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ON THE POSSIBILITY OF HEDGING OPTIONS IN THE PRESENCE OF TRANSACTION COSTS

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We study the continuous-time problem of hedging a generalized call option of the European and of the American type, in the presence of transaction costs. We show that if the price process of the relevant stock fluctuates with positive probability, then the only hedge that is possible for the American option is the trivial one. If the price of the stock, in addition to fluctuating with positive probability, is also stable with positive probability, then the same is true for the European option. We also show that in some sense, stable price with positive probability is a necessary condition for having only a trivial hedge for the European option. Our basic idea is to work with an appropriate discrete-time version of the problem which is transaction costs free. The mathematical tools that we use are elementary. A related result appears in Soner, Shreve and Cvitanic.

1. Introduction. In their fundamental paper, Black and Scholes (1973) discovered how to price options in continuous-time financial markets where stock prices follow the geometric Brownian motion model and the markets are free of transaction costs. By "option," we mean here a contract between a buyer and a seller whose value in some future date, "exercise time," will be equal to a given function of the underlying stock. The value of the option when it will be exercised will be transferred from the seller to the buyer. For the right to receive that transfer of wealth in the future, the buyer pays the seller a certain amount of money which is the option price. The main idea in Black and Scholes (1973) is that the option price should be the exact difference between the value of the option at the exercise time and the "capital gain" achieved from some "replicating portfolio." This replicating portfolio is based on the underlying stock and a money market account. By using the replicating portfolio, the seller is able to "hedge" his or her liability; namely, the seller will not lose any money from the option contract. The Black–Scholes theory was extended considerably and put on a more solid mathematical foundation in Harrison and Pliska (1981). By now, there are some good textbooks that review the Black–Scholes theory, for example Cox and Rubinstein (1985), Duffie (1992) and Merton (1992).

The main problem in the Black–Scholes theory is that the replicating portfolio demands continuous trading. This makes the theory not practical in

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the presence of transaction costs since the replicating portfolio will create an infinite amount of trading and hence an infinite amount of transaction costs. One possible compromise was suggested in Leland (1985) whose idea was to trade only periodically and by doing that create a trade-off: the transaction costs will be kept finite at the expense of having less than perfect hedging.

The problem of pricing options in the presence of transaction costs looks somehow easier in discrete-time finance where the stock price follows the binomial model and the number of trading dates is finite and constant (see Cox and Rubinstein (1985) for a presentation of that model). Indeed, the discrete-time problem with transaction costs was handled successfully in some recent papers: Bensaid, Lesne, Pages and Scheinkman (1992), Boyle and Vorst (1992), and Edirisinghe, Naik and Uppal (1993). It turns out that in the discrete-time set-up there is a replicating portfolio in spite of the transaction costs. In addition to that, it was discovered that if the requirement of exact replication is relaxed, it is sometimes possible to lower the option price. The idea is to insist that the option price will be greater than or equal to the difference between the value of the option at exercise time and the capital gain achieved from a "superreplicating" portfolio. The reason that this approach works better than exact replication is that exact replication demands too much trading, which is expensive due to the transaction costs. Obviously both the buyer and the seller will agree on that lower option price option; the buyer is always happy to pay less and the seller is still capable of hedging the option.

After understanding the discrete-time situation, we have found it quite natural to ask if the idea of superreplication can be carried out to the continuous-time set-up. This paper is an attempt to answer that question.

In our model the stock price follows a continuous, positive semimartingale which is nondegenerate in the sense that it fluctuates with positive probability [see (2.2) for the exact definition]. The option that we use is a generalized version of a "call option" [see (2.17)]. One has to pay transaction costs which are proportional to the value of the trades. The transaction costs are called "two-sided" when they are being charged in both buying and selling of shares. They are called "one-sided" when they are being charged only in buying shares (or only in selling shares). There is always a "trivial hedge": buy a share right away and do no more trading.

The main result of this paper is Theorem 3.22, which says that the only possible hedge for the American option, in the case of one-sided transaction costs, is the trivial one. We prove this result by observing the price process and the hedging portfolio each time the stock price goes up or down by a factor of e^{δ}, where $\delta > 0$ is a constant that is related to the size of the transaction costs. We show that in the presence of transaction costs, unless we use the trivial hedge, there is a positive probability that our hedge will not work at one of the observation times.

We also prove, under several sets of conditions, that the only possible hedge for the European option is the trivial one. The idea is to show that a European hedge is actually an American one, hence the result. In Theorem

4.9 the price process is a martingale (with respect to a risk-neutral probability measure), the transaction costs are one-sided and only "tame portfolios" (limited loss potential) are allowed. The result follows from the optional stopping theorem, a standard martingale method. In Theorem 4.14 the price process is both fluctuating and stable with positive probability, the transaction costs are two-sided and all portfolios are allowed. The result follows from price stability through integration by parts. In Theorem 4.26 we show that if the European option has only a trivial hedging, then, necessarily, there is some price stability.

In terms of financial interpretation, results like Theorem 3.22 and Theorem 4.14 tell us that one cannot use superreplicating portfolios for the purpose of option pricing, in the presence of transaction costs. On one hand, the seller will have to charge for the option the market value of a share. On the other hand, the buyer will not agree to pay that price, since the intrinsic value of a typical call option is always smaller than that of a share.

Now we describe the organization of the paper: Section 2 describes the model and contains all the assumptions, definitions and notations that are used throughout. It also contains precise statements of our main results. Section 3 deals with the American option. Section 4 deals with the European option. In Section 5 we bring two examples which have some relevance to our results about the European option. Section A.1 of the Appendix compares hedging options with portfolios restricted to finite variation sample paths and portfolios with no such restriction. This has relevance only when there are no transaction costs, and is not used in the rest of the paper. Section A.2 of the Appendix contains a detailed comparison between our paper and the preprint of Soner, Shreve and Cvitanic (1995) (henceforth referred to as SSC), which we received as we were about to write down our results. SSC deals with the European option when the stock price process is a geometric Brownian motion. Our results about the European option can be considered as a generalization of SSC in two directions: we remove the restriction on possible capital losses of the hedge portfolio, and we deal with the one-sided case. Also, and perhaps more important, we believe that the mathematical techniques that we use have more financial flavor to them. We also want to mention that we have learned that Davis and Clark (1994) have formally conjectured the result that was proved in SSC.

2. The model, basic definitions and main results. We consider a financial market in which one stock is traded in the time interval $0 \leq t \leq 1$. The price of this stock is represented by a stochastic process, $Z = \{Z(t): 0 \leq t \leq 1\}$, which is defined on a complete probability space (Ω, F, P). We will take Z to be a continuous semimartingale with respect to a filtration $\{F_t: 0 \leq t \leq 1\}$ that is right continuous, and such that F_t contains all P-null sets, $0 \leq t \leq 1$, and F_0 is a trivial σ-algebra. Since Z represents a price of a stock, we will assume that Z is a positive (> 0) process. We will also assume, without loss of generality as we show in Remark 2.19, that $Z(0) = 1$.

Assumption about interest rate. The interest rate will be 0 in our model. Since one can always work with discounted prices rather than the actual ones, this entails no loss of generality.

We want to assume that Z fluctuates enough. To define it precisely we need some notation. Let $0 \le \tau \le 1$ be a stopping time and let $\delta > 0$. We define the following stopping time.

$$\tau_\delta = \begin{cases} \inf\{\tau \le t \le 1: Z(t) = Z(\tau)e^{\delta} \text{ or } Z(t) = Z(\tau)e^{-\delta}\}, \\ 1, \quad \text{if no such } t \text{ exists.} \end{cases} \tag{2.1}$$

The following basic assumption on Z will hold throughout this paper.

Assumption 2.2. There exist $\delta_0 > 0$ so that for all stopping times $0 \le \tau \le 1$ and $0 < \delta < \delta_0$, we have, on the event $\{\tau < 1\}$, a.s.

$$P\big(\tau_\delta < 1, Z(\tau_\delta) = Z(\tau)e^{\delta} \,|\, F_\tau\big) > 0 \quad \text{and}$$
$$P\big(\tau_\delta < 1, Z(\tau_\delta) = Z(\tau)e^{-\delta} \,|\, F_\tau\big) > 0.$$

The number of shares that the investor owns, $M = \{M(t): 0 \le t \le 1\}$, is what is known as a portfolio.

Definition 2.3. A portfolio is an adapted stochastic process $M = \{M(t): 0 \le t \le 1\}$, which has a.s. right continuous sample paths with left limits, and has finite variation paths, namely

$$P\left(\int_0^1 |dM|(t) < \infty\right) = 1. \tag{2.4}$$

We denote the class of all portfolios by FV.

Let $M \in FV$. We define now two processes M^+ and M^-, which are associated with M:

$$\begin{aligned} M^+(t) &= \frac{M(0) + M(t) + \int_0^t |dM|(s)}{2}, \\ M^-(t) &= \frac{M(0) - M(t) + \int_0^t |dM|(s)}{2}. \end{aligned} \tag{2.5}$$

The processes M^+ and M^- are nondecreasing a.s. and satisfy:

(2.6) (i) $M = M^+ - M^-$,

$|dM| = dM^+ + dM^-$;

(2.6)(ii) $M^+(0) = M(0)$,

$M^-(0) = 0$.

The process $M^+(t)$ [respectively, $M^-(t)$] represents the accumulated number of shares that the owner of the M portfolio has bought (respectively, sold) up to time t. In terms of financial interpretation, this representation rules out the possibility of buying and selling shares at the same time.

Let $0 \le \lambda < 1$ and $0 \le \mu < 1$. Here λ (respectively, μ) represents the fractional transaction costs when buying (respectively, selling) shares. Namely, when one buys (respectively, sells) some shares at time $t > 0$, then one has to pay $\lambda Z(t)\, dM^+(t)$ [respectively, $\mu Z(t)\, dM^-(t)$] as transaction costs.

REMARK. It is assumed that no transaction costs are paid due to the holding of $M(0)$ shares at time $t = 0$.

The capital gain generated by a portfolio M is a stochastic process $\{S_M(t): 0 \le t \le 1\}$ defined by

$$S_M(t) = \int_0^t M(s)\, dZ(s) - \lambda \int_0^t Z(s)\, dM^+(s) - \mu \int_0^t Z(s)\, dM^-(s). \tag{2.7}$$

We review first the definition of the stochastic integral $\int_0^t M(s)\, dZ(s)$. For more details see Chapter 4 of Revuz and Yor (1991). Since Z is a continuous semimartingale we have the unique representation

$$Z = A + B, \tag{2.8}$$

where A and B are both adapted and continuous processes, A is a local martingale, $A(0) = 0$ and B has a.s. finite variation paths. Since M is bounded a.s., we have

$$\int_0^1 M^2(s) d[A, A](s) < \infty \quad \text{a.s.},$$

where $[A, A]$ is the quadratic variation of A, and

$$\int_0^1 |M(s)|\, |dB|(s) < \infty \quad \text{a.s.}$$

So we can define

$$\int_0^t M(s)\, dZ(s) = \int_0^t M(s)\, dA(s) + \int_0^t M(s)\, dB(s). \tag{2.9}$$

The financial interpretation of (2.7) is the following: $\int_0^t M(s)\, dZ(s)$, $\lambda \int_0^t Z(s)\, dM^+(s)$ and $\mu \int_0^t Z(s)\, dM^-(s)$ represent the capital gain before paying transaction costs, the transaction costs which are due to buying and the transaction costs which are due to selling, respectively, up to time t.

Next we describe what we mean by an option in this paper. With every option we associate a "payoff function" $g: (0, \infty) \to \mathbf{R}$. The option is a contract between two persons: a seller and a buyer. When the option is exercised at time t, the seller has the obligation to pay the buyer $g(Z(t))$ dollars. If $g(Z(t)) < 0$, then the buyer is actually paying the seller $-g(Z(t)) > 0$. We will deal with two types of options: a "European option" which is always exercised at time $t = 1$, and an "American option" in which the buyer has the right to decide at what stopping time τ, $0 \le \tau \le 1$, the option will be exercised.

The question that we are asking here is: what price should the seller charge the potential buyer, at $t = 0$, for the right to own the option? The idea is that the seller will charge the minimal amount of money that will allow him or her to hedge his or her liability. Namely, the seller will skillfully create a portfolio in such a way that whenever the option is exercised, the portfolio's capital gain plus the money received for the option from the buyer will be at least as large as the payment that the seller has to transfer to the buyer. In that way there is a certainty that the seller will not lose any money.

