On the potential theory of one-dimensional subordinate Brownian motions with continuous components

Panki Kim*
Department of Mathematics and Research Institute of Mathematics
Seoul National University
Seoul 151-747, Republic of Korea
Email: pkim@snu.ac.kr,

Renming Song
Department of Mathematics
University of Illinois
Urbana, IL 61801, USA
Email: rsong@math.uiuc.edu

and

Zoran Vondraček†
Department of Mathematics
University of Zagreb
Zagreb, Croatia
Email: vondra@math.hr

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Abstract

Suppose that S is a subordinator with a nonzero drift and W is an independent 1-dimensional Brownian motion. We study the subordinate Brownian motion X defined by $X_t = W(S_t)$. We give sharp bounds for the Green function of the process X killed upon exiting a bounded open interval and prove a boundary Harnack principle. In the case when S is a stable subordinator with a positive drift, we prove sharp bounds for the Green function of X in $(0, \infty)$, and sharp bounds for the Poisson kernel of X in a bounded open interval.

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1 Introduction

A one-dimensional subordinate Brownian motion is a Lévy process obtained by subordinating a one-dimensional Brownian motion by an independent subordinator. In this paper we will be concerned with the case when the subordinator has a drift. This leads to a Lévy process with both a continuous and a jumping component. A typical example is the independent sum of a Brownian motion and a symmetric α-stable process. The difficulty in studying the potential theory of such a process stems from the fact that the process runs on two different scales: on the small scale one expects the continuous component to be dominant, while on the large scale the jumping component of the process should be the dominant one. Furthermore, upon exiting an open set, the process can both jump out of the set and exit continuously through the boundary.

The literature on the potential theory of Markov processes with both continuous and jumping components is rather scarce. Green function estimates (for the whole space) and the Harnack inequality for some of these processes were established in [12] and [14]. The parabolic Harnack inequality and heat kernel estimates were studied in [16] for the independent sum of a d-dimensional Brownian motion and a rotationally invariant α-stable process, and in [6] for much more general diffusions with jumps. There are still a lot of open questions about subordinate Brownian motions with both continuous and jumping components. Some of these questions are as follows: Can one establish sharp two-sided estimates for the Green functions of these processes in open sets? Can one prove a boundary Harnack principle for these processes?

The goal of this paper is to answer the above questions in the case of a subordinate Brownian motion with a continuous component in the one-dimensional setting. The results obtained in this paper should provide a guideline for the more difficult d-dimensional case. Our method relies on two main ingredients: one is the fluctuation theory of one-dimensional Lévy processes (which has already proved very useful in [10]), and the other is a comparison of the killed subordinate Brownian motion with the subordinate killed Brownian motion where we will use some of the results obtained in [18].

The paper is organized as follows: In the next section we set up notations, introduce our basic process X - the subordinate Brownian motion with a continuous component, and give some auxiliary results. In Section 3 we prove sharp two-sided estimates for the Green function of X killed upon exiting a bounded open interval. Not surprisingly, the estimates are given by the Green function of the Brownian motion killed upon exiting that interval. Those estimates are used in Section 4 to prove the boundary Harnack principle for X, which is then used in Section 5 to prove sharp estimates for the Green function of X killed upon exiting a bounded open set D which is the union of finitely many disjoint open intervals such that the distance between any two of them is strictly positive. In the last section we consider the special case when X is the independent sum of a Brownian motion and a symmetric α-stable process, and we give sharp bounds for the Green function of X killed upon exiting $(0, \infty)$ and sharp bounds of the Poisson kernel of a bounded open interval.

Throughout the paper we use the following notations: For functions f and g, $f \sim g$, $t \to 0$ (respectively $t \to \infty$) means that $\lim_{t\to 0} f(t)/g(t) = 1$ (respectively $\lim_{t\to\infty} f(t)/g(t) = 1$), while $f \asymp g$ means that the quotient $f(t)/g(t)$ is bounded and bounded away from zero. The uppercase constants $C_1, C_2, \dots$ will appear in the statements of results and will stay fixed throughout the paper, while the lowercase constants $c_1, c_2, \dots$ will be used in proofs (and will change from one proof to another).

Throughout this paper, we will use dx to denote the Lebesgue measure in $\mathbb{R}$. We will use ":=" to denote a definition, which is read as "is defined to be". For a Borel set $A \subset \mathbb{R}$, we also use $|A|$ to denote the Lebesgue measure of A. For $a, b \in \mathbb{R}$, $a \wedge b := \min\{a, b\}$ and $a \vee b := \max\{a, b\}$. We will use ∂ to denote a cemetery point and for every function f, we extend its definition to ∂ by setting $f(\partial) = 0$.

2 Setting and notation

Let $S = (S_t : t \geq 0)$ be a subordinator with a positive drift. Without loss of generality, we shall assume that the drift of S is equal to 1. The Laplace exponent of S can be written as

$$\phi(\lambda) = \lambda + \psi(\lambda),$$

where

$$\psi(\lambda) = \int_{(0,\infty)} (1 - e^{-\lambda t})\, \mu(dt).$$

The measure μ in the display above satisfies $\int_{(0,\infty)} (1 \wedge t)\, \mu(dt) < \infty$ and is called the Lévy measure of S. Let $W = (W_t : t \geq 0)$ be a 1-dimensional Brownian motion independent of S. The process $X = (X_t : t \geq 0)$ defined by $X_t = W(S_t)$ is called a subordinate Brownian motion. We denote by $\mathbb{P}_x$ the law of X started at $x \in \mathbb{R}$. The process X is a one-dimensional Lévy process with the characteristic exponent Φ given by

$$\Phi(\theta) = \phi(\theta^2) = \theta^2 + \psi(\theta^2), \quad \theta \in \mathbb{R}\,. \tag{2.1}$$

The Lévy measure of X has a density j with respect to the Lebesgue measure given by

$$j(x) = \int_0^\infty (4\pi t)^{-1/2} e^{-x^2/4t}\, \mu(dt)\,, \quad x \in \mathbb{R}\,. \tag{2.2}$$

Note that $j(-x) = j(x)$, and that j is decreasing on $(0, \infty)$.

Let $\overline{X} = (\overline{X}_t : t \geq 0)$ be the supremum process of X defined by $\overline{X}_t = \sup\{0 \vee X_s : 0 \leq s \leq t\}$, and let $\overline{X} - X$ be the reflected process at the supremum. The local time at zero of $\overline{X} - X$ is denoted by $L = (L_t : t \geq 0)$ and the inverse local time by $L^{-1} = (L_t^{-1} : t \geq 0)$. The inverse local time is a (possibly killed) subordinator. The (ascending) ladder height process of X is the process

$H = (H_t : t \geq 0)$ defined by $H_t = X(L_t^{-1})$. The ladder height process is again a (possibly killed) subordinator. We denote by χ the Laplace exponent of H. It follows from [7, Corollary 9.7] that

$$\chi(\lambda) = \exp\left(\frac{1}{\pi}\int_0^\infty \frac{\log(\Phi(\lambda\theta))}{1+\theta^2} d\theta\right) = \exp\left(\frac{1}{\pi}\int_0^\infty \frac{\log(\theta^2\lambda^2 + \psi(\theta^2\lambda^2))}{1+\theta^2}\, d\theta\right), \quad \lambda > 0. \tag{2.3}$$

In the next lemma we show that the ladder height process H has a drift, and give a necessary and sufficient condition for its Lévy measure to be finite.

