delta|^2} \tag{16}$$

A direct proof is not difficult, but it is rather cumbersome and not particularly enlightening. It is preferable to rely on the link between $S[\eta]$ and the Laplacian [Eq. (1)] and realize that $SL(2,\mathbb{C})$ is the largest connected group of conformal transformations of S^2 onto itself, Eq. (14) giving the transformation rule for η. The spectrum of the Laplacian is clearly the same for η and $T_g\eta$.

Now, without changing the value of $S[\eta]$, we can look for a $g \in SL(2,\mathbb{C})$ such that Eq. (8) is satisfied by $T_g\eta$. If such a g exists then, by Eq. (11),

$$S[\eta] = S[T_g\eta] \geqslant 0 \tag{17}$$

and $S[\eta] = 0 \Rightarrow T_g\eta = 0$ for some g which is the assertion of the theorem. So everything is reduced to the problem of finding a root of the equation

$$\int_{S^2} e^{(T_g\eta)(\xi)}\,\vec{x}(\xi)\,d\mu_0 = 0 \tag{18}$$

A simple topological argument will show that such a root actually exists, and the proof of the theorem will be complete. By inserting the definition of $T_g\eta$ and changing the integration variable to $g^{-1}\xi$, we get the equation

$$\int_{S^2} e^{\eta(\xi)}\,\vec{x}(g\xi)\,d\mu_0 = 0 \tag{19}$$

where g is the unknown. The function

$$\vec{X}(g) = \int_{S^2} e^{\eta(\xi)}\,\vec{x}(g\xi)\,d\mu_0 \tag{20}$$

defines a continuous map $\vec{X} : SL(2,\mathbb{C}) \to \mathbb{R}^3$ the image being contained in the unit ball $\|\vec{X}\| < 1$. For any $\lambda > 1$ let B_λ denote a sphere in $SL(2,\mathbb{C})$ defined by

$$\mathcal{B}_\lambda = \left\{ g \in SL(2,\mathbb{C}) \,\middle|\, g = u \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} u^\dagger ,\ u \in SU(2) \right\} \tag{21}$$

If λ is taken sufficiently large the image of B_λ under the map $\vec{X}$ is close to the sphere $\|\vec{X}\| = 1$; in fact,

$$\vec{x}\left(u\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}u^\dagger \xi\right) = \mathcal{D}(u)\, \vec{x}\left(\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} u^\dagger \xi\right) \tag{22}$$

$\mathcal{D} : SU(2) \to O(3)$ being the three-dimensional representation of $SU(2)$; but

$$\lim_{\lambda \to +\infty} \vec{x}\left(\lambda^2 (u^\dagger \xi)\right) = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \tag{23}$$

except for a set of measure zero $(u^+\xi = 0)$ which does not contribute to the integral. Hence

$$\lim_{\lambda \to +\infty} \vec{X}\left(u \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} u^\dagger\right) = \mathcal{D}(u) \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \tag{24}$$

This shows that for sufficiently large λ the map $\vec{X} : B_\lambda \to \mathbb{R}^3 - \{0\}$ is homotopically non-trivial. Since B_λ is contractible (it shrinks to the identity as $\lambda \to 1$) this implies the existence of a root. [A similar argument holds in a much more general setting (Gluck [8])].

3. - CONCLUDING REMARKS

We have shown that the action functional introduced in [1] in the context of Polyakov's theory of random surfaces [9] is indeed bounded from below and attains its absolute minimum at the "classical solutions" Eq. (5). Let us recall that the symmetry of $S[n]$ under conformal transformations is a reflection of the fact that Polyakov's "gauge choice" $g_{ab} = \rho\delta_{ab}$ does not completely fix the gauge in the case of simply connected surfaces. Our result shows that the residual gauge freedom can be consistently eliminated by imposing the additional constraint $\int e^{\eta}\vec{x}\,d\mu_0 = 0$, which near $\eta = 0$ reduces to the condition that η be orthogonal to the zero modes. All these problems are peculiar of the simply connected surfaces. For surfaces with Euler characteristic $\chi \leq 0$ there is no residual gauge freedom, no zero modes and the effective action is manifestly positive definite.

From a mathematical point of view, we have obtained the best constant in the Moser-Trudinger inequality, which now reads

$$\int_{S^2} e^{\psi} \frac{d\mu_0}{4\pi} \leq \exp\left\{ \frac{1}{4\pi}\int \left[\psi + \frac{1}{4}|\nabla_0\psi|^2 \right] d\mu_0 \right\} \tag{25}$$

If ψ is independent of ϕ, this reduces to the elementary inequality

$$\int_0^1 e^{\psi(t)}dt \leq \exp\left\{ \int_0^1 \psi(t)\,dt + \frac{1}{4}\int_0^1 t(1-t)\,\psi'(t)^2\,dt \right\} \tag{26}$$

the equality sign implying

$$\psi(t) = \ln\left[\frac{c_1}{(1+c_2 t)^2}\right], \quad (c_1>0,\ c_2>-1) \tag{27}$$

The inequality (26) is "complementary" to the arithmetic-geometric-mean inequality [10].

Finally, the result of the theorem implies the following bound on the spectrum of Δ, which does not seem to have been noticed previously

$$\lim_{n\to\infty} \prod_{k=1}^{n} \frac{\lambda_k}{\lambda_k^0} = e^{-S[\eta]} \leq 1 \tag{28}$$

the bound being saturated only by the standard metric (up to isometries).

ACKNOWLEDGEMENTS

The content of the paper is a corollary to a joint paper with M. Virasoro, whose constant encouragement is gratefully acknowledged. I warmly thank T. Aubin and J. Moser for useful correspondence and the CERN Theory Division for the kind hospitality in the years 1981-82.

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