More precisely we define for each $M \in FV$,

$$\begin{aligned} x_M &= \inf\{x \in \mathbf{R}\colon x + S_M(1) \ge g(Z(1)) \text{ a.s.}\}. \\ y_M &= \inf\{y \in \mathbf{R}\colon y + S_M(\tau) \ge g(Z(\tau)) \text{ a.s.}, \\ &\qquad \text{all stopping times } 0 \le \tau \le 1\}. \end{aligned} \tag{2.10}$$

Both x_M and y_M will be taken to be ∞ if the sets above are empty.

We define the selling price of the option, in the case of European option, to be

$$b_E = \inf\{x_M\colon M \in FV\}. \tag{2.11}$$

In the case of the American option the selling price will be

$$b_A = \inf\{y_M\colon M \in FV\}. \tag{2.12}$$

Observe that since $x_M \le y_M$ for every $M \in FV$, it follows that

$$b_E \le b_A. \tag{2.13}$$

We will also be interested here in a variation of the European option where the set of portfolios that the seller can use to hedge his liability is restricted to a subset of FV.

DEFINITION 2.14. A portfolio M is called a tame portfolio if there exists a constant $D > -\infty$ so that $S_M(t) \ge D$, $0 \le t \le 1$, a.s.

When we restrict ourselves to tame portfolios, the appropriate definition of the selling price of the European option will be

$$\bar{b}_E = \inf\{x_M\colon M \text{ is a tame portfolio}\}. \tag{2.15}$$

We obviously have $b_E \le \bar{b}_E$, because the infimum in the definition of $\bar{b}_E$ is taken over a subset of FV. Together with (2.13), this gives

$$b_E \le b_A \wedge \bar{b}_E. \tag{2.16}$$

In this paper the payoff function $g\colon (0,\infty) \to \mathbf{R}$ will always have the following three properties:

(2.17) (i) $g(0+) = 0$,

(2.17) (ii) $\lim_{z\to\infty} \frac{g(z)}{z} = 1$ and

(2.17)(iii) $g(z) \ge C$ for all $z > 0$, where $C > -\infty$ is a constant.

An important example is $g(z) = (z-K)^+$, where $K \ge 0$ is a constant. An option with this payoff function is called a call option. In this case $b_A \le 1$ and

$b_E \le 1$, because when we take $M = 1$ identically, then

$$Z(t) = 1 + S_M(t) \ge (Z(t) - K)^+, \qquad 0 \le t \le 1.$$

Now we describe our main results. We start with the American option. (See Section 3.)

THEOREM 3.22. *Let $\lambda > 0$, $\mu = 0$ or $\lambda = 0$, $\mu > 0$. Then $b_A \ge 1$.*

We continue with results about the European option. (See Section 4.) First, we present a result about $\bar{b}_E$.

THEOREM 4.9. *Let $\lambda > 0$, $\mu = 0$ or $\lambda = 0$, $\mu > 0$. Assume that Z is a martingale under a probability measure Q which is equivalent to P. Then $\bar{b}_E \ge 1$.*

Now we present a result about b_E. In order to state the result, we define, for each $\delta > 0$, a sequence of stopping times

$$\tau_\delta^0 = 0, \qquad \tau_\delta^n = \left(\tau_\delta^{n-1}\right)_\delta, \qquad n \ge 1. \tag{2.18}$$

THEOREM 4.14. *Let $\lambda > 0$, $\mu > 0$. Assume that for every $\varepsilon > 0$, $\delta > 0$, $n \ge 0$, we have*

$$P\left((\tau_\delta^n)_\varepsilon = 1 \,|\, F_{\tau_\delta^n}\right) > 0 \quad a.s.,$$

Then $b_E \ge 1$. In particular, if for every stopping time $0 \le \tau \le 1$, $\varepsilon > 0$, we have $P(\tau_\varepsilon = 1 \,|\, F_\tau) > 0$, a.s. then $b_E \ge 1$.

Finally we offer a necessary condition for $b_E \ge 1$. We add an assumption on g that holds in the classical case.

THEOREM 4.26. *Assume that for every $0 < a < b$ there is $\eta > 0$ so that $g(z) + \eta < z$ if $a \le z \le b$. If $b_E \ge 1$ for all $\lambda > 0$, $\mu > 0$, then for every $\varepsilon > 0$, $\delta > 0$, $n \ge 0$, we have*

$$P\left((\tau_\delta^n)_\varepsilon = 1 \,|\, Z\left(\tau_\delta^k\right): 0 \le k \le n\right) > 0 \quad a.s.$$

Observe that the σ-algebra that appears in the conditional probability of (4.26) is smaller than the one in (4.14). So the combination of (4.14) and (4.26) does not give us a necessary and sufficient condition.

REMARK 2.19. Assume that $Z(0) = p > 0$, where p is not necessarily 1. We claim that all the results will be as before with only one difference: we will have $b_A \ge p$, $b_E \ge p$ and $\bar{b}_E \ge p$, rather than $b_A \ge 1$, $b_E \ge 1$ and $\bar{b}_E \ge 1$, respectively. In order to see that, we look at the process $\hat{Z}(t) = Z(t)/p$ and the payoff function $\hat{g}(z) = g(zp)/p$. Observe that $\hat{Z}$ and $\hat{g}$ satisfy the assumptions (2.2) and (2.17), respectively. We denote by $S_M^{\hat{Z}}$ and S_M^Z the capital gains associated with the price processes $\hat{Z}$ and Z, respectively.

Obviously $S_M^{\hat{Z}}(t) = S_M^Z(t)/p$. We conclude that for every $x \in R$,

$$x + S_M^Z(t) \geq g(Z(t)) \quad \text{iff } \frac{x}{p} + S_M^{\hat{Z}}(t) \geq \hat{g}(\hat{Z}(t))$$

and the claim follows when we apply our results to the pair $(\hat{Z}, \hat{g})$.

3. The American option. In this section we will prove our results about the American option. First we will prove the following.

THEOREM 3.1. *Let* $\lambda > 0$, $\mu > 0$. *Then* $b_A \geq 1$.

In order to prove Theorem 3.1 we will need two lemmas. The first, Lemma 3.2, is a simple result of the integration by parts formula. It allows us to create a discrete-time version of the problem by looking at the hedging portfolio when Z goes up or down by a factor of e^{δ}, where $\delta > 0$ is related to the order of λ and μ. The second lemma, Lemma 3.6, shows how to deal with that discrete-time version.

LEMMA 3.2. *Let* $\delta > 0$, $\lambda \geq e^{2\delta} - 1$. *Let* $0 \leq \tau_1 \leq \tau_2 \leq 1$ *be two stopping times. Assume that* $M \in FV$, *and that* $e^{-\delta} \leq Z(t)/Z(\tau_1) \leq e^{\delta}$, $\tau_1 \leq t \leq \tau_2$, *a.s. Then we have*

$$\int_{\tau_1}^{\tau_2} M(s)\, dZ(s) - \lambda \int_{\tau_1}^{\tau_2} Z(s) |dM|(s) \leq M(\tau_1)(Z(\tau_2) - Z(\tau_1)) \quad a.s.$$

PROOF. We will use the integration by parts formula:

$$M(b)Z(b) - M(a)Z(a) = \int_a^b Z(s)\, dM(s) + \int_a^b M(s)\, dZ(s) \quad \text{a.s.} \tag{3.3}$$

where $0 \leq a \leq b \leq 1$.

REMARK. $\int_a^b Z(s)\, dM(s) = \int_{]a,b]} Z(s)\, dM(s)$, so that if M has a jump at $t = a$, namely $M(a) \neq M(a-)$, it is not reflected in $\int_a^b Z(s)\, dM(s)$.

Formula (3.3) follows from the general integration by parts formula for semimartingales; see Protter (1990):

$$\begin{aligned} &M(b)Z(b) - M(a)Z(a) \\ &\quad = \int_a^b Z_-(s)\, dM(s) + \int_a^b M_-(s)\, dZ(s) + \int_a^b d[M, Z](s), \end{aligned} \tag{3.3*}$$

where M_- and Z_- are the left continuous versions of M and Z, respectively, and $[M, Z]$ is the coquadratic variation of M and Z. Equation (3.3) follows from (3.3*). Indeed, $Z_- = Z$, because Z is continuous; $M_-\, dZ = M\, dZ$ because

$$\int_0^1 (M(s) - M_-(s))^2 d[A, A](s) = 0 = \int_0^1 |M(s) - M_-(s)|\, |dB|(s) \quad \text{a.s.,}$$

where $Z = A + B$ as in (2.8), and finally $d[M, Z] = 0$ because Z is continuous and $M \in FV$. By using formula (3.3) we get:

$$\int_{\tau_1}^{\tau_2} (M(s) - M(\tau_1))\, dZ(s) = \int_{\tau_1}^{\tau_2} (Z(\tau_2) - Z(s))\, dM(s). \tag{3.4}$$

The right-hand side of (3.4) should be understood as a path-by-path integration. Next we observe that

$$\begin{aligned} |Z(\tau_2) - Z(t)| &\le Z(\tau_1)(e^{\delta} - e^{-\delta}) \\ &\le Z(t)(e^{2\delta} - 1), \qquad \tau_1 \le t \le \tau_2. \end{aligned}$$

From the simple estimate

$$\int_{\tau_1}^{\tau_2} (Z(\tau_2) - Z(s))\, dM(s) \le \int_{\tau_1}^{\tau_2} |Z(\tau_2) - Z(s)|\, |dM|(s),$$

we get

$$\begin{aligned} \int_{\tau_1}^{\tau_2} M(s)\, dZ(s) &\le M(\tau_1)(Z(\tau_2) - Z(\tau_1)) \\ &\quad + (e^{2\delta} - 1) \int_{\tau_1}^{\tau_2} Z(s) |dM|(s), \end{aligned} \tag{3.5}$$

and the result follows because $\lambda \ge e^{2\delta} - 1$. □

LEMMA 3.6. *Let $0 < c < 1$, $\delta > 0$. There exist an integer $N = N(c, \delta) \ge 1$ and a sequence of measurable functions $Z_k: \mathbf{R}^k \to R^+$, $k \ge 0$, so that $Z_0 = 1$, and for every sequence of numbers M_k, $k \ge 0$, we have*

$$\begin{aligned} &\frac{Z_k(M_0, \dots, M_{k-1})}{Z_{k-1}(M_0, \dots, M_{k-2})} = e^{\pm\delta}, \qquad k \ge 1 \quad \textit{and} \\ &\sup_{0 \le n \le N} \left\{ g(z_n) - \sum_{k=0}^{n-1} M_k (z_{k+1} - z_k) \right\} > c, \end{aligned} \tag{3.7}$$

where $z_k = Z_k(M_0, \dots, M_{k-1})$, $k \ge 0$.

PROOF. Let $f: R^+ \to (0, 1)$ be a strictly increasing function so that

$$c < f(0) < f(+\infty) < 1. \tag{3.8}$$

Next we define $z_k = Z_k(M_0, \dots, M_{k-1})$, $k \ge 0$, as follows: $z_0 = 1$, and

$$z_{k+1} = \begin{cases} z_k e^{\delta}, & \text{if } M_k < f(z_k), \\ z_k e^{-\delta}, & \text{if } M_k \ge f(z_k), \end{cases} \qquad k \ge 0. \tag{3.9}$$

Now let

$$\begin{aligned} S_0 &= 0, \\ S_0 &= \sum_{k=0}^{n-1} M_k (z_{k+1} - z_k), \qquad n \ge 1. \end{aligned}$$

The sequence $\{S_n\}$ satisfies the following:

(3.10) (i) If $m > k$ and $z_k > z_m$, then

$$S_m - S_k < f(0)(z_m - z_k).$$

(3.10) (ii) If $m > k$ and $z_k < z_m$, then

$$S_m - S_k < f(+\infty)(z_m - z_k).$$

(3.10)(iii) Let $a = e^{-r\delta}$, $b = e^{s\delta}$, and r, s are positive integers. If $a \le z_k \le b$, $0 \le k \le n$ and $r + s < n$, then

$$S_n \le f(+\infty)(b - a) - (n - (s + r))\frac{\theta}{2},$$

where

$$\theta = \inf_{-r \le k \le s} \left\{ \left(f(e^{(k+1)\delta}) - f(e^{k\delta}) \right) \left(e^{(k+1)\delta} - e^{k\delta} \right) \right\}.$$

We will prove (3.10) later. First, however, we will use (3.10) to finish the proof. We need to choose $0 < a < 1 < b$. There are three cases.