Lemma 2.1 *(a) It holds that*

$$\lim_{\lambda\to\infty} \frac{\chi(\lambda)}{\lambda} = 1\,.$$

(b) The Lévy measure of H is finite if and only if

$$\int_0^\infty \log\left(1 + \frac{\psi(t^2)}{t^2}\right)\, dt < \infty\,. \tag{2.4}$$

Proof. (a) Note first that the following identity is valid for $\lambda > 0$:

$$\lambda = \exp\left\{\frac{1}{\pi}\int_0^\infty \frac{\log(\theta^2\lambda^2)}{1+\theta^2}\, d\theta\right\}\,. \tag{2.5}$$

Therefore

$$\begin{aligned}
\frac{\chi(\lambda)}{\lambda} &= \frac{\exp\{\frac{1}{\pi}\int_0^\infty \log(\theta^2\lambda^2 + \psi(\theta^2\lambda^2))\,\frac{d\theta}{1+\theta^2}\}}{\exp\{\frac{1}{\pi}\int_0^\infty \log(\theta^2\lambda^2)\,\frac{d\theta}{1+\theta^2}\}} \\
&= \exp\left\{\frac{1}{\pi}\int_0^\infty (\log(\theta^2\lambda^2 + \psi(\theta^2\lambda^2)) - \log(\theta^2\lambda^2))\,\frac{d\theta}{1+\theta^2}\right\} \\
&= \exp\left\{\frac{1}{\pi}\int_0^\infty \log\left(1 + \frac{\psi(\theta^2\lambda^2)}{\theta^2\lambda^2}\right)\,\frac{d\theta}{1+\theta^2}\right\} \\
&= \exp\left\{\frac{1}{\pi}\int_0^\infty \log\left(1 + \frac{\psi(\theta^2\lambda^2)}{\theta^2\lambda^2}\right)(1_{\{\theta\leq 1/\lambda\}} + 1_{\{\theta > 1/\lambda\}})\frac{d\theta}{1+\theta^2}\right\}\,.
\end{aligned}$$

Since there exists a constant $c_1 > 0$ such that $\log(1+x) \leq c_1 x^{\frac{1}{4}}$ for $x \geq 1$, we have, for any $\theta \leq \frac{1}{\lambda}$,

$$\log\left(1 + \frac{\psi(\theta^2\lambda^2)}{\theta^2\lambda^2}\right) \leq \log\left(1 + \frac{\psi(1)}{\theta^2\lambda^2}\right) \leq \frac{c_2}{\theta^{1/2}\lambda^{1/2}},$$

for some $c_2 > 0$. Consequently,

$$\lim_{\lambda\to\infty}\int_0^\infty \log\left(1 + \frac{\psi(\theta^2\lambda^2)}{\theta^2\lambda^2}\right) 1_{\{\theta\leq 1/\lambda\}}\frac{d\theta}{1+\theta^2} = 0.$$

Since

$$\psi(x) \leq \int_{(0,\infty)} (xt \wedge 1)\mu(dt) \leq x\int_{(0,\infty)} (t\wedge 1)\mu(dt) = c_3 x, \qquad \text{for all } x \in (1,\infty),$$

we know that
$$\log\left(1+\frac{\psi(\theta^2\lambda^2)}{\theta^2\lambda^2}\right)\frac{1}{1+\theta^2}1_{\{\theta>1/\lambda\}}\le\frac{\log(1+c_3)}{1+\theta^2},$$
thus by the dominated convergence theorem
$$\lim_{\lambda\to\infty}\int_0^\infty\log\left(1+\frac{\psi(\theta^2\lambda^2)}{\theta^2\lambda^2}\right)1_{\{\theta>1/\lambda\}}\frac{d\theta}{1+\theta^2}=0.$$
Therefore we have shown
$$\lim_{\lambda\to\infty}\frac{\chi(\lambda)}{\lambda}=1\,. \tag{2.6}$$
(b) Since the drift coefficient of χ is equal to 1, the function $\chi(\lambda)-\lambda$ is the Laplace exponent of the jump part of H. The corresponding Lévy measure will be finite if and only if $\lim_{\lambda\to\infty}(\chi(\lambda)-\lambda)<\infty$. First note that by a change of variables we have
$$\int_0^\infty\log\left(1+\frac{\psi(\theta^2\lambda^2)}{\theta^2\lambda^2}\right)\frac{d\theta}{1+\theta^2}=\lambda\int_0^\infty\log\left(1+\frac{\psi(t^2)}{t^2}\right)\frac{dt}{\lambda^2+t^2}\,.$$
By (2.6) this integral converges to 0 as $\lambda\to\infty$. Therefore,
$$\begin{aligned}\lim_{\lambda\to\infty}(\chi(\lambda)-\lambda) &= \lim_{\lambda\to\infty}\lambda\left[\exp\left\{\frac{1}{\pi}\int_0^\infty\log\left(1+\frac{\psi(\theta^2\lambda^2)}{\theta^2\lambda^2}\right)\frac{d\theta}{1+\theta^2}\right\}-1\right]\\ &= \lim_{\lambda\to\infty}\frac{\lambda}{\pi}\int_0^\infty\log\left(1+\frac{\psi(\theta^2\lambda^2)}{\theta^2\lambda^2}\right)\frac{d\theta}{1+\theta^2}\\ &= \lim_{\lambda\to\infty}\frac{1}{\pi}\int_0^\infty\log\left(1+\frac{\psi(t^2)}{t^2}\right)\frac{\lambda^2}{\lambda^2+t^2}\,dt\\ &= \frac{1}{\pi}\int_0^\infty\log\left(1+\frac{\psi(t^2)}{t^2}\right)dt\,.\end{aligned}$$
□

Remark 2.2 It is easy to see that, in the case when $\psi(\lambda)=\lambda^{\alpha/2}$, the integral in (2.4) converges if and only if $0<\alpha<1$.

The potential measure (or the occupation measure) of the subordinator H is the measure on $[0,\infty)$ defined by
$$V(A)=\mathbb{E}\left[\int_0^\infty 1_{\{H_t\in A\}}\,dt\right],$$
where A is a Borel subset of $[0,\infty)$.