Case 1. If there is $n > 1$ such that $z_n = a$, then by (3.10)(i) we have

$$\begin{aligned} g(z_n) - S_n &\ge g(a) - f(0)(a - 1) \\ &= f(0) + \{g(a) - f(0)a\} > c, \end{aligned}$$

whenever a is close enough to 0 since $g(0+) = 0$ and $f(0) > c$.

Case 2. If there is $n > 1$ such that $z_n = b$, then by (3.10)(ii) we have

$$\begin{aligned} g(z_n) - S_n &\ge g(b) - f(+\infty)(b - 1) \\ &\ge g(b) - f(+\infty)b > c, \end{aligned}$$

when b is large enough since $\lim_{b \to \infty} g(b)/b = 1$ and $f(+\infty) < 1$.

Case 3. If $a = e^{-r\delta} < z_k < b = e^{s\delta}$, $k \ge 0$, then by (3.10)(iii) and by using that g is bounded below by $C > -\infty$, we have

$$\begin{aligned} g(z_n) - S_n &\ge C - S_n \\ &\ge C - f(+\infty)(b - a) + (n - (s + r))\frac{\theta}{2} \\ &\ge c, \end{aligned}$$

whenever $n > r + s$, and

$$n > \frac{(c - C + f(+\infty)(b - a))2}{\theta} + s + r.$$

So we choose $a = e^{-r\delta}$ small enough as described in Case 1, $b = e^{s\delta}$ large enough as described in Case 2, and we define

$$N = 1 + \left[\frac{(c - C + f(+\infty)(b - a))2}{\theta} + s + r \right]. \tag{3.11}$$

With this choice of N we obviously get (3.7).

We are now going back to the proof of (3.10). We first observe that for every $k \geq 0$ we have

$$
\begin{aligned}
S_{k+1} - S_k &= M_k(z_{k+1} - z_k) \\
&\leq f(z_k)(z_{k+1} - z_k) \\
&\leq \begin{cases} f(0)(z_{k+1} - z_k), & \text{if } z_{k+1} < z_k, \\ f(+\infty)(z_{k+1} - z_k), & \text{if } z_{k+1} > z_k. \end{cases}
\end{aligned} \tag{3.12}
$$

We also observe that

$$
\begin{aligned}
&\text{If } z_k = z_{m+1} \text{ and } z_{k+1} = z_m,\ k \neq m, \text{ then} \\
&\quad (S_{k+1} - S_k) + (S_{m+1} - S_m) \\
&\qquad < -[f(z_{k+1}) - f(z_k)](z_{k+1} - z_k) < 0.
\end{aligned} \tag{3.13}
$$

To see (3.13) we calculate

$$
\begin{aligned}
&(S_{k+1} - S_k) + (S_{m+1} - S_m) \\
&\quad \leq f(z_k)(z_{k+1} - z_k) + f(z_m)(z_{m+1} - z_m) \\
&\quad = f(z_k)(z_{k+1} - z_k) + f(z_{k+1})(z_k - z_{k+1}) \\
&\quad = -[f(z_{k+1}) - f(z_k)](z_{k+1} - z_k).
\end{aligned}
$$

Next we define, for every integer v and $0 \leq k < m$,

$$
\begin{aligned}
u(v, k, m) &= \sum_{n=k}^{m-1} \{(z_n, z_{n+1}) = (e^{v\delta}, e^{(v+1)\delta})\} \\
d(v, k, m) &= \sum_{n=k}^{m-1} \{(z_n, z_{n+1}) = (e^{(v+1)\delta}, e^{v\delta})\},
\end{aligned} \tag{3.14}
$$

where we identify sets with their indicator functions. In words, $u(v, k, m)$ and $d(v, k, m)$ are the number of changes $e^{v\delta} \uparrow e^{(v+1)\delta}$ and $e^{(v+1)\delta} \downarrow e^{v\delta}$, respectively, of the sequence $(z_k, \dots, z_m)$.

Now we verify (3.10)(i). We define for each v that satisfies $z_k > e^{v\delta} \geq z_m$:

$$
n(v) = \min\{n \geq k : (z_n, z_{n+1}) = (e^{(v+1)\delta}, e^{v\delta})\}.
$$

We get

$$
\begin{aligned}
S_m - S_k &= \sum_{n=k}^{m-1} (S_{n+1} - S_n) \\
&< \sum_{z_k > e^{v\delta} \geq z_m} S_{n(v)+1} - S_{n(v)} \\
&= \sum_{z_k > e^{v\delta} \geq z_m} M_{n(v)}(z_{n(v)+1} - z_{n(v)}) \\
&\leq \sum_{z_k > e^{v\delta} \geq z_m} f(0)(z_{n(v)+1} - z_{n(v)}) \\
&= f(0)(z_m - z_k).
\end{aligned}
$$

The first inequality follows from (3.13) and the fact that $z_k > e^{v\delta} \geq z_m$ implies $d(v,k,m) = u(v,k,m) + 1$, while $e^{v\delta} \geq z_k$ or $z_m \geq e^{v\delta}$ implies $d(v,k,m) = u(v,k,m)$. The second inequality follows from (3.12).

The proof of (3.10)(ii) is similar to the proof of (3.10)(i) and will be omitted.

Next we prove (3.10)(iii). We have

$$\begin{aligned} n &= \sum_{t=-r}^{s-1} d(t,0,n) + u(t,0,n) \\ &\leq \sum_{t=-r}^{s-1} 2\big[d(t,0,n) \wedge u(t,0,n)\big] + 1. \end{aligned}$$

So

$$\sum_{t=-r}^{s-1} \big[d(t,0,n) \wedge u(t,0,n)\big] \geq \frac{n-(r+s)}{2}. \tag{3.15}$$

Next we define

$$A_t = \big\{0 \leq k \leq n-1 \colon (z_k, z_{K+1}) = (e^{t\delta}, e^{(t+1)\delta}) \text{ or } (e^{(t+1)\delta}, e^{t\delta})\big\}, \qquad -r \leq t \leq s-1.$$

By using (3.12) and (3.13) we get

$$\sum_{k \in A_t} (S_{k+1} - S_k) \leq f(+\infty)(e^{(t+1)\delta} - e^{t\delta}) - \big[d(t,0,n) \wedge u(t,0,n)\big]\theta.$$

Using (3.15) we now have

$$\begin{aligned} S_n &= \sum_{t=-r}^{s-1} \sum_{k \in A_t} (S_{k+1} - S_k) \\ &\leq f(+\infty)(e^{s\delta} - e^{-r\delta}) - (n - (s+r))\frac{\theta}{2}. \end{aligned}$$ □

We will use now Lemmas 3.2 and 3.6 to prove Theorem 3.1.

Proof of Theorem 3.1. Let $M \in FV$ and $0 < c < 1$. We select $\delta > 0$ so that $\lambda \wedge \mu \geq e^{2\delta} - 1$. Let $\tau_n = \tau_\delta^n$, $n \geq 0$, be the sequence of stopping times defined by (2.18). By Lemma 3.2 and the remark after (3.3), we have for every $n \geq 1$, a.s.,

$$\begin{aligned} &\int_0^{\tau_n} M(s)\,dZ(s) - (\lambda \wedge \mu)\int_0^{\tau_n} Z(s)|dM|(s) \\ &\qquad \leq \sum_{k=0}^{n-1} M(\tau_k)(Z(\tau_{k+1}) - Z(\tau_k)). \end{aligned} \tag{3.16}$$

This leads to

$$\begin{aligned} &g(Z(\tau_n)) - S_M(\tau_n) \\ &\qquad \geq g(Z(\tau_n)) - \sum_{k=0}^{n-1} M(\tau_k)(Z(\tau_{k+1}) - Z(\tau_k)) \quad \text{a.s.} \end{aligned} \tag{3.17}$$

By Lemma 3.6 there is an integer $N \geq 1$, and there are measurable functions $Z_k: \mathbf{R}^k \to \mathbf{R}^+$, $k \geq 0$, so that

$$\sup_{0 \leq n \leq N} \left\{ g(z_n) - \sum_{k=0}^{n-1} M(\tau_k)(z_{k+1} - z_k) \right\} > c \quad \text{a.s.}, \tag{3.18}$$

where $z_k = Z_k(M(\tau_0), \dots, M(\tau_{k-1}))$, and $z_{k+1} = z_k e^{\pm\delta}$, $k \geq 0$. We will show that

$$P(Z(\tau_k) = z_k, 1 \leq k \leq N, \tau_N < 1) > 0. \tag{3.19}$$

This follows because the price process Z fluctuates according to Assumption 2.2. We will prove it formally by induction. To start the induction, we assume that $P(A_n) > 0$, for some n between 1 and $N - 1$ and where $A_n = \{Z(\tau_k) = z_k, 1 \leq k \leq n, \tau_n < 1)\} \in F_{\tau_n}$. Since

$$P(A_n) = P(A_n, z_{n+1} = z_n e^{\delta}) + P(A_n, z_{n+1} = z_n e^{-\delta}),$$

we will assume without loss of generality, that $P(A_n, z_{n+1} = z_n e^{\delta}) > 0$. Since $z_{n+1} \in F_{\tau_n}$ it follows that $\{A_n, z_{n+1} = z_n e^{\delta}\} \in F_{\tau_n}$. From Assumption 2.2 we get

$$P(A_n, z_{n+1} = z_n e^{\delta}, Z(\tau_{n+1}) = Z(\tau_n) e^{\delta}, \tau_{n+1} < 1) > 0.$$

But

$$A_{n+1} \supseteq \{A_n, z_{n+1} = z_n e^{\delta}, Z(\tau_{n+1}) = Z(\tau_n) e^{\delta}, \tau_{n+1} < 1\},$$

so $P(A_{n+1}) > 0$. By induction we now get (3.19). From (3.18) and (3.19) we conclude that

$$P\left(\sup_{0 \leq n \leq N} \left\{ g(Z(\tau_n)) - \sum_{k=0}^{n-1} M(\tau_k)(Z(\tau_{k+1}) - Z(\tau_k)) \right\} > c, \tau_N < 1 \right) > 0. \tag{3.20}$$

From (3.20) we learn that there is $0 \leq n \leq N$ so that

$$P\left(g(Z(\tau_n)) - \sum_{k=0}^{n-1} M(\tau_k)(Z(\tau_{k+1}) - Z(\tau_k)) > c, \tau_n < 1 \right) > 0. \tag{3.21}$$

By (3.17) it follows now that $y_M \geq c$. Since c is arbitrarily close to 1, we have $y_M \geq 1$. Since $M \in FV$ was arbitrary, we have $b_A \geq 1$. □

We will now extend Theorem 3.1 to the case of one-sided transaction costs.

THEOREM 3.22. *Let $\lambda > 0$, $\mu = 0$ or $\lambda = 0$, $\mu > 0$. Then $b_A \geq 1$.*

We need the following lemma for the case $\lambda > 0$, $\mu = 0$. The lemma has two parts. The first part will be used to show that, when the hedge is not trivial, there is a positive probability that the hedge will not work at one of the stopping times that we are interested in unless the number of shares at that time is negative. The second part of the lemma is used when the number of shares at that stopping time is negative. It gives a bound on the capital gain of the hedge when the stock price goes up. Basically the idea is that bad things happen when the hedge has a short position, $\lambda > 0$, and the stock price goes up. If one buys shares there are transaction costs to pay, and if one does nothing there is a capital loss. The hedge fails when the stock price, starting at the critical stopping time, goes up by a factor of $e^{3\delta}$, where $\delta > 0$ is chosen appropriately.

LEMMA 3.23. *Let* $\delta > 0$, $\lambda > 2\alpha/1 - \alpha$, *and* $\lambda' = (\lambda - \alpha)e^{-\delta}$ *where* $\alpha = e^{2\delta} - 1 < 1$. *Let* $0 \le \tau_1 \le \tau_2 \le 1$ *be two stopping times. Assume that* $e^{-\delta} \le Z(t)/z(\tau_1) \le e^{\delta}$, $\tau_1 \le t \le \tau_2$, *a.s. Then a.s.*,

(i)
$$\int_{\tau_1}^{\tau_2} M(s)\,dZ(s) - \lambda \int_{\tau_1}^{\tau_2} Z(s)\,dM^+(s) \le \frac{M(\tau_1)}{1-\alpha}(Z(\tau_2) - Z(\tau_1)) - \frac{\alpha}{1-\alpha}(Z(\tau_2)M(\tau_2) - Z(\tau_1)M(\tau_1)).$$

(ii) *Assume in addition that* $Z(\tau_2) = Z(\tau_1)e^{\delta}$. *Then*
$$\int_{\tau_1}^{\tau_2} M(s)\,dZ(s) - \lambda \int_{\tau_1}^{\tau_2} Z(s)\,dM^+(s) \le M(\tau_1)(Z(\tau_2) - Z(\tau_1)) - \lambda' Z(\tau_1)(M^+(\tau_2) - M^+(\tau_1)).$$

PROOF. (i). Using (3.5) and $|dM| = 2\,dM^+ - dM$ we get
$$\int_{\tau_1}^{\tau_2} M(s)\,dZ(s) \le M(\tau_1)(Z(\tau_2) - Z(\tau_1)) + \alpha\left(2\int_{\tau_1}^{\tau_2} Z(s)\,dM^+(s) - \int_{\tau_1}^{\tau_2} Z(s)\,dM(s)\right).$$
Since, by integration by parts,
$$\int_{\tau_1}^{\tau_2} Z(s)\,dM(s) = Z(\tau_2)M(\tau_2) - Z(\tau_1)M(\tau_1) - \int_{\tau_1}^{\tau_2} M(s)\,dZ(s),$$
we get
$$\int_{\tau_1}^{\tau_2} M(s)\,dZ(s) \le \frac{M(\tau_1)}{1-\alpha}(Z(\tau_2) - Z(\tau_1)) + \frac{\alpha}{1-\alpha}\left(2\int_{\tau_1}^{\tau_2} Z(s)\,dM^+(s) - [Z(\tau_2)M(\tau_2) - Z(\tau_1)M(\tau_1)]\right),$$
which leads to the result because $\lambda > 2\alpha/1 - \alpha$.

(ii). Since $0 < Z(\tau_2) - Z(t) \le \alpha Z(t)$, $\tau_1 \le t \le \tau_2$, we get

$$\int_{\tau_1}^{\tau_2} (Z(\tau_2) - Z(s))\, dM(s) \le \alpha \int_{\tau_1}^{\tau_2} Z(s)\, dM^+(s).$$

From integration by parts formula (3.4) we get

$$\int_{\tau_1}^{\tau_2} M(s)\, dZ(s) \le M(\tau_1)(Z(\tau_2) - Z(\tau_1)) + \alpha \int_{\tau_1}^{\tau_2} Z(s)\, dM^+(s).$$

So

$$\int_{\tau_1}^{\tau_2} M(s)\, dZ(s) - \lambda \int_{\tau_1}^{\tau_2} Z(s)\, dM^+(s)$$
$$\le M(\tau_1)(Z(\tau_2) - Z(\tau_1)) - (\lambda - \alpha) \int_{\tau_1}^{\tau_2} Z(s)\, dM^+(s)$$

and the result follows because $e^{-\delta} Z(\tau_1) \le Z(t)$, $\tau_1 \le t \le \tau_2$. □

We now state the lemma that is needed for the case $\lambda = 0$, $\mu > 0$. The proof is similar to that of Lemma 3.23 and will be omitted.

LEMMA 3.24. *Let $\delta > 0$, $\mu > 2\alpha/1 + \alpha$, and $\mu' = (\mu - \alpha)e^{-\delta}$ where $1 > \alpha = e^{2\delta} - 1$. Let $0 \le \tau_1 \le \tau_2 \le 1$ be two stopping times. Assume that $e^{-\delta} \le Z(t)/z(\tau_1) \le e^{\delta}$, $\tau_1 \le t \le \tau_2$, a.s. Then a.s.,*

(i)
$$\int_{\tau_1}^{\tau_2} M(s)\, dZ(s) - \mu \int_{\tau_1}^{\tau_2} Z(s)\, dM^-(s)$$
$$\le \frac{M(\tau_1)}{1 + \alpha}(Z(\tau_2) - Z(\tau_1))$$
$$+ \frac{\alpha}{1 + \alpha}(Z(\tau_2) M(\tau_2) - Z(\tau_1) M(\tau_1)).$$

(ii) *Assume in addition that $Z(\tau_2) = Z(\tau_1)e^{-\delta}$. Then*

$$\int_{\tau_1}^{\tau_2} M(s)\, dZ(s) - \mu \int_{\tau_1}^{\tau_2} Z(s)\, dM^-(s)$$
$$\le M(\tau_1)(Z(\tau_2) - Z(\tau_1)) - \mu' Z(\tau_1)(M^-(\tau_2) - M^-(\tau_1)).$$

PROOF OF THEOREM 3.22. We will first deal with the case $\lambda > 0$, $\mu = 0$. Let $M \in FV$, $0 < c < 1$ and $\varepsilon > 0$. We select $\delta > 0$, so that $\lambda > 2\alpha/1 - \alpha$ where $\alpha = e^{2\delta} - 1 < 1$, and the following hold:

$$\frac{\alpha}{1 - \alpha} M(0) < \varepsilon, \tag{3.25}$$

$$\frac{\alpha}{1 - \alpha} < e^{3\delta} - 1 \quad \text{and} \quad \lambda' = (\lambda - \alpha)e^{-\delta} > e^{3\delta} - e^{\delta}. \tag{3.26}$$

We define

$$h(z) = \min\{g(z), g(ze^{3\delta})\}, \qquad z > 0. \tag{3.27}$$

It is easy to see that h, like g, satisfies the three basic properties of (2.17). Let $\tau_n = \tau_\delta^n$, $n \geq 0$, be the sequence of stopping times defined in (2.18). From Lemma 3.23(i) we get, for every $n \geq 1$, a.s.

$$\begin{aligned} S_M(\tau_n) \leq & \sum_{k=0}^{n-1} \frac{M(\tau_k)}{1-\alpha}(Z(\tau_{k+1}) - Z(\tau_k)) \\ & - \frac{\alpha}{1-\alpha} Z(\tau_n) M(\tau_n) + \frac{\alpha}{1-\alpha} M(0), \end{aligned} \tag{3.28}$$

namely

$$\begin{aligned} & g(Z(\tau_n)) - S_M(\tau_n) + \frac{\alpha}{1-\alpha} M(0) \\ & \quad \geq g(Z(\tau_n)) - \sum_{k=0}^{n-1} \frac{M(\tau_k)}{1-\alpha}(Z(\tau_{k+1}) - Z(\tau_k)) \\ & \qquad + \frac{\alpha}{1-\alpha} Z(\tau_n) M(\tau_n). \end{aligned} \tag{3.29}$$

Let us define, for every $n \geq 1$,

$$B_n = \left\{ h(Z(\tau_n)) - \sum_{k=0}^{n-1} \frac{M(\tau_k)}{1-\alpha}(Z(\tau_{k+1}) - Z(\tau_k)) > c \right\}. \tag{3.30}$$

Using (3.21) with h and $M/(1-\alpha)$ replacing g and M, respectively, we get that there is $n \geq 1$ with

$$P(B_n, \tau_n < 1) > 0. \tag{3.31}$$

From (3.31) it follows that either $P(M(\tau_n) \geq 0, B_n, \tau_n < 1) > 0$ or $P(M(\tau_n) < 0, B_n, \tau_n < 1) > 0$.

We claim that $P(M(\tau_n) \geq 0, B_n, \tau_n < 1) > 0$ implies

$$P\big(g(Z(\tau_n)) - S_M(\tau_n) > c - \varepsilon\big) > 0. \tag{3.32}$$

We also claim that $P(M(\tau_n) < 0, B_n, \tau_n < 1) > 0$ implies

$$P\big(g(Z(\tau_{n+3})) - S_M(\tau_{n+3}) > c - \varepsilon\big) > 0. \tag{3.33}$$

To see (3.32) all we need is to put together (3.25), (3.27) and (3.29). To see (3.33) is more difficult. We start by observing that $P(M(\tau_n) < 0, B_n, \tau_n < 1) > 0$ and Assumption 2.2 imply that

$$P\big(M(\tau_n) < 0, B_n, \tau_{n+3} < 1, Z(\tau_{n+k}) = e^{k\delta} Z(\tau_n), 1 \leq k \leq 3\big) > 0. \tag{3.34}$$

Now we use Lemma 3.23(ii) and we get that if $Z(\tau_{n+k}) = e^{k\delta} Z(\tau_n)$, $1 \leq k \leq 3$, then

$$\begin{aligned} S_M(\tau_{n+3}) - S_M(\tau_n) \leq & \sum_{k=n}^{n+2} M(\tau_k)(Z(\tau_{k+1}) - Z(\tau_k)) \\ & - \lambda' \sum_{k=n}^{n+2} Z(\tau_k)\big(M^+(\tau_{k+1}) - M^+(\tau_k)\big). \end{aligned} \tag{3.35}$$

Now we use (3.27), (3.28) and (3.35), and we get

$$\begin{aligned} &g(Z(\tau_{n+3})) - S_M(\tau_{n+3}) + \frac{\alpha}{1-\alpha}M(0) \\ &\quad \geq h(Z(\tau_n)) - \sum_{k=0}^{n-1} \frac{M(\tau_k)}{1-\alpha}(Z(\tau_{k+1}) - Z(\tau_k)) \\ &\qquad + \frac{\alpha}{1-\alpha}Z(\tau_n)M(\tau_n) - \sum_{k=n}^{n+2} M(\tau_k)(Z(\tau_{k+1}) - Z(\tau_k)) \\ &\qquad + \lambda' \sum_{k=n}^{n+2} Z(\tau_k)(M^+(\tau_{k+1}) - M^+(\tau_k)). \end{aligned} \tag{3.36}$$

Finally (3.33) follows from (3.25), (3.36) and the following calculation:

$$\begin{aligned} &\frac{\alpha}{1-\alpha}Z(\tau_n)M(\tau_n) - \sum_{k=n}^{n+2} M(\tau_k)(Z(\tau_{k+1}) - Z(\tau_k)) \\ &\quad + \lambda' \sum_{k=n}^{n+2} Z(\tau_k)(M^+(\tau_{k+1}) - M^+(\tau_k)) \\ &\qquad \geq \frac{\alpha}{1-\alpha}Z(\tau_n)M(\tau_n) \\ &\qquad\quad - \sum_{k=n}^{n+2} [M(\tau_n) + (M^+(\tau_k) - M^+(\tau_n))](Z(\tau_{k+1}) - Z(\tau_k)) \\ &\qquad\quad + \lambda' \sum_{k=n}^{n+2} Z(\tau_k)(M^+(\tau_{k+1}) - M^+(\tau_k)) \\ &\qquad \geq \left[\frac{\alpha}{1-\alpha} - (e^{3\delta} - 1)\right] Z(\tau_n)M(\tau_n) \\ &\qquad\quad - (e^{3\delta} - e^{\delta})Z(\tau_n)(M^+(\tau_{n+2}) - M^+(\tau_n)) \\ &\qquad\quad + \lambda' Z(\tau_n)(M^+(\tau_{n+2}) - M^+(\tau_n)). \end{aligned}$$

The last expression is positive when $M(\tau_n) < 0$ because of (3.26).

From (3.32) and (3.33) we can conclude that $y_M \geq c - \varepsilon$. Since c and ε are arbitrarily close to 1 and 0, respectively, we have $y_M \geq 1$. Since $M \in FV$ was arbitrary we have $b_A \geq 1$.

We will now sketch the proof of the case $\lambda = 0$, $\mu > 0$. We define

$$h(z) = \min\{g(z), g(ze^{-3\delta})\}, \qquad z > 0. \tag{3.37}$$

It is easy to see that $\lim_{z\to\infty} h(z)/z = e^{-3\delta}$. So h satisfies (2.17)(ii) with $e^{-3\delta}$ instead of with 1. Nonetheless, Lemma 3.6 will still be valid when applied to h, with only one difference: we need to assume now that $0 < c < e^{-3\delta}$ instead of $0 < c < 1$. But since δ can be arbitrarily close to 0, it does not matter.

The proof now will be similar to that of the case $\lambda > 0$, $\mu = 0$. Here, Lemma 3.24 will replace Lemma 3.23, $Z(\tau_{n+k}) = e^{-k\delta}Z(\tau_n)$, $1 \le k \le 3$, will replace $Z(\tau_{n+k}) = e^{k\delta}Z(\tau_n)$, $1 \le k \le 3$. □

4. The European option. In this section we will prove some results about the European option. We need first the following lemma. The lemma shows that when we want to prove that only trivial hedging is possible, we may assume, without loss of generality, that g is a convex function.