By [1, Theorem 5, page 79] and our Lemma 2.1(a), V is absolutely continuous and has a continuous and strictly positive density v such that $v(0+)=1$. Thus
$$V(x)\;:=\;V([0,x])\;=\;\int_0^x v(t)\,dt\sim x\qquad\text{as } x\to 0.$$

Lemma 2.3 *Let $R > 0$. There exists a constant $C_1 = C_1(R) \in (0,1)$ such that for all $x \in (0,R]$,*

$$C_1 \le v(x) \le C_1^{-1} \quad \text{and} \quad C_1 x \le V(x) \le C_1^{-1} x\,.$$

Proof. Let $c_1 = \inf_{0<t\le R} v(t) > 0$ and $c_2 = \sup_{0<t\le R} v(t)$. Since $v(0+) = 1$, we have that $c_1 \le 1$. Choose $C_1 = C_1(R) \in (0,1)$ such that $C_1 \le c_1 \le c_2 \le C_1^{-1}$. Since $V(x) = \int_0^x v(t)\,dt$, the claim follows immediately. □

For any open set D, we use τ_D to denote the first exit time from D, i.e., $\tau_D = \inf\{t > 0 : X_t \notin D\}$. Given an open set $D \subset \mathbb{R}$, we define $X_t^D(\omega) = X_t(\omega)$ if $t < \tau_D(\omega)$ and $X_t^D(\omega) = \partial$ if $t \ge \tau_D(\omega)$, where ∂ is a cemetery state. We now recall the definition of harmonic functions with respect to X.

Definition 2.4 *Let D be an open subset of $\mathbb{R}$. A function h defined on $\mathbb{R}$ is said to be*

(1) harmonic in D for X if

$$\mathbb{E}_x\left[|h(X_{\tau_B})|\right] < \infty \qquad \text{and} \qquad h(x) = \mathbb{E}_x\left[h(X_{\tau_B})\right], \qquad x \in B,$$

for every open set B whose closure is a compact subset of D;

(2) regular harmonic in D for X if it is harmonic in D with respect to X and for each $x \in D$,

$$h(x) = \mathbb{E}_x\left[h(X_{\tau_D})\right];$$

(3) invariant in D for X if for each $x \in D$ and each $t \ge 0$,

$$h(x) = \mathbb{E}_x\left[h(X_t)\right];$$

(4) harmonic for X^D if it is harmonic for X in D and vanishes outside D.

We are now going to use some results from [11]. It is assumed there that the Lévy process satisfies the condition ACC: the resolvent kernels are absolutely continuous with respect to the Lebesgue measure. This is true in our case since X has transition densities. Another assumption in [11] is that 0 is regular for $(0,\infty)$ which is also satisfied here, since X is of unbounded variation. Further, since X is symmetric, the notions of coharmonic and harmonic functions coincide. In [11, Theorem 2] it is proved that V is invariant, hence harmonic, for X in $(0,\infty)$. In particular, for $0 < \epsilon < r < \infty$, let $\tau_{(\epsilon,r)} = \inf\{t > 0 : X_t \notin (\epsilon, r)\}$ be the exit time from (ϵ, r) and let $T_{(-\infty,0]} = \inf\{t > 0 : X_t \in (-\infty, 0]\}$ be the hitting time to $(-\infty, 0]$. Then by harmonicity

$$V(x) = \mathbb{E}_x\left[V(X(\tau_{(\epsilon,r)}); \tau_{(\epsilon,r)} < T_{(-\infty,0]}\right], \quad x > 0\,. \tag{2.7}$$

By letting $\epsilon \to 0$ in (2.7) and using that V is continuous at zero and $V(0) = 0$, it follows that

$$V(x) = \mathbb{E}_x\left[V(X(\tau_{(0,r)}); \tau_{(0,r)} < T_{(-\infty,0]}\right], \quad x > 0\,. \tag{2.8}$$

Formula (2.8) also reads

$$V(x)=\mathbb{E}_x\left[V(X^{(0,\infty)}(T_r))\right]=\int_{[r,\infty)}V(y)\mathbb{P}_x(X^{(0,\infty)}(T_r)\in dy)\,,\quad x>0\,, \tag{2.9}$$

where $T_r=\inf\{t>0: X^{(0,\infty)}\geq r\}$. Let $\zeta=T_{(-\infty,0]}$ be the lifetime of $X^{(0,\infty)}$. Since V is nondecreasing, it follows from (2.9) that

$$V(x)\geq V(r)\mathbb{P}_x(T_r<\zeta)\,,\quad 0<x<r<\infty\,. \tag{2.10}$$

We end this section by noting that the function v is also harmonic for X in $(0,\infty)$. This is shown in [11, Theorem 1].

3 Green function estimates

Let $G^{(0,\infty)}$ be the Green function of $X^{(0,\infty)}$, the process X killed upon exiting $(0,\infty)$. By using [1, Theorem 20, p. 176] which was originally proved in [11], the following formula for $G^{(0,\infty)}$ was shown in [10, Proposition 2.8]:

$$G^{(0,\infty)}(x,y)=\begin{cases}\int_0^x v(z)v(y+z-x)dz, & x\leq y,\\ \int_0^y v(z)v(x+z-y)dz, & x>y.\end{cases} \tag{3.1}$$

The goal of this section is to obtain the sharp bounds for the Green function $G^{(0,r)}$ of $X^{(0,r)}$, the process $X^{(0,\infty)}$ killed upon exiting $(0,r)$ (which is the same as X killed upon exiting $(0,r)$). Note that by symmetry, for all $x,y\in(0,r)$,

$$G^{(0,r)}(x,y) = G^{(0,r)}(y,x)\,, \tag{3.2}$$

$$G^{(0,r)}(r-x,r-y) = G^{(0,r)}(x,y)\,. \tag{3.3}$$

Proposition 3.1 *Let $R>0$. There exists a constant $C_2=C_2(R)>0$ such that for all $r\in(0,R]$,*

$$G^{(0,r)}(x,y)\leq C_2\,\frac{x(r-y)\wedge(r-x)y}{r}\,,\quad 0<x,y<r\,.$$

Proof. Assume first that $0<x\leq y\leq r/2$, and note that $x(r-y)\wedge(r-x)y=x(r-y)\geq xr/2$. Therefore, by Lemma 2.3

$$G^{(0,r)}(x,y)\;\leq\;G^{(0,\infty)}(x,y)\;=\;\int_0^x v(z)v(y+z-x)dz\;\leq\;C_1^{-2}x\;\leq\;2C_1^{-2}\,\frac{x(r-y)\wedge(r-x)y}{r}. \tag{3.4}$$

Now we consider the case $0<x<r/2<y<r$ and use an idea from [8]. Let $\tau_{(0,r/2)}$ be the exit time of $X^{(0,r)}$ from $(0,r/2)$. Note that this is the same as the exit time of $X^{(0,\infty)}$ from $(0,r/2)$.