LEMMA 4.1. *Assume that* $g\colon (0,\infty) \to \mathbf{R}$ *satisfies* (2.17). *Then there is* $h\colon (0,\infty) \to \mathbf{R}$ *with the following properties*: $h(z) \le g(z)$, $z > 0$, *and*

(4.2) (i) $h(0+) = 0;$

(4.2) (ii) $\lim_{z\to\infty} \frac{h(z)}{z} = 1;$

(4.2)(iii) *h is a convex function.*

PROOF. Observe that the constant C that appears in (2.17)(iii) is not positive since $g(0+) = 0$. Let $C_k = C/2^k$, $k \ge 0$. Let $a_k > 0$, $k \ge 0$ satisfy

$$a_{k+1} \le \frac{a_k}{6} \quad \text{and} \quad g(z) \ge C_{k+1}, \qquad 0 < z \le a_k.$$

We now define $h(z)$, $0 < z \le a_0$, inductively:

$$h(a_0) = C_0,$$

$$h(z) = C_{k+1} + \left(\frac{h(a_k) - C_{k+1}}{a_k}\right) z, \qquad a_{k+1} \le z < a_k,\ k \ge 0. \tag{4.3}$$

Using induction it is easy to see that

$$C_k \ge h(a_k) \ge 2C_k \qquad k \ge 0. \tag{4.4}$$

From (4.4) it follows that

$$\frac{h(a_k) - C_{k+1}}{a_k} \ge \frac{h(a_{k+1}) - C_{k+2}}{a_{k+1}}, \qquad k \ge 0.$$

We conclude that h is convex on $0 < z \le a_0$. We also have

$$g(z) \ge C_{k+1} \ge h(z), \qquad a_{k+1} \le z < a_k \quad \text{and}$$
$$h(0+) = 0, \text{because } C_k \uparrow 0. \tag{4.5}$$

Let $b_k \uparrow \infty$ and $0 \le \varepsilon_0 < 1$, $\varepsilon_k \downarrow 0$ so that $b_0 > a_0$, and

$$g(z) \ge (1 - \varepsilon_k)z, \qquad z \ge b_k,\ k \ge 0. \tag{4.6}$$

We now define $h(z)$, $a_0 < z < \infty$, inductively:

$$h(z) = C, \qquad a_0 < z \le b_0,$$
$$h(z) = h(b_k) + (1 - \varepsilon_k)(z - b_k), \qquad b_k \le z < b_{k+1},\ k \ge 0. \tag{4.7}$$

Obviously h is convex on $a_0 < z < \infty$. We observe that for each $k \geq 0$,

$$h(z) \geq h(b_k) + (1 - \varepsilon_k)(z - b_k), \qquad z \geq b_k \quad \text{and}$$

$$h(z) \leq (1 - \varepsilon_k)z, \qquad b_k \leq z < b_{k+1}.$$

So $\lim_{z \to \infty} h(z)/z = 1$. It follows from (4.5) and (4.6) that $h(z) \leq g(z)$, $z > 0$. □

REMARK 4.8. The function h satisfies (2.17) and is a legitimate payoff function.

We state now the first result. It follows from Theorem 3.22.

THEOREM 4.9. *Let $\lambda > 0$, $\mu = 0$ or $\lambda = 0$, $\mu > 0$. Assume that Z is a martingale under a probability measure Q which is equivalent to P. Then* $\bar{b}_E \geq 1$.

PROOF. Assume that $\bar{b}_E < 1$. So there exists a tame portfolio M and $x < 1$, so that $x \geq g(Z(1)) - S_M(1)$, a.s. P. Let h be a function as described in Lemma 4.1. Since $h \leq g$, we get $x \geq h(Z(1)) - S_M(1)$, a.s. P. Since Q is equivalent to P we get

$$x \geq h(Z(1)) - S_M(1) \quad \text{a.s. } Q. \tag{4.10}$$

The process Z is a Q-martingale. Hence $\int_0^t M(s)\,dZ(s)$ is a Q-local martingale which is actually a Q-supermartingale, since M is a tame portfolio. This implies that $S_M(t)$ is a Q-supermartingale. Also, since h is convex, $h(Z(t))$ is a Q-submartingale. So we conclude that $h(Z(t)) - S_M(t)$ is a Q-submartingale. By the optional stopping theorem, it follows that for every stopping time $0 \leq \tau \leq 1$ we have

$$x \geq E_Q[h(Z(1)) - S_M(1) \mid F_\tau] \geq h(Z(\tau)) - S_M(\tau) \quad \text{a.s. } Q. \tag{4.11}$$

Hence, for every stopping time $0 \leq \tau \leq 1$ we have

$$x \geq h(Z(\tau)) - S_M(\tau) \quad \text{a.s. } P. \tag{4.12}$$

This contradicts Theorem (3.22) when applied to the payoff function h. □

Our second result is Theorem 4.14. First we will prove the following.

PROPOSITION 4.13. *Let $\lambda > 0$, $\mu > 0$. Assume that there exists $\delta > 0$ so that $\lambda \wedge \mu \geq e^{2\delta} - 1$, and for each $n \geq 1$ either* (i) *or* (ii) *holds, a.s.*:

(i)
$$P\big(\tau_\delta^n = 1, Z(1) \geq Z(\tau_\delta^{n-1}) \mid F_{\tau_\delta^{n-1}}\big) > 0 \quad \textit{and}$$
$$P\big(\tau_\delta^n = 1, Z(1) \leq Z(\tau_\delta^{n-1}) \mid F_{\tau_\delta^{n-1}}\big) > 0.$$

(ii) *For every $\varepsilon > 0$,*
$$P\big(\tau_\delta^n = 1, |Z(1) - Z(\tau_\delta^{n-1})| < \varepsilon \mid F_{\tau_\delta^{n-1}}\big) > 0.$$

Then $b_E \geq 1$.

The next theorem follows immediately from part (ii) of Proposition 4.13. We will omit the proof.

THEOREM 4.14. *Let* $\lambda > 0$, $\mu > 0$. *Assume that for every* $\varepsilon > 0$, $\delta > 0$, $n \geq 0$, *we have*

$$P\big((\tau_\delta^n)_\varepsilon = 1 \,|\, F_{\tau_\delta^n}\big) > 0 \quad a.s.,$$

then $b_E \geq 1$. *In particular, if for every stopping time* $0 \leq \tau \leq 1$, $\varepsilon > 0$, *we have*: $P(\tau_\varepsilon = 1 \,|\, F_\tau) > 0$ *a.s., then* $b_E \geq 1$.

PROOF OF PROPOSITION 4.13. Let $M \in FV$ and $0 < c < 1$. Denote $\tau_n = \tau_\delta^n$, $n \geq 0$. From (3.31) we learn that there is $n \geq 0$ so that $P(B) > 0$, where

$$B = \left\{ g(Z(\tau_n)) - \sum_{k=0}^{n-1} M(\tau_k)\big(Z(\tau_{k+1}) - Z(\tau_k)\big) > c \right\}. \tag{4.15}$$

By Lemma 4.1 we can assume without loss of generality that g is a convex function.

We first assume that (4.13)(i) holds. Denote

$$A = \big\{ g(Z(1)) - g(Z(\tau_n)) \geq M(\tau_n)\big(Z(1) - Z(\tau_n)\big) \big\}. \tag{4.16}$$

From Lemma 3.2 we get that on $\{\tau_{n+1} = 1\}$ we have

$$S_M(1) - S_M(\tau_n) \leq M(\tau_n)\big(Z(1) - Z(\tau_n)\big) \quad \text{a.s.} \tag{4.17}$$

Now we use (3.17) and (4.17). We conclude that on $\{\tau_{n+1} = 1\} \cap A$ we have

$$\begin{aligned} g(Z(1)) - S_M(1) &\geq g(Z(\tau_n)) \\ &\qquad - \sum_{k=0}^{n-1} M(\tau_k)\big(Z(\tau_{k+1}) - Z(\tau_k)\big) \quad \text{a.s.} \end{aligned} \tag{4.18}$$

We need to show that

$$P\big(\{\tau_{n+1} = 1\} \cap A \cap B\big) > 0. \tag{4.19}$$

We denote by D^+ and D^- the derivative from the right and left, respectively. The convexity of g implies that

$$\text{(4.20) (i)} \quad A \supseteq \big\{ M(\tau_n) \leq D^+ g(Z(\tau_n)),\, Z(1) \geq Z(\tau_n) \big\} \quad \text{and}$$

$$\text{(4.20)(ii)} \quad A \supseteq \big\{ M(\tau_n) \geq D^- g(Z(\tau_n)),\, Z(1) \leq Z(\tau_n) \big\}.$$

It also implies that $D^+ g(z) \geq D^- g(z)$, $z > 0$. Since $P(B) > 0$ we get that either $P(B, M(\tau_n) \leq D^+ g(Z(\tau_n)), > 0$ or $P(B, M(\tau_n) \geq D^- g(Z(\tau_n))) > 0$. First we assume that $P(B, M(\tau_n) \leq D^+ g(Z(\tau_n))) > 0$. From (4.20)(i) we have $\{\tau_{n+1} = 1\} \cap A \cap B \supseteq \{\tau_{n+1} = 1,\ Z(1) \geq Z(\tau_n),\ M(\tau_n) \leq D^+ g(Z(\tau_n)), B\}$ and (4.19) follows from assumption (i), because

$$\big\{ M(\tau_n) \leq D^+ g(Z(\tau_n)), B \big\} \in F_{\tau_n}.$$

When we assume $P(B, M(\tau_n) \geq D^- g(Z(\tau_n))) > 0$, we use (4.20)(ii) and (4.19) follows again. From (4.19) we conclude that $x_M \geq c$.

Now we assume that (4.13)(ii) holds.

From (3.17) and (4.17) we get on $\{\tau_{n+1} = 1\}$:

$$g(Z(1)) - S_M(1) \geq \big[g(Z(1)) - g(Z(\tau_n)) - M(\tau_n)(Z(1) - Z(\tau_n))\big] + g(Z(\tau_n)) - \sum_{k=0}^{n-1} M(\tau_k)(Z(\tau_{k+1}) - Z(\tau_k)). \tag{4.21}$$

There is $K > 0$ so that

$$P\big(B \cap \{|M(\tau_n)| \leq K\}\big) > 0. \tag{4.22}$$

Since $\{B \cap |M(\tau_n)| \leq K\} \in F_{\tau_n}$ it follows from (ii) that for every $\varepsilon > 0$,

$$P\big(\{\tau_{n+1} = 1\} \cap \{|Z(1) - Z(\tau_n)| < \varepsilon\} \cap B \cap \{|M(\tau_n)| \leq K\}\big) > 0. \tag{4.23}$$

Fix $\eta > 0$. We observe that Z is bounded on the event $\{\tau_{n+1} = 1\}$, and, by Lemma 4.1, we may assume, without loss of generality, that g is continuous on $\mathbf{R}^+$. We conclude that there is $\varepsilon > 0$ so that on the event $\{\tau_{n+1} = 1\} \cap \{|Z(1) - Z(\tau_n)| < \varepsilon\} \cap \{|M(\tau_n)| \leq K\}$ we have

$$\big|g(Z(1)) - g(Z(\tau_n)) - M(\tau_n)(Z(1) - Z(\tau_n))\big| < \eta. \tag{4.24}$$

Using (4.23) and (4.24) we get that (4.21) implies

$$P\big(g(Z(1)) - S_M(1) \geq c - \eta\big) > 0. \tag{4.25}$$

From (4.25) we conclude that $x_M \geq c$ since η is arbitrary close to 0. □

Next we give a necessary condition for $b_E \geq 1$. We will add an extra assumption on g that holds in the typical case of call option.

THEOREM 4.26. *Assume that for every $0 < a < b$ there is $\eta > 0$ so that $g(z) + \eta < z$, $a \leq z \leq b$. If $b_E \geq 1$ for all $\lambda > 0$, $\mu > 0$, then for every $\varepsilon > 0$, $\delta > 0$, $n \geq 0$, we have*

$$P\big((\tau_\delta^n)_\varepsilon = 1 \,|\, Z(\tau_\delta^k): 0 \leq k \leq n\big) > 0 \quad a.s.$$

PROOF. First we observe that it will be enough to prove that for every $\delta > 0$, $n \geq 0$, we have

$$P\big(\tau_\delta^{n+1} = 1 \,|\, Z(\tau_\delta^k): 0 \leq k \leq n\big) > 0 \quad \text{a.s.} \tag{4.27}$$

The reason is that if there are $\varepsilon > 0$, $\delta > 0$, $n > 0$ for which (4.26) does not hold, then we can choose $\varepsilon > \eta > 0$ so that δ/η is a positive integer and we get a contradiction when we apply (4.27) with η instead of δ, and with an appropriate (larger) n.