Since $w \mapsto G^{(0,r)}(w,y)$ is regular harmonic in $(0,r/2)$ for $X^{(0,r)}$, we have

$$\begin{aligned}
G^{(0,r)}(x,y) &= \mathbb{E}_x\left[G^{(0,r)}(X^{(0,r)}(\tau_{(0,r/2)}),y);\ X^{(0,r)}(\tau_{(0,r/2)}) > r/2\right] \\
&= \mathbb{E}_x\left[G^{(0,r)}(r - X^{(0,r)}(\tau_{(0,r/2)}), r-y);\ X^{(0,r)}(\tau_{(0,r/2)}) > r/2\right] \\
&\leq \frac{2C_1^{-2}}{r}\,\mathbb{E}_x\left[X^{(0,r)}(\tau_{(0,r/2)})(r-y);\ X^{(0,r)}(\tau_{(0,r/2)}) > r/2\right] \\
&\leq \frac{2C_1^{-2}}{r}\,r(r-y)\mathbb{P}_x\left(X^{(0,r)}(\tau_{(0,r/2)}) > r/2\right) \\
&\leq 2C_1^{-2}(r-y)\frac{V(x)}{V(r/2)} \leq 2C_1^{-2}(r-y)2C_1^{-2}\frac{x}{r} \\
&= C_2\frac{x(r-y)\wedge(r-x)y}{r}\,.
\end{aligned}$$

Here the second line follows from (3.3), the third from the first part of the proof, and the fifth from (2.10) and Lemma 2.3.

All other cases follow by (3.2) and (3.3). □

For $x \in (0,r)$, let $\delta(x) = \mathrm{dist}(x,(0,r)^c)$ be the distance of the point x to the boundary of the interval $(0,r)$: $\delta(x) = x$ for $x \leq r/2$, and $\delta(x) = r - x$ for $r/2 \leq x < r$.

Remark 3.2 The upper bound in Proposition 3.1 can be written in a different way. Suppose, first, that $0 < x \leq r/2 < y$. Then

$$G^{(0,r)}(x,y) \leq C_2\frac{x(r-y)}{r} = C_2\frac{\delta(x)\delta(y)}{r} \leq C_2\frac{\delta(x)\delta(y)}{|y-x|}\,,$$

and since $\delta(x)^{1/2}\delta(y)^{1/2} < r$, we also have

$$\frac{\delta(x)\delta(y)}{r} \leq (\delta(x)\delta(y))^{1/2}\,.$$

Therefore,

$$G^{(0,r)}(x,y) \leq C_2\left((\delta(x)\delta(y))^{1/2} \wedge \frac{\delta(x)\delta(y)}{|y-x|}\right)\,. \tag{3.5}$$

Assume now that $0 < x < y \leq r/2$. It follows from (3.4) that $G^{(0,r)}(x,y) \leq C_1^{-2}x$. Clearly, $x \leq \delta(x)^{1/2}\delta(y)^{1/2}$, and also,

$$x < \frac{xy}{y-x} = \frac{\delta(x)\delta(y)}{|y-x|}\,.$$

Hence, (3.5) is valid in this case too.

In order to obtain the lower bound for $G^{(0,r)}$ we recall the notion of a subordinate killed Brownian motion. Let $W^{(0,r)}$ be the Brownian motion W killed upon exiting $(0,r)$, then the process $Z^{(0,r)}$ defined by $Z_t^{(0,r)} = W^{(0,r)}(S_t)$ is called a subordinate killed Brownian motion. The precise relationship between $X^{(0,r)}$ - the killed subordinate Brownian motion, and $Z^{(0,r)}$ - the subordinate killed Brownian motion, was studied in [18]. Let $U^{(0,r)}$ denote the Green function of $Z^{(0,r)}$. Since $Z^{(0,r)}$ is a subprocess of $X^{(0,r)}$, it holds that $G^{(0,r)}(x,y) \ge U^{(0,r)}(x,y)$ for all $x,y \in (0,r)$. Hence, it suffices to obtain a lower bound for $U^{(0,r)}$. For this we use a slight modification of the proof of the lower bound in [17], page 87.

Recall that

$$U^{(0,r)}(x,y) = \int_0^\infty p^{(0,r)}(t,x,y)u(t)\,dt\,,$$

where $p^{(0,r)}(t,x,y)$ is the transition density of the Brownian motion $W^{(0,r)}$ and u is the potential density of the subordinator S. Since the drift of S is equal to 1, it follows from [1, Theorem 5, page 79] that the density u exists, is continuous, strictly positive and $u(0+) = 1$.

Proposition 3.3 *Let $R > 0$. There exists a constant $C_3 = C_3(R) > 0$ such that for all $r \in (0,R]$,*

$$U^{(0,r)}(x,y) \ge C_3\, \frac{x(r-y) \wedge (r-x)y}{r}\,, \quad 0 < x,y < r\,.$$

Proof. Let $r > 0$ be such that $r < R$. Since $U^{(0,r)}$ is symmetric and $U^{(0,r)}(r-x,r-y) = U^{(0,r)}(x,y)$, we only need to consider the case $0 < x \le r/2$ and $x \le y < r$. It follows from [13, Theorem 3.9] and the scaling property that there exist $c_1 > 0$ and $c_2 > 0$ independent of r such that for all $t \in (0,r^2]$ and all $x,y \in (0,r)$

$$p^{(0,r)}(t,x,y) \ge c_2 \left(\frac{\delta(x)\delta(y)}{t} \wedge 1\right) t^{-1/2} \exp\left\{-\frac{c_1|x-y|^2}{t}\right\}.$$

For convenience, we put $A := 2r^2$. Let $c_3 = c_3(R) := \inf_{0<t\le 2R^2} u(t)$. We consider two cases:
Case (i): $|x-y|^2 < \delta(x)\delta(y)$. Then,

$$\begin{aligned} U^{(0,r)}(x,y) &\ge c_2 \int_0^A \left(\frac{\delta(x)\delta(y)}{t} \wedge 1\right) t^{-1/2} \exp\left\{-\frac{c_1\delta(x)\delta(y)}{t}\right\} u(t)\,dt \\ &\ge c_2c_3 \int_0^{\delta(x)\delta(y)} t^{-1/2} \exp\left\{-\frac{c_1\delta(x)\delta(y)}{t}\right\} dt \\ &= c_2c_3 \int_1^\infty \left(\frac{\delta(x)\delta(y)}{s}\right)^{-1/2} e^{-c_1 s}\frac{\delta(x)\delta(y)}{s^2}\,ds \\ &= c_2c_3(\delta(x)\delta(y))^{1/2} \int_1^\infty s^{-3/2}e^{-c_1 s}\,ds \;=\; c_4(\delta(x)\delta(y))^{1/2}\,. \end{aligned} \tag{3.6}$$

Assume that $0 < x \le y < r/2$. Then $\delta(y) \ge \delta(x)$, and hence

$$U^{(0,r)}(x,y) \ge c_4(\delta(x)\delta(y))^{1/2} \ge c_4\delta(x) \ge c_4(1/r)x(r-y) = c_4(x(r-y)\wedge(r-x)y)/r\,.$$

Now assume that $0 < x \le r/2 \le y < r$. Then

$$U^{(0,r)}(x,y) \;\ge\; c_4(\delta(x)\delta(y))^{1/2} \;=\; c_4\frac{x(r-y)}{(x(r-y))^{1/2}} \;\ge\; c_4\frac{x(r-y)}{r} = c_4(x(r-y)\wedge(r-x)y)/r\,.$$