We start by fixing $\delta > 0$, and selecting $\lambda = \mu > 0$ so that $\delta > \log((1 + \lambda)/(1 - \lambda))$.

Step 1 ($n = 0$). We assume $P(\tau_\delta^1 = 1) = 0$. We will get a contradiction by proving that $b_E = -\infty$. Fix a constant $H > -\infty$. Since $\delta > \log((1 + \lambda)/(1 - \lambda))$, we can select $\delta > \varepsilon > 0$ so that

$$(1 - \lambda)e^{\delta} - (1 + \lambda)e^{\varepsilon} > 0. \tag{4.28}$$

Let us define

$$m = \inf\{1 \le k: Z(\alpha_k) = e^{\pm\delta}\}, \tag{4.29}$$

where α_k, $k \ge 0$, is a sequence of stopping times defined inductively as

$$\alpha_0 = 0_\varepsilon,$$

$$\alpha_{k+1} = \begin{cases} \inf\{\alpha_k < t: Z(t) = e^{-\varepsilon} \text{ or } e^{\delta}\}, & \text{if } Z(\alpha_k) = e^{\varepsilon}, \\ \inf\{\alpha_k < t: Z(t) = e^{\varepsilon} \text{ or } e^{-\delta}\}, & \text{if } Z(\alpha_k) = e^{-\varepsilon}, \\ 1, & \text{if } Z(\alpha_k) = e^{\pm\delta}. \end{cases}$$

Since $\tau_\delta^1 < 1$, a.s., and Z is continuous, we have $\alpha_0 < 1$, $m < \infty$, a.s. Next we define a portfolio M:

$$M(t) = \begin{cases} 0, & \text{if } 0 \le t < \alpha_0, \\ M_k, & \text{if } \alpha_k \le t < \alpha_{k+1}, Z(\alpha_k) = e^{\varepsilon}, \\ -M_k, & \text{if } \alpha_k \le t < \alpha_{k+1}, Z(\alpha_k) = e^{-\varepsilon}, \\ 1, & \text{if } \alpha_m \le t \le 1, \end{cases} \tag{4.30}$$

where M_k, $k \ge 0$, is a sequence of constants that we will choose. In the case $Z(\alpha_m) = e^{\delta}$, we get

$$\begin{aligned} &S_M(1) - Z(1) \\ &\quad = ((1-\lambda)e^{\delta} - (1+\lambda)e^{-\varepsilon})M_{m-1} \\ &\qquad -(e^{\varepsilon} - e^{-\varepsilon} + \lambda(e^{\varepsilon} + e^{-\varepsilon}))\left(\sum_{k=0}^{m-2} M_k\right) - (1-\lambda)e^{\delta}, \end{aligned} \tag{4.31}$$

while in the case $Z(\alpha_m) = e^{-\delta}$, we get

$$\begin{aligned} &S_M(1) - Z(1) \\ &\quad = ((1-\lambda)e^{-\varepsilon} - (1+\lambda)e^{-\delta})M_{m-1} \\ &\qquad -(e^{\varepsilon} - e^{-\varepsilon} + \lambda(e^{\varepsilon} + e^{-\varepsilon}))\left(\sum_{k=0}^{m-2} M_k\right) - (1+\lambda)e^{-\delta}. \end{aligned} \tag{4.32}$$

The crucial point is that in both (4.31) and (4.32), the coefficient of M_{m-1} is positive due to (4.28). So for every constant H, we can choose a sequence (M_k), so that $\mathbf{S}_M(1) - Z(1) > -H$, a.s. Since $g(z) < z$, $z > 0$, we get

$$\mathbf{S}_M(1) - g(Z(1)) > -H \quad \text{a.s.} \tag{4.33}$$

So $x_M \le H$, and since H is arbitrary we get $x_M = -\infty$.

Remark 4.34. Step 1 has an obvious extension. Let $0 \le \alpha \le 1$ be a stopping time. Assume $P(A \cap \{\alpha_\delta = 1\}) = 0$, for some $A \in F_\alpha$. Then there is a portfolio M, which is defined arbitrarily on $[0, \alpha)$, so that

$$g(Z(1)) \le \mathbf{S}_M(1) \quad \text{a.s. on } A. \tag{4.34}$$

Step 2 ($n \geq 1$). Denote: $\tau_n = \tau_\delta^n$, $n \geq 0$. We assume that there is $n \geq 1$ and a sequence of constants $\zeta_k \in \{-1, 1\}$, so that

$$(4.35) \qquad P\big(\tau_{n+1} = 1, Z(\tau_k) = Z(\tau_{k-1})e^{\zeta_{k-1}\delta}, 1 \leq k \leq n\big) = 0.$$

We need to get a contradiction by creating a portfolio M with $x_M < 1$. To start with, let

$$(4.36) \qquad m = \inf\{1 \leq k : \tau_k = 1 \text{ or } Z(\tau_k) = Z(\tau_{k-1})\exp(-\zeta_{k-1}\delta)\}.$$

We define now a portfolio M:

$$(4.37) \qquad M(t) = \begin{cases} 1 - \zeta_k \varepsilon_k, & \text{if } \tau_k \leq t < \tau_{k+1}, 0 \leq k < m \wedge n, \\ 1, & \text{if } \tau_m \leq t < 1, m \leq n, \\ M^*(t), & \text{if } \tau_n \leq t < 1, n < m. \end{cases}$$

We want to choose M^* and $0 < \varepsilon_k < 1$, $0 \leq k \leq n-1$, so that there will be $\varepsilon > 0$ with

$$(4.38) \qquad g(Z(1)) < 1 - \varepsilon + S_M(1) \quad \text{a.s.,}$$

namely, $x_M \leq 1 - \varepsilon$. We divide the calculations into three cases.

Case 1 ($n < m$). This is the case where $\tau_n < 1$, and the stock price follows the $\{\zeta_k\}$ path: $Z(\tau_k) = Z(\tau_{k-1})\exp(\zeta_{k-1}\delta)$, $1 \leq k \leq n$. In this case, we see from Remark 4.34, that for every $\varepsilon > 0$, regardless of how we choose $\{\varepsilon_k\}$, there is M^* so that (4.38) holds.

Before we proceed with the other two cases, we let $a = e^{-n\delta}$, $b = e^{n\delta}$, $\gamma_1 = b(e^\delta - 1 + 2\lambda)$ and $\gamma_2 = a \min_{\zeta = \pm 1}\{|e^{-\zeta\delta} - 1| - \lambda(1 + e^{-\zeta\delta})\}$. We have

$$(4.39) \text{ (i)} \quad a \leq Z(t) \leq b, \qquad 0 \leq t \leq \tau_n,$$

$$(4.39) \text{ (ii)} \quad |Z(\tau_{k+1}) - Z(\tau_k)| \leq b(e^\delta - 1), \qquad 0 \leq k \leq n-1,$$

$$(4.39)\text{(iii)} \quad \gamma_2 > 0,$$

where (4.39)(iii) follows from $\delta > \log((1 + \lambda)/(1 - \lambda))$.

Case 2 ($m \leq m$ and $\tau_m = 1$). In this case the time expires before or as the stock price completes the $\{\zeta_k\}$ path. We will use our extra assumption on g: there is $\eta > 0$ so that $g(z) < z - \eta$, $a \leq z \leq b$. Since the transaction costs that are paid due to M are bounded by $\lambda 2b\sum_{i=0}^{m-1}\varepsilon_i$, (4.38) will follow from

$$(4.40) \quad Z(1) - \eta < 1 - \varepsilon + \sum_{i=0}^{m-1}(1 - \zeta_i\varepsilon_i)(Z(\tau_{i+1}) - Z(\tau_i)) - \lambda 2b\sum_{i=0}^{m-1}\varepsilon_i,$$

which in turn will follow from

$$(4.41) \qquad \varepsilon \leq \eta - \sum_{i=0}^{m-1}\varepsilon_i|Z(\tau_{i+1}) - Z(\tau_i)| - \lambda 2b\sum_{i=0}^{m-1}\varepsilon_i.$$

By using (4.39) we see that (4.41) will follow from

$$(4.42) \qquad \varepsilon \leq \eta - \gamma_1 \sum_{i=0}^{m-1}\varepsilon_i.$$

To ensure that there is $\varepsilon > 0$ for which (4.42) holds for all $m \le n$, we need that $\{\varepsilon_k\}$ will satisfy

$$\frac{\gamma_1}{\eta} < \frac{1}{\sum_{i=0}^{n-1}\varepsilon_i}. \tag{4.43}$$

Case 3 ($m \le n$ and $\tau_m < 1$). In this case the stock price deviates from the $\{\zeta_k\}$ path to another branch of the δ tree: $Z(\tau_m) = Z(\tau_{m-1})\exp(-\zeta_{m-1}\delta)$. Since $g(z) < z$, $z > 0$, and the transaction costs due to M are bounded by $\lambda[2b\sum_{i=0}^{m-2}\varepsilon_i + \varepsilon_{m-1}Z(\tau_{m-1})(1 + \exp(-\zeta_{m-1}\delta))]$, we get that (4.38) will follow from

$$\begin{aligned} Z(1) &\le 1 - \varepsilon + Z(1) - Z(\tau_m) \\ &\quad + \sum_{i=0}^{m-1}(1 - \zeta_i\varepsilon_i)(Z(\tau_{i+1}) - Z(\tau_i)) \\ &\quad - \lambda\left[2b\sum_{i=0}^{m-2}\varepsilon_i + \varepsilon_{m-1}Z(\tau_{m-1})(1 + \exp(-\zeta_{m-1}\delta))\right], \end{aligned} \tag{4.44}$$

which will follow from

$$\begin{aligned} \varepsilon &\le -\sum_{i=0}^{m-2}\varepsilon_i|Z(\tau_{i+1}) - Z(\tau_i)| \\ &\quad + \varepsilon_{m-1}Z(\tau_{m-1})|\exp(-\zeta_{m-1}\delta) - 1| \\ &\quad - \lambda\left[2b\sum_{i=0}^{m-2}\varepsilon_i + \varepsilon_{m-1}Z(\tau_{m-1})(1 + \exp(-\zeta_{m-1}\delta))\right]. \end{aligned} \tag{4.45}$$

By using (4.39) we see that (4.45) will follow from

$$\varepsilon \le -\gamma_1\sum_{i=0}^{m-2}\varepsilon_i + \gamma_2\varepsilon_{m-1}. \tag{4.46}$$

We need, therefore, to find $\{\varepsilon_k\}$ so that

$$\frac{\gamma_1}{\gamma_2} < \frac{\varepsilon_k}{\sum_{i=0}^{k-1}\varepsilon_i}, \qquad 1 \le k \le n-1. \tag{4.47}$$

To satisfy both (4.43) and (4.47) we will choose

$$\varepsilon_k = y^{k-n}, \qquad 0 \le k \le n-1. \tag{4.48}$$

With this choice (4.43) and (4.47) will follow from

$$\frac{\gamma_1}{\eta} < \frac{y^n}{\sum_{i=0}^{n-1}y^i} \quad \text{and} \quad \frac{\gamma_1}{\gamma_2} < \frac{y^k}{\sum_{i=0}^{k-1}y^i}, \qquad 1 \le k \le n-1,$$

which in turn follows from

$$\left(\frac{\gamma_1}{\gamma_2} \vee \frac{\gamma_1}{\eta}\right) < \min_{1 \le k \le N}\left\{\frac{y^k}{\sum_{i=0}^{k-1}y^i}\right\}. \tag{4.49}$$

The last inequality will hold when y is large enough since

$$\frac{y}{n} \le \min_{0 \le k \le n} \left\{ \frac{y^k}{\sum_{i=0}^{k-1} y^i} \right\}, \qquad y \ge 1. \qquad \square$$

5. Two examples. In this section we bring two examples which are relevant to our results about the European option.

Example 1 shows that, in the two-sided case, Assumption 2.2 by itself does not imply $\bar{b}_E \ge 1$.