Case (ii): $|x-y|^2 \geq \delta(x)\delta(y)$. Then

$$\begin{aligned}
U^{(0,r)}(x,y) &\geq c_2 \int_{\delta(x)\delta(y)}^{A} \left(\frac{\delta(x)\delta(y)}{t} \wedge 1\right) t^{-1/2} \exp\left\{-\frac{c_1|x-y|^2}{t}\right\} u(t)\, dt \\
&\geq c_2 c_3 \delta(x)\delta(y) \int_{\delta(x)\delta(y)}^{A} t^{-3/2} \exp\left\{-\frac{c_1|x-y|^2}{t}\right\} dt \\
&= c_2 c_3 |x-y|^{-1} \delta(x)\delta(y) \int_{c_1|x-y|^2/A}^{c_1|x-y|^2/(\delta(x)\delta(y))} s^{-1/2} e^{-s}\, ds \\
&\geq c_2 c_3 |x-y|^{-1} \delta(x)\delta(y) \int_{c_1/2}^{c_1} s^{-1/2} e^{-s}\, ds \\
&= c_6 |x-y|^{-1} \delta(x)\delta(y)\,. \qquad (3.7)
\end{aligned}$$

Assume that $0 < x \leq y < r/2$. Then

$$U^{(0,r)}(x,y) \geq c_6 x \frac{y}{y-x} \geq c_6 x \frac{r-y}{r} \geq c_6 (x(r-y) \wedge (r-x)y)/r.$$

Now assume that $0 < x \leq r/2 \leq y < r$. Then

$$U^{(0,r)}(x,y) \geq c_6 |x-y|^{-1} \delta(x)\delta(y) \geq c_6 (1/r) x(r-y) = c_6 (x(r-y) \wedge (r-x)y)/r\,.$$

□

Remark 3.4 It follows from (3.6) and (3.7) that

$$U^{(0,r)}(x,y) \geq C_3 \left((\delta(x)\delta(y))^{1/2} \wedge \frac{\delta(x)\delta(y)}{|y-x|} \right)\,. \qquad (3.8)$$

By combining Propositions 3.1 and 3.3 with $G^{(0,r)}(x,y) \geq U^{(0,r)}(x,y)$ we arrive at the following

Theorem 3.5 *Let $R > 0$. There exist a constant $C_4 = C_4(R) > 1$ such that for all $r \in (0,R]$ and all $x, y \in (0,r)$,*

$$\begin{aligned}
C_4^{-1} \frac{x(r-y) \wedge (r-x)y}{r} &\leq G^{(0,r)}(x,y) \leq C_4 \frac{x(r-y) \wedge (r-x)y}{r}\,, \\
C_4^{-1} \frac{x(r-y) \wedge (r-x)y}{r} &\leq U^{(0,r)}(x,y) \leq C_4 \frac{x(r-y) \wedge (r-x)y}{r}\,.
\end{aligned}$$

Remark 3.6 From Remarks 3.2 and 3.4 it follows that

$$G^{(0,r)}(x,y) \asymp (\delta(x)\delta(y))^{1/2} \wedge \frac{\delta(x)\delta(y)}{|y-x|}\,.$$

The bounds written in this way can be generalized to some disconnected open sets (see Theorem 5.1).

Corollary 3.7 *Let $R > 0$. There exist a constant $C_5 = C_5(R) > 1$ such that for all $r \in (0, R]$ and all $x \in (0, r)$,*

$$C_5^{-1}\delta(x) \le \mathbb{E}_x[\tau_{(0,r)}] \le C_5\delta(x)\,.$$

Proof. This follows immediately by integrating the bounds for $G^{(0,r)}$ in the formula $\mathbb{E}_x\tau_{(0,r)} = \int_0^r G^{(0,r)}(x, y)\, dy$. □

4 Boundary Harnack principle

We start this section by looking at how the process X exits the interval $(0, r)$. By use of Ikeda-Watanabe formula (see [9]), it follows that for any Borel set $A \subset [0, r]^c$,

$$\mathbb{P}_x(X(\tau_{(0,r)}) \in A) = \int_A P^{(0,r)}(x, z)\, dz\,, \quad x \in (0, r)\,,$$

where $P^{(0,r)}(x, z)$ is the Poisson kernel for X in $(0, r)$ given by

$$P^{(0,r)}(x, z) = \int_0^r G^{(0,r)}(x, y)j(y - z)\, dy, \quad z \in [0, r]^c. \tag{4.1}$$

Recall that the function j is the density of the Lévy measure of X and is given by (2.2). The function $z \mapsto P^{(0,r)}(x, z)$ is the density of the exit distribution of $X^{(0,r)}$ starting at $x \in (0, r)$ by jumping out of $(0, r)$. This type of exit from an open set is well-studied. In the last section we will give sharp bounds on $P^{(0,r)}$ in the case when $\psi(\lambda) = \lambda^{\alpha/2}, \alpha \in (0, 2)$. On the other hand, the process X can also exit the interval $(0, r)$ continuously. By a slight abuse of notation, for $x \in (0, r)$ and $z \in \{0, r\}$, let

$$P^{(0,r)}(x, z) = \mathbb{P}_x\left(X(\tau_{(0,r)}) = z\right)\,.$$

Note that if $\zeta = T_{(-\infty,0]}$, then $\mathbb{P}_x(T_r < \zeta) = \mathbb{P}_x\left(X(\tau_{(0,r)}) \ge r\right)$. Hence, (2.10) can be rewritten as

$$\mathbb{P}_x\left(X(\tau_{(0,r)}) \ge r\right) \le \frac{V(x)}{V(r)} \le C_1^{-2}\frac{x}{r}\,,$$

where we have used Lemma 2.3 in the second inequality. Suppose that $0 < x < 5r/6$. Then

$$P^{(0,r)}(x, r) \le \mathbb{P}_x\left(X(\tau_{(0,r)}) \ge r\right) \le C_1^{-2}\frac{x}{r}\,. \tag{4.2}$$

By symmetry, for $r/6 < x < r$,

$$P^{(0,r)}(x, 0) \le C_1^{-2}\frac{r - x}{r}\,. \tag{4.3}$$

We prove now the lower bound corresponding to (4.2).

Lemma 4.1 *Let $R > 0$. There exists $C_6 = C_6(R) > 0$ such that for all $r \in (0, R]$ and all $x \in (0, r)$,*

$$P^{(0,r)}(x, r) \ge C_6\frac{x}{r}\,. \tag{4.4}$$

Proof. Let $Z^{(0,r)}$ be the subordinate killed Brownian motion and let $\tau^Z_{(0,r)}$ be its lifetime. From the results in [18, Section 3], it follows immediately that

$$\mathbb{P}_x\left(X(\tau_{(0,r)})=r\right)\geq \mathbb{P}_x\left(Z^{(0,r)}(\tau^Z_{(0,r)}-)=r\right).$$

By [18, Corollary 4.4] (although it was assumed that the Lévy measure μ of S is infinite there, what was really used there was the condition that the potential measure of S has no atoms which is obviously satisfied in the present case),

$$\mathbb{P}_x\left(Z^{(0,r)}(\tau^Z_{(0,r)}-)=r\right)=\mathbb{E}_x\left[u(\rho);\,W_\rho=r\right],$$

where $\rho=\inf\{t>0: W_t\notin(0,r)\}$ and u is the potential density of the subordinator S. Let $c_1=c_1(R):=\inf_{0<t\leq R^2}u(t)$.

For every $t>0$ we have that $t\,\mathbb{P}_x(\rho>t)\leq \mathbb{E}_x[\,\rho\,]=x(r-x)/2$, hence

$$\mathbb{P}_x(W_\rho=r,\rho>t)\leq \mathbb{P}_x(\rho>t)\leq \frac{1}{2t}x(r-x)\leq \frac{r}{2t}x\,.$$

Choose $t=t(r)=r^2$. Then

$$\mathbb{P}_x(W_\rho=r,\rho\leq t)=\mathbb{P}_x(W_\rho=r)-\mathbb{P}_x(W_\rho=r,\rho>t)\geq \frac{x}{r}-\frac{r}{2r^2}x=\frac{1}{2}\frac{x}{r}\,.$$

Therefore,

$$\mathbb{E}_x\left[u(\rho);\,W_\rho=r\right]\geq \mathbb{E}_x\left[u(\rho);\,W_\rho=r,\rho\leq t\right]\geq c_1\mathbb{P}_x(W_\rho=r,\rho\leq t)\geq \frac{c_1}{2}\frac{x}{r}=C_6\,\frac{x}{r}\,.$$

This concludes the proof. □

Proposition 4.2 (Harnack inequality) *Let $R>0$. There exists a constant $C_7=C_7(R)>0$ such that for all $r\in(0,R)$ and every nonnegative function h on $\mathbb{R}$ which is harmonic with respect to X in $(0,3r)$,*

$$h(x)\leq C_7 h(y)\,,\quad \textit{for all } x,y\in(r/2,5r/2)\,.$$

Proof. It follows from Theorem 3.5 that there exists $c_1=c_1(R)>0$ such that

$$G^{(\frac{r}{4},\frac{11r}{4})}(x_1,y)\leq c_1 G^{(\frac{r}{4},\frac{11r}{4})}(x_2,y),\quad \text{for all } x_1,x_2\in(\frac{r}{2},\frac{5r}{2}),\ y\in(\frac{r}{4},\frac{11r}{4}),$$

consequently by (4.1) we have

$$P^{(\frac{r}{4},\frac{11r}{4})}(x_1,z)\leq c_1 P^{(\frac{r}{4},\frac{11r}{4})}(x_2,z),\quad \text{for all } x_1,x_2\in(\frac{r}{2},\frac{5r}{2}),\ z\in[\frac{r}{4},\frac{11r}{4}]^c.$$

It follows from (4.2)–(4.4) that there exists $c_2=c_2(R)>0$ such that

$$P^{(\frac{r}{4},\frac{11r}{4})}(x_1,z)\leq c_2 P^{(\frac{r}{4},\frac{11r}{4})}(x_2,z),\quad \text{for all } x_1,x_2\in(\frac{r}{2},\frac{5r}{2}),\ z\in\{\frac{r}{4},\frac{11r}{4}\}.$$

The conclusion of the proposition follows immediately from the last two displays. □

We are ready now to prove a boundary Harnack principle.

Theorem 4.3 (Boundary Harnack principle) *Let $R>0$. There exists a constant $C_8 = C_8(R) > 0$ such that for all $r \in (0, R)$, and every $h : \mathbb{R} \to [0, \infty)$ which is harmonic in $(0, 3r)$ and vanishes continuously on $(-r, 0]$ it holds that*

$$\frac{h(x)}{h(y)} \le C_8\, \frac{x}{y}$$

for all $x, y \in (0, r/2)$.

Proof. Let $x \in (0, r/2)$. Since h is harmonic in $(0, 3r)$ and vanishes continuously on $(-r, 0]$ we have

$$\begin{aligned} h(x) \;&=\; \lim_{\varepsilon\downarrow 0} \mathbb{E}_x \left[h(X_{\tau_{(\varepsilon,r)}}) \right] \;=\; \mathbb{E}_x \left[h(X_{\tau_{(0,r)}}) \right] \\ &= \mathbb{E}_x \left[h(X_{\tau_{(0,r)}});\, X_{\tau_{(0,r)}} \in [r, 2r) \right] + \mathbb{E}_x \left[h(X_{\tau_{(0,r)}});\, X_{\tau_{(0,r)}} \ge 2r \right] + \mathbb{E}_x \left[h(X_{\tau_{(0,r)}});\, X_{\tau_{(0,r)}} \le -r \right] \\ &= I_1 + I_2 + I_3\,. \end{aligned} \tag{4.5}$$

We estimate each term separately. By use of (2.10) and the Harnack inequality (Proposition 4.2), we have

$$I_1 \le C_7 h(r/2) \mathbb{P}_x(X_{\tau_{(0,r)}} \ge r) \le C_7 h(r/2) \frac{V(x)}{V(r)} \le C_7 C_1^{-2}\, \frac{x}{r}\, h(r/2)\,. \tag{4.6}$$

In the last inequality we used Lemma 2.3. For the lower bound we use Lemma 4.1 and the Harnack inequality (Proposition 4.2):

$$h(x) \ge I_1 \ge \mathbb{E}_x \left[h(X_{\tau_{(0,r)}});\, X_{\tau_{(0,r)}} = r \right] = h(r) P^{(0,r)}(x, r) \ge C_7^{-1} C_6 \frac{x}{r}\, h(r/2)\,. \tag{4.7}$$

In order to deal with I_2 and I_3 we use Theorem 3.5. Since $x \in (0, r/2)$, by Theorem 3.5, we have

$$\begin{aligned} G^{(0,r)}(x, y) \;&\le\; C_4 \frac{x(r-y) \wedge (r-x)y}{r} \;=\; C_4 \frac{x}{r} \left((r-y) \wedge (\frac{r}{x} - 1) y \right) \\ &\le\; C_4 \frac{x}{r} \left((r-y) \wedge y \right) \;\le\; 2C_4^2\, \frac{x}{r}\, G^{(0,r)}(r/2, y)\,. \end{aligned}$$

Thus

$$\begin{aligned} I_2 \;&=\; \int_{2r}^{\infty} P^{(0,r)}(x, z) h(z)\, dz = \int_{2r}^{\infty} \int_0^r G^{(0,r)}(x, y) j(z-y) h(z)\, dy\, dz \\ &\le\; 2C_4^2 \frac{x}{r} \int_{2r}^{\infty} \int_0^r G^{(0,r)}(r/2, y) j(z-y) h(z)\, dy\, dz \le 2C_4^2 \frac{x}{r}\, h(r/2). \end{aligned}$$