Example 2 shows that Assumption 2.2 and the assumption $P(\tau_\delta = 1 \mid F_\tau) > 0$, for each stopping time $0 \le \tau \le 1$ and $\delta > 0$, do not imply $b_E \ge 1$ in the one-sided case. In other words, we cannot extend Theorem 4.14 to the case of one-sided transaction costs. Since the price process in this example is a martingale, we also conclude that we cannot replace $\bar{b}_E \ge 1$ by $b_E \ge 1$ in Theorem 4.9.

In this section, W stands for a standard Brownian motion. Also, the payoff function g will satisfy $g(z) \le z$, $z > 0$.

EXAMPLE 1. In this example $\lambda > 0$, $\mu > 0$, Assumption 2.2 holds, but $\bar{b}_E < 1$. Let Z_1 be the following process:

$$(5.1) \quad Z_1(t) = \exp\left\{ \int_0^t \frac{1}{\sqrt{1-s}}\, dW(s) + \int_0^t \frac{1}{(1-s)}\, ds \right\}, \qquad 0 \le t \le 1.$$

We observe that

$$(5.2) \qquad Z_1(t) \to \infty \quad \text{as } t \to 1, \text{ a.s.}$$

Indeed, $\int_0^t (1/\sqrt{1-s})\, dW(s)$ is a local martingale whose quadratic variation is $\int_0^t (1/(1-s))\, ds$. By a time-change argument [Karatzas and Shreve (1987), page 173] we learn that there exists a Brownian motion, $B(t)$, $0 \le t < \infty$, so that

$$B\left(\int_0^t \frac{1}{(1-s)}\, ds \right) = \int_0^t \frac{1}{\sqrt{1-s}}\, dW(s), \qquad 0 \le t < 1.$$

The result follows now from the fact that $B(t) + t \to \infty$ as $t \to \infty$, a.s.

Next we define the following stopping time:

$$(5.3) \qquad \tau = \inf\{0 \le t \le 1 \colon Z_1(t) = 2\}.$$

We conclude from (5.2) that $\tau < 1$ a.s. Now we define the price process:

$$(5.4) \qquad Z(t) = \begin{cases} Z_1(t), & 0 \le t < \tau, \\ 2\exp\{W(t) - W(\tau)\}, & \tau \le t \le 1. \end{cases}$$

Obviously Z is a continuous positive semimartingale and Assumption 2.2 holds.

We select a portfolio M:

$$M(t) = \begin{cases} 2, & 0 \le t < \tau, \\ 1, & \tau \le t \le 1. \end{cases} \tag{5.5}$$

This portfolio pays 2μ at $t = \tau$ as transaction costs. So

$$S_M(t) = \begin{cases} 2(Z(t) - 1), & 0 \le t < \tau, \\ Z(t) - 2\mu, & \tau \le t \le 1. \end{cases} \tag{5.6}$$

We get that M is a tame portfolio, $x_M \le 2\mu$, and $\overline{b}_E < 1$ if $\mu < \frac{1}{2}$.

EXAMPLE 2. In this example $\lambda > 0$, $\mu = 0$, Assumption 2.2 holds, $P(\tau_\delta = 1 \mid F_\tau) > 0$, for each stopping time τ and $\delta > 0$, but $b_E < 1$. This example shows that we cannot extend Theorem 4.14 to the case of one-sided transaction costs. It also shows that we cannot replace $\overline{b}_E \ge 1$ by $b_E \ge 1$ in Theorem 4.9.

The price process, Z, that we will use is the martingale

$$Z(t) = \exp\left\{W(t) - \frac{t}{2}\right\}, \qquad 0 \le t \le 1. \tag{5.7}$$

Let us define

$$M_1(t) = \begin{cases} e^{1/2}, & 0 \le t < 1 - e^{-1}, \\ \dfrac{1}{\sqrt{(1-t)\log(1/(1-t))}}, & 1 - e^{-1} \le t < 1. \end{cases} \tag{5.8}$$

Let $0 < a_k < 1$, $k \ge 1$ be a decreasing sequence of constants so that $a_k \to 0$. We now define, inductively, a sequence of stopping times τ_k, $k \ge 0$:

$$\begin{aligned} \tau_0 &= 0, \\ \tau_1 &= \inf\left\{1 - a_1 \le t \le 1: \int_0^t M_1(s)\, dZ(s) = -2\right\}, \\ \tau_{2k} &= \inf\{\tau_{2k-1} \le t \le 1: \\ &\qquad M_1(\tau_{2k-1})(Z(t) - Z(\tau_{2k-1})) = k^2\}, \qquad k \ge 1. \\ \tau_{2k+1} &= \inf\left\{(1 - a_{k+1}) \vee \tau_{2k} \le t \le 1: \right. \\ &\qquad \left. \int_{\tau_{2k}}^t M_1(s)\, dZ(s) = -(k+1)^2\right\}, \qquad k \ge 1. \end{aligned} \tag{5.9}$$

These stopping times are taken to be 1 if no such t exists. Since $\int_0^t M_1(s)\, dZ(s)$ is a time-changed Brownian motion, we have

$$P(\tau_{2k+1} < 1 \mid \tau_{2k} < 1) = 1, \qquad k \ge 0. \tag{5.10}$$

We claim that

$$P(\tau_{2k} < 1) \to 0. \tag{5.11}$$

We will prove (5.11) later. It follows from (5.11) that $\tau_{2k} = 1$ eventually, a.s. We define, now, the process M_2:

$$M_2(t) = \begin{cases} M_1(t), & \tau_{2k} \le t < \tau_{2k+1},\ k \ge 0, \\ M_1(\tau_{2k-1}), & \tau_{2k-1} \le t < \tau_{2k},\ k \ge 1. \end{cases} \tag{5.12}$$

From the definition of $\{\tau_k\}$ and from the fact that $\tau_{2k} = 1$ eventually, we get that

$$\int_0^1 M_2(t)\, dZ(t) \le -1 \quad \text{a.s.} \tag{5.13}$$

To see (5.13) observe that when $\tau_2 = 1$ we have $\int_0^1 M_2(t)\, dZ(t) \le -2 + 1^2 = -1$. While if $\tau_{2k-1} < \tau_{2k} = 1$, $k > 1$, then

$$\int_0^{\tau_2} M_2(t)\, dZ(t) = -1, \qquad \int_{\tau_2}^{\tau_{2k-2}} M_2(t)\, dZ(t) = 0 \quad \text{and}$$

$$\int_{\tau_{2k-2}}^{\tau_{2k}} M_2(t)\, dZ(t) \le -k^2 + k^2 = 0.$$

So (5.13) follows.

Finally we define the portfolio M:

$$M(t) = 1 - M_2(t), \qquad 0 \le t < 1. \tag{5.14}$$

Since M_1 is increasing it follows that M_2 is increasing and M is decreasing, so the M portfolio does not pay any transaction costs. Using (5.13) we calculate

$$\begin{aligned} S_M(1) &= \int_0^1 M(t)\, dZ(t) \\ &= Z(1) - 1 - \int_0^1 M_2(t)\, dZ(t) \\ &\ge Z(1) \quad \text{a.s.} \end{aligned} \tag{5.15}$$

We conclude that $b_E \le 0$.

Now we go back to the proof of (5.11). We have

$$\begin{aligned} P(\tau_{2k} < 1) &= P\Big(\sup_{\tau_{2k-1} \le t \le 1} \{M_1(\tau_{2k-1})(Z(t) - Z(\tau_{2k-1}))\} \ge k^2 \Big) \\ &\le P\Big(\max_{0 \le t \le 1} \{Z(t)\} \ge k^2 \Big) \\ &\quad + P\left(\sup_{\tau_{2k-1} \le t \le 1} \left\{ M_1(\tau_{2k-1}) \left(\frac{Z(t)}{Z(\tau_{2k-1})} - 1 \right) \right\} \ge 1 \right) \\ &= (\mathbf{I}) + (\mathbf{II}). \end{aligned}$$

We first deal with (**I**). Since Z is a L_2 martingale, we get, by Doob's inequality:

$$(\mathbf{I}) \le \frac{E(Z(1))}{k^2} = \frac{1}{k^2} \to 0.$$

Next we deal with (**II**):

$$(\mathbf{II}) = P\left(\sup_{\tau_{2k-1}\le t\le 1}\left\{\frac{Z(t)}{Z(\tau_{2k-1})}\right\} \ge 1 + \frac{1}{M_1(\tau_{2k-1})}\right)$$
$$\le P\left(\sup_{\tau_{2k-1}\le t\le 1}\{W(t) - W(\tau_{2k-1})\} \ge \log\left(1 + \frac{1}{M_1(\tau_{2k-1})}\right)\right).$$

Now we condition on $F_{\tau_{2k-1}}$ and use the reflection principle:

$$\le 2E\left(P\left(W(1) - W(\tau_{2k-1}) \ge \log\left(1 + \frac{1}{M_1(\tau_{2k-1})}\right)\middle| F_{\tau 2k-1}\right)\right)$$
$$= 2E\left(P\left(N(0,1) \ge \frac{\log(1 + 1/(M_1(\tau_{2k-1})))}{\sqrt{1-\tau_{2k-1}}}\middle| F_{\tau_{2k-1}}\right)\right).$$

Now we use the inequality $\log(1+x) \ge (x/2)$, $0 \le x \le 1$. Since $M_1 \ge 1$ we get

$$\le 2E\left(P\left(N(0,1) \ge \frac{1}{2M_1(\tau_{2k-1})\sqrt{1-\tau_{2k-1}}}\middle| F_{\tau_{2k-1}}\right)\right).$$

Since $\tau_{2k-1} \ge 1 - a_k$, we get that $\tau_{2k-1} \ge 1 - e^{-1}$ when $a_k < e^{-1}$. Now we use the definition of M_1 under the assumption $a_k < e^{-1}$:

$$= 2E\left(P\left(N(0,1) \ge \frac{\sqrt{\log(1/1-\tau_{2k-1})}}{2}\middle| F_{\tau_{2k-1}}\right)\right)$$
$$\le 2P\left(N(0,1) \ge \frac{\sqrt{\log(1/a_k)}}{2}\right),$$

because $\tau_{2k-1} \ge 1 - a_k$.

Now we use $P(N(0,1) > a) \le e^{-a^2/2}$, $a > 1$. So when $a_k \le e^{-4}$ we get

$$\le 2\exp(-\log(1/a_k)/8) = 2(a_k)^{1/8}.$$

APPENDIX

A.1. A remark on the case λ = 0, μ = 0. Let *RCLL* denote the set of adapted stochastic processes which have right continuous sample paths with left limits. When there are no transaction costs, the capital gain process $S_M(t)$ is well defined when $M \in RCLL$. Let us define

$$\begin{aligned} B_E &= \inf\{x_M : M \in RCLL\},\\ B_A &= \inf\{y_M : M \in RCLL\} \quad \text{and} \\ \bar{B}_E &= \inf\{x_M : M \in RCLL \text{ and } M \text{ is tame}\}. \end{aligned} \tag{A.1}$$

Since $RCLL \supset FV$, we get immediately that $B_E \le b_E$, $B_A \le b_A$, and $\overline{B}_E \le \overline{b}_E$. We will prove the following remark.

REMARK A.2. We have $B_E = b_E$, $B_A = b_A$ and $\overline{B}_E = \overline{b}_E$.

The interpretation that we give to (A.2) is that restricting the portfolios to the class of processes with finite variation paths is not really a disadvantage in terms of hedging. The proof of (A.2) follows immediately from the lemma.

LEMMA A.3. *Let $M \in RCLL$. For every $\varepsilon > 0$ there is a simple process H such that*

$$\sup_{0 \le t \le 1} \left\{ \left| \int_0^t (H(s) - M(s))\, dZ(s) \right| \right\} < \varepsilon \quad a.s.$$

By a "simple process" we mean here a process whose sample paths are step functions.

DEFINITION A.4. A process $H \in FV$ is called a simple process if it has a representation

$$H(t) = \sum_{k=0}^{\infty} H_k 1_{[\tau_k, \tau_{k+1})}(t), \qquad 0 \le t < 1,$$

where $0 = \tau_0 \le \tau_1 \le \cdots \le \tau_k \le \tau_{k+1} \le \cdots$ is a sequence of stopping times, $P(\tau_k = 1 \text{ eventually}) = 1$ so the sum is finite a.s. and $H_k \in F_{\tau_k}$.