Similarly,

$$I_3 \;\le\; 2C_4^2 \frac{x}{r}\, h(r/2). \tag{4.8}$$

By putting together (4.5)–(4.8) we obtain

$$\frac{1}{c_1} \frac{x}{r}\, h(r/2) \le h(x) \le c_1 \frac{x}{r}\, h(r/2)$$

for some constant $c_1 = c_1(R) > 1$. If, now, $x, y \in (0, r/2)$, then it follows from the last display that

$$\frac{h(x)}{h(y)} \le c_1^2 \frac{x\, h(r/2)}{y\, h(r/2)} = c_1^2\, \frac{x}{y}\,,$$

which completes the proof. □

5 Green function revisited

In this section we assume that $D \subset \mathbb{R}$ is a bounded open set that can be written as the union of finitely many disjoint intervals at a positive distance from each other. More precisely, let $a_1 < b_1 < a_2 < b_2 < \cdots < a_n < b_n$, $n \in \mathbb{N}$, $I_j := (a_j, b_j)$, and $D = \cup_{j=1}^n I_j$. Such a set D is sometimes called a bounded $C^{1,1}$ open set in $\mathbb{R}$ (see [5]). For a point $x \in D$, let $\delta(x) := \mathrm{dist}(x, D^c)$ be the distance of x to the boundary of D. Further, let $R := \mathrm{diam}(D) = b_n - a_1$, $\rho := \min_{1\le j\le n} |I_j|$, $\xi := \min_{1\le j\le n-1} \mathrm{dist}(I_j, I_{j+1}) = \min_{1\le j\le n-1}(a_{j+1} - b_j)$, and let

$$\eta := \frac{\rho}{6} \wedge \frac{\xi}{2} \wedge 1\,.$$

Note that R and η depend on the geometry of D.

Let X^D be the process X killed upon exiting the set D, and let G^D be the corresponding Green function. The goal of this section is to prove the following sharp estimates for G^D corresponding to the estimates in Remark 3.6.

Theorem 5.1 *There exists a constant $C_9 = C_9(D)$ such that for all $x, y \in D$,*

$$C_9^{-1}\left((\delta(x)\delta(y))^{1/2} \wedge \frac{\delta(x)\delta(y)}{|y-x|}\right) \le G^D(x,y) \le C_9\left((\delta(x)\delta(y))^{1/2} \wedge \frac{\delta(x)\delta(y)}{|y-x|}\right). \tag{5.1}$$

Proof. We first prove the lower bound. If the points x and y are in the same interval, say $x, y \in I_j$, then by monotonicity,

$$G^D(x,y) \ge G^{I_j}(x,y) \ge c_1\left((\delta(x)\delta(y))^{1/2} \wedge \frac{\delta(x)\delta(y)}{|y-x|}\right), \tag{5.2}$$

where the last estimate follows from Remark 3.6 and the constant c_1 depends only on R.

Assume, now, that $x \in I_j$, $y \in I_k$, $j \ne k$. Choose $w \in I_j$ such that $\delta(w) = \delta(x)/2$, and let $B := (w - \delta(x)/4, w + \delta(x)/4)$. Note that the function $G^D(x, \cdot)$ is regular harmonic in I_k, hence

$$G^D(x,y) = \mathbb{E}_y\left[G^D(x, X(\tau_{I_k}))\right] \ge \mathbb{E}_y\left[G^D(x, X(\tau_{I_k}));\, X(\tau_{I_k}) \in B\right] \ge c_2 G^D(x,w)\mathbb{P}_y(X(\tau_{I_k}) \in B)\,.$$

The last inequality follows from the Harnack inequality applied to $G^D(x, \cdot)$ in the interval B. The constant c_2 is independent of the size of B, i.e., independent of $\delta(x)$. Next, by (5.2),

$$G^D(x,w) \ge c_1\left((\delta(x)\delta(w))^{1/2} \wedge \frac{\delta(x)\delta(w)}{|w-x|}\right) = c_1\left((\delta(x)\delta(x)/2)^{1/2} \wedge \frac{\delta(x)\delta(x)/2}{\delta(x)/2}\right) \ge c_3\delta(x)\,.$$

Further,

$$\begin{aligned}
\mathbb{P}_y(X(\tau_{I_k}) \in B) &= \int_B P^{I_k}(y,z)\,dz = \int_B \int_{I_k} G^{I_k}(y,y')j(y'-z)\,dy'\,dz \\
&\ge \int_B \int_{I_k} G^{I_k}(y,y')j(R)\,dy'\,dz = j(R)|B|\,\mathbb{E}_y[\tau_{I_k}] \ge c_4\delta(y)|B| = (c_4/2)\delta(x)\delta(y)\,,
\end{aligned}$$

where the last estimate follows from Corollary 3.7. By combining the last two estimates, we get

$$G^D(x,y) \geq c_2 c_3 (c_4/2)\delta(x)^2\delta(y) = c_5\delta(x)^2\delta(y)\,.$$

Assume now that $\delta(x) \geq \eta$. Then

$$G^D(x,y) \geq c_5\eta\delta(x)\delta(y) = c_6\delta(x)\delta(y)\,, \quad x \in I_j, \delta(x) > \eta, y \in I_k\,, j \neq k\,,$$

with the constant c_6 depending on D. To extend the above estimate to all $x \in I_j$ we will use the boundary Harnack principle. For $y \in I_k$, the function $G^D(\cdot, y)$ is harmonic in I_j and vanishes continuously in the set $(a_j - \eta, a_j] \cup [b_j, b_j + \eta)$. Choose $w \in I_j$ such that $\delta(w) = \eta/2$. Then by the boundary Harnack principle, for all $x \in I_j$ such that $\delta(x) < \eta$,

$$\frac{G^D(w,y)}{G^D(x,y)} \leq C_8 \frac{\delta(w)}{\delta(x)}\,,$$

implying

$$G^D(x,y) \geq C_8^{-1} G^D(w,y) \frac{\delta(x)}{\delta(w)} \geq C_8^{-1} c_6 \delta(w)\delta(y)\frac{\delta(x)}{\delta(w)} = c_7\delta(x)\delta(y) \geq c_8 \frac{\delta(x)\delta(y)}{|x-y|}\,.$$

Next we prove the upper bound. Let $I := (a_1, b_n)$. Then for $x, y \in D$, $G^D(x,y) \leq G^I(x,y) \leq c_9$ with a constant c_9 depending on R. If $\delta(x) \geq \eta/2$ and $\delta(y) \geq \eta/2$, then

$$G^D(x,y) \leq c_9 = \frac{c_9}{\eta^2}\,\eta^2 \leq \frac{4c_9}{\eta^2}\,\delta(x)\delta(y) = c_{10}\delta(x)\delta(y)\,. \tag{5.3}$$

Assume that $y \in I_k$ such that $\delta(y) \geq \eta/2$, and choose $w \in I_j$, $j \neq k$, such that $\delta(w) = \eta/2$. By the boundary Harnack principle, for all $x \in I_j$ such that $\delta(x) < \eta$,

$$\frac{G^D(x,y)}{G^D(w,y)} \leq C_8 \frac{\delta(x)}{\delta(w)}\,,$$

implying

$$G^D(x,y) \leq C_8 G^D(w,y)\frac{\delta(x)}{\delta(w)} \leq c_{11}\delta(x)\delta(y)\,.$$

Now we fix $x \in I_j$ and use the same method to show that for all $y \in I_k$, $k \neq j$,

$$G^D(x,y) \leq c_{12}\delta(x)\delta(y) \leq c_{13}\left((\delta(x)\delta(y))^{1/2} \wedge \frac{\delta(x)\delta(y)}{|y-x|}\right)$$

for some constants c_{12}, c_{13} depending only on D. This completes the proof of the upper bound for the case when x and y are in different intervals.