PROOF OF LEMMA A.3. For every $\delta > 0$, we construct a simple process M_δ that approximates M uniformly a.s.; that is,

$$\text{(A.5)} \qquad \sup_{0 \le t \le 1} \{ |M_\delta(t) - M(t)| \} < \delta \quad \text{a.s.}$$

We define M_δ by

$$\text{(A.6)} \qquad M_\delta(t) = \sum_{k=0}^{\infty} M(\alpha_k) 1_{[\alpha_k, \alpha_{k+1})}(t), \qquad 0 \le t < 1,$$

where $\alpha_0 = 0$, $\alpha_{k+1} = \inf\{\alpha_k \le t \colon |M(t) - M(\alpha_k)| \ge \delta\}$, $\alpha_{k+1} = 1$ if no such t exists.

We have the representation $Z = A + B$, where A and B are both adapted and continuous processes, A is a local martingale, $A(0) = 0$ and B has finite variation paths, a.s. With every $L > 0$, we associate a stopping time, β, defined by

$$\text{(A.7)} \qquad \beta = \inf\left\{ t \colon |A^2(t)| \ge L \quad \text{or} \quad \int_0^t |dB|(s) \ge L \right\}.$$

Fix $\varepsilon > 0$. We now construct a sequence of simple processes, S_n, $n \geq 1$, such that for every $n \geq 1$,

$$P\left(\sup_{0 \leq t \leq 1}\left\{\left|\int_0^t (S_n(s) - M(s))\,dZ(s)\right|\right\} > \frac{\varepsilon}{2^{n+1}}\right) < \frac{2}{n^2}. \tag{A.8}$$

We start by choosing a sequence of constants $L_n > 0$, so that the sequence of stopping times β_n, that is associated with it by (A.7), satisfies

$$P(\beta_n < 1) \leq \frac{1}{n^2}. \tag{A.9}$$

For every $\delta > 0$, the following holds for all $n \geq 1$. The first step is Doob's inequality, and the second step is the basic isometry of stochastic integrals:

$$\begin{aligned} E \sup_{0 \leq t \leq \beta_n} &\left(\int_0^t (M_\delta(s) - M(s))\,dA(s)\right)^2 \\ &\leq 4E\left(\int_0^{\beta_n}(M_\delta(t) - M(t))\,dA(t)\right)^2 \\ &= 4E\int_0^{\beta_n}(M_\delta(t) - M(t))^2\,d[A, A](t) \\ &\leq \delta^2 E([A, A](\beta_n)) \\ &= \delta^2 E\left(A^2(\beta_n)\right) \\ &\leq \delta^2 L_n. \end{aligned} \tag{A.10}$$

So we get, for every $\delta > 0$,

$$P\left(\sup_{0 \leq t \leq \beta_n}\left\{\left|\int_0^t (M_\delta(s) - M(s))\,dA(s)\right|\right\} > \frac{\varepsilon}{2^{n+2}}\right) < \frac{\delta^2 L_n 4^{n+2}}{\varepsilon^2} \tag{A.11}$$

and

$$\sup_{0 \leq t \leq \beta}\left|\int_0^t (M_\delta(s) - M(s))\,dB(s)\right| \leq \delta L_n \quad \text{a.s.} \tag{A.12}$$

Next we choose a sequence

$$\delta_n = \min\left\{\frac{\varepsilon}{n\sqrt{L_n}\,2^{n+2}}, \frac{\varepsilon}{L_n 2^{n+2}}\right\}, \qquad n \geq 1.$$

It follows that

$$\frac{\delta_n^2 L_n 4^{n+2}}{\varepsilon^2} \leq \frac{1}{n^2} \quad \text{and} \quad \delta_n L_n \leq \frac{\varepsilon}{2^{n+2}}, \qquad n \geq 1.$$

We now define

$$S_n = M_{\delta_n}, \qquad n \geq 1. \tag{A.13}$$

It is easy to see that (A.8) will be satisfied because of (A.9), (A.11) and (A.12).

Next we define, inductively, an increasing sequence of stopping times

$$\tau_0 = 0,$$

$$\tau_n = \inf\left\{ t > \tau_{n-1} : \left| \int_{\tau_{n-1}}^{t} (S_n(s) - M(s))\, dZ(s) \right| = \frac{\varepsilon}{2^n} \right\}, \qquad n \geq 1.$$

Since by (A.8) we have $P(\tau_n < 1) < (2/n^2)$, it follows that $P(\tau_n = 1$ eventually$) = 1$. Finally we define

$$H(t) = \sum_{n=1}^{\infty} S_n(t) 1_{[\tau_{n-1}, \tau_n)}(t), \qquad 0 \leq t < 1. \tag{A.14}$$

It is easy to see that H is a simple process. It follows now from the definition of H that

$$\sup_{0 \leq t \leq 1} \left\{ \left| \int_0^t (H(s) - M(s))\, dZ(s) \right| \right\} \leq \sum_{n=1}^{\infty} \frac{\varepsilon}{2^n} = \varepsilon \quad \text{a.s.} \qquad \square$$

EXAMPLE A.15. Here we take $Z(t) = \exp\{W(t) - (t/2)\}$, where W is a standard Brownian motion, and $g(z) = (z-1)^+$. We will show that (1) $b_E = -\infty$, and (2) $\overline{b}_E = E(Z(1) - 1)^+$, so $0 < \overline{b}_E < 1$.

PROOF OF (1). Fix $a > 0$. Define

$$M(t) = 1 + \frac{1}{\sqrt{1-t}}\{t < \tau\}, \qquad 0 \leq t \leq 1, \tag{A.16}$$

where τ is a stopping time defined by

$$\tau = \inf\left\{ t > 0 : \int_0^t \frac{1}{\sqrt{1-s}}\, dZ(s) = a + 1 \right\}. \tag{A.17}$$

Since the local martingale $\int_0^t (1/\sqrt{1-s})\, dZ(s)$ is a time-changed Brownian motion [Karatzas and Shreve (1987), page 173], we conclude that $\tau < 1$, a.s. and $M \in FV$. Since

$$-a + S_M(1) = -a + \int_0^1 M(t)\, dZ(t) = Z(1) \geq g(Z(1)), \tag{A.18}$$

we get that $x_M \leq -a$. Since $a > 0$ is arbitrary we get $b_E = -\infty$.

PROOF OF (2). We use the representation

$$E\left[(Z(1) - 1)^+ | F_t \right] = E(Z(1) - 1)^+ + \int_0^t M(s)\, dZ(s), \tag{A.19}$$

where

$$M(t) = \Phi\left(\frac{\log(Z(t)) + (1/2)(1-t)}{\sqrt{1-t}} \right)$$

and Φ is the distribution function of a standard normal random variable. Obviously $M \in RCLL$. Also M is a tame portfolio because for each $0 \le t \le 1$ we have

$$\int_0^t M(s)\,dZ(s) = E\big((Z(1)-1)^+ | F_t\big) - E(Z(1)-1)^+ > -1 \quad \text{a.s.}$$

Since $M \in RCLL$ and M is tame, we get that $\overline{B}_E \le E(Z(1)-1)^+$. To see that $\overline{B}_E \ge E(Z(1)-1)^+$, we use the fact that, if M is a tame portfolio, then S_M is a supermartingale because it is a local martingale which is bounded from below. Hence $E(S_M(1)) \le E(S_M(0)) = 0$. Therefore for every $x \in \mathbf{R}$ we get that $x + S_M(1) \ge (Z(1)-1)^+$, a.s. implies $x \ge E(Z(1)-1)^+$. Hence $\overline{B}_E \ge E(Z(1)-1)^+$. The result now follows from (A.2).

A.2. Comparison with SSC. Our model is slightly different than the one in SSC. In order to compare the two results we need first to translate the notation of SSC into our notation: $(\int_0^t Z(s)\,dM^+(s))/1-\lambda$, $\int_0^t Z(s)\,dM^-(s)$ and $\int_0^t M(s)\,dZ(s)$ in our notation stand for $L(t)$, $M(t)$ and $\int_0^t \sigma Y(s)\,dW(s)$ in SSC notation, respectively. The problem that is presented in SSC is, in our notation, the following: Let $\lambda > 0$, $\mu > 0$ and let $q > 0$. Given $x \in R$, find

$$y_0 = \inf\{ y \in R: \text{(I), (II), (III) hold a.s. for some } M \in FV\},$$

where

(I) $$(1-\lambda)x + y + \int_0^1 M(s)\,dZ(s) - (\lambda+\mu-\lambda\mu)\int_0^1 Z(s)\,dM^-(s) \ge g_1(Z(1)) \equiv (Z(1)-(1-\lambda)q)\{Z(1) > (1-\lambda)q\};$$

(II) $$\frac{x}{1-\mu} + y + \int_0^1 M(s)\,dZ(s) - \left(\frac{1}{(1-\mu)(1-\lambda)} - 1\right)\int_0^1 Z(s)\,dM^+(s) \ge g_2(Z(1)) \equiv \left(Z(1) - \frac{q}{1-\mu}\right)\{Z(1) > (1-\lambda)q\};$$

(III) for all $0 \le t \le 1$,

$$(1-\lambda)x + y + \int_0^t M(s)\,dZ(s) - (\lambda+\mu-\lambda\mu)\int_0^t Z(s)\,dM^-(s) \ge 0$$

and

$$\frac{x}{1-\mu} + y + \int_0^1 M(s)\,dZ(s) - \left(\frac{1}{(1-\mu)(1-\lambda)} - 1\right)\int_0^1 Z(s)\,dM^+(s) \ge 0.$$

Under the assumption that Z is a geometric Brownian motion with constant coefficients, it is proved in SSC that

(A.20) $$y_0 = p - (1-\lambda)x, \qquad x \ge 0,$$
$$y_0 = p - \frac{x}{1-\mu}, \qquad x \le 0,$$

where $Z(0) = p$. Actually (A.20) follows from

$$y_0 \ge p - (1-\lambda)x, \qquad x \ge 0,$$
$$y_0 \ge p - \frac{x}{1-\mu}, \qquad x \le 0, \tag{A.21}$$

as we see when we choose the portfolio $M = 1$.

Inequality (A.21) follows from our Theorem 4.14. In fact we can get, using our Theorem 4.14, a stronger result in two ways. First, we only need to assume $\lambda > 0$, $\mu = 0$ or $\lambda = 0$, $\mu > 0$. Second, we can achieve

$$y_1 \ge p - (1-\lambda)x, \qquad x \ge 0,$$
$$y_1 \ge p - \frac{x}{1-\mu}, \qquad x \le 0, \tag{A.21*}$$

where y_1 is defined by

$$y_1 = \inf\{y \in R: (\mathrm{I}), (\mathrm{II}) \text{ hold a.s. for some } M \in FV\}. \tag{A.22}$$

Obviously $y_0 \ge y_1$ since requirement (III) that appears in the definition of y_0 does not appear in (A.22).

In order to see how to conclude (A.21*) from Theorem 4.14, we let $0 < \alpha < 1$ and look at the inequality that is generated by a weighted combination of (I) and (II), namely: α(I) + $(1 - \alpha)$(II). This inequality has the form

$$\left[\alpha(1-\lambda) + (1-\alpha)\frac{1}{1-\mu}\right]x + y + S_M(1)$$
$$\ge \alpha g_1(Z(1)) + (1-\alpha) g_2(Z(1)), \tag{A.23}$$

where the fractional transaction costs that appear in $S_M(1)$ are

$$(1-\alpha)((1/(1-\mu)(1-\lambda)) - 1)$$

for buying and $\alpha(\lambda + \mu - \lambda\mu)$ for selling, both strictly positive even if $\lambda > 0$, $\mu = 0$ or $\lambda = 0$, $\mu > 0$. Now suppose that there is $y \in R$ and $M \in FV$ so that (I) and (II) hold. Then (A.23) holds with the same y and M. Since the payoff function $\alpha g_1(z) + (1-\alpha) g_2(z)$ satisfies (2.17) and the price process is a geometric Brownian motion and hence satisfies both (2.2) and the assumption of Theorem 4.14, it follows from that theorem and Remark 2.19 that

$$\left[\alpha(1-\lambda) + (1-\alpha)\frac{1}{1-\mu}\right]x + y \ge p. \tag{A.24}$$

Since α can be made as close to 0 and 1 as we want, (A.21*) follows.

Inequality (A.21) itself also follows from our Theorem 4.9. Indeed, if M satisfies requirement (III), then it is a tame portfolio. Now, if there is $y \in R$ and $M \in FV$ so that (I), (II) and (III) hold, then (A.23) holds with the same y and M. As before we will get (A.24) from (A.23), but now we apply Theorem 4.9.

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