If $x, y \in I_j$ and $|x-y| \geq \rho/2$, then starting from (5.3), in the same way as before, by using the boundary Harnack principle twice, we get that $G^D(x,y) \leq c_{14}\delta(x)\delta(y)$. A similar argument as above then gives the upper bound in (5.1).

It remains to consider the case when x and y are in the same interval, close to each other and close to the boundary. Suppose $x, y \in I_j$, $\mathrm{dist}(x, a_j) < \mathrm{dist}(x, b_j)$, $\mathrm{dist}(x, a_j) < \eta$, $\mathrm{dist}(y, a_j) < \mathrm{dist}(y, b_j)$ and $\mathrm{dist}(y, a_j) < \eta$. Let $D_1 := (a_1, b_{j-1})$, $D_2 := (a_j, b_n)$ and $\widetilde{D} := D_1 \cup D_2$. Then $G^D(x, y) \le G^{\widetilde{D}}(x, y)$, $\delta(x) = \delta_{\widetilde{D}}(x) = \delta_{D_2}(x)$ and $\delta(y) = \delta_{\widetilde{D}}(y) = \delta_{D_2}(y)$. By considering the first exit from D_2, it follows that

$$G^{\widetilde{D}}(x, y) \le G^{D_2}(x, y) + \int_{D_1} P^{D_2}(x, z) G^{\widetilde{D}}(z, y)\, dz\,,$$

where $P^{D_2}(x, z)$ is the Poisson kernel for X in D_2. The first term above is estimated by use of Remark 3.6:

$$G^{D_2}(x, y) \le c_{15} \left((\delta(x)\delta(y))^{1/2} \wedge \frac{\delta(x)\delta(y)}{|y - x|} \right).$$

The Poisson kernel is estimated as

$$P^{D_2}(x, z) = \int_{D_2} G^{D_2}(x, x') j(x' - z)\, dx' \le j(\xi) \mathbb{E}_x[\tau_{D_2}] \le c_{16} \delta(x)\,.$$

For $z \in D_1$ and $y \in D_2$, we know that $G^{\widetilde{D}}(z, y) \le c_{17} \delta_{\widetilde{D}}(y) \delta_{\widetilde{D}}(z) = c_{17} \delta(y) \delta_{\widetilde{D}}(z)$. Hence,

$$\int_{D_1} P^{D_2}(x, z) G^{\widetilde{D}}(z, y)\, dz \le c_{16} c_{17} \delta(x) \int_{D_1} \delta(y) \delta_{\widetilde{D}}(z)\, dz \le c_{18} \delta(x) \delta(y)\,,$$

for some constant $c_{18} = c_{18}(D)$. Since $|x - y| < \eta < 1$, $\delta(x)\delta(y) \le \delta(x)\delta(y)/|x - y|$. Also, since $\max\{\delta(x), \delta(y)\} < \eta < 1$, it holds that $\delta(x)\delta(y) \le (\delta(x)\delta(y))^{1/2}$. Therefore,

$$\begin{aligned} G^D(x, y) &\le G^{\widetilde{D}}(x, y) \le c_{15} \left((\delta(x)\delta(y))^{1/2} \wedge \frac{\delta(x)\delta(y)}{|y - x|} \right) + c_{18} \delta(x)\delta(y) \\ &\le c_{19} \left((\delta(x)\delta(y))^{1/2} \wedge \frac{\delta(x)\delta(y)}{|y - x|} \right). \end{aligned}$$

This finishes the proof of the upper bound. □

Remark 5.2 In case of one-dimensional symmetric α-stable process, $0 < \alpha < 2$, and D as above, the sharp bounds for the Green function G^D_α are given in [5]. When $1 < \alpha < 2$ they read

$$G^D_\alpha(x, y) \asymp (\delta(x)\delta(y))^{(\alpha-1)/2} \wedge \frac{\delta(x)^{\alpha/2} \delta(y)^{\alpha/2}}{|y - x|}\,.$$

6 The case of stable subordinator

In this section we assume that $\psi(\lambda) = \lambda^{\alpha/2}$, $0 < \alpha < 2$. Thus the subordinator S is the sum of a unit drift and an $\alpha/2$-stable subordinator, while X is the sum of a Brownian motion and a symmetric α-stable process. We will use the fact that S is a special subordinator, that is, the restriction to $(0, \infty)$ of the potential measure of S has a decreasing density with respect to the

Lebesgue measure (for more details see [15] or [17]). It follows from [10, Proposition 2.1] and [15, Corollary 2.3] that H is a special (possibly killed) subordinator. Thus the density v of the potential measure V is decreasing, and since $v(0+) = 1$, it holds that $v(t) \le 1$ for all $t > 0$.

By applying the Tauberian theorem (Theorem 1.7.1 in [2]) and the monotone density theorem (Theorem 1.7.2 in [2]) one easily gets that

$$v(t) \sim \frac{t^{\alpha/2-1}}{\Gamma(\alpha/2)}\,, \quad t \to \infty\,.$$

Together with $v(t) \sim 1$, as $t \to 0+$, we obtain the following estimates

$$v(t) \asymp \begin{cases} 1\,, & 0 < t < 2\,, \\ t^{\alpha/2-1}\,, & 1/2 < t < \infty\,. \end{cases} \tag{6.1}$$

We recall now the Green function formula (3.1) for the process $X^{(0,\infty)}$:

$$G^{(0,\infty)}(x,y) = \int_0^x v(t)v(y-x+t)\,dt\,, \quad 0 < x \le y < \infty\,. \tag{6.2}$$

The next result provides sharp bounds for the Green function $G^{(0,\infty)}$.

Theorem 6.1 *Assume that $\phi(\lambda) = \lambda + \lambda^{\alpha/2}$, $0 < \alpha < 2$. Then the Green function $G^{(0,\infty)}$ of the killed process $X^{(0,\infty)}$ satisfies the following sharp bounds:*

(a) For $1 < \alpha < 2$,

$$G^{(0,\infty)}(x,y) \asymp (x \wedge x^{\alpha/2})(y^{\alpha/2-1} \wedge 1)\,, \quad 0 < x < y < \infty\,.$$

(b) For $\alpha = 1$,

$$G^{(0,\infty)}(x,y) \asymp \begin{cases} x(y^{-1/2} \wedge 1)\,, & 0 < x < 1\,, \\ \log \frac{1+x^{1/2}y^{1/2}}{1+y-x}\,, & 1 \le x < y < 2x\,, \\ x^{1/2}y^{-1/2}\,, & 1 \le x < 2x < y\,. \end{cases}$$

(c) For $0 < \alpha < 1$,

$$G^{(0,\infty)}(x,y) \asymp \begin{cases} x(y^{\alpha/2-1} \wedge 1)\,, & 0 < x < 1\,, \\ 1\,, & 1 \le x < y < x+1\,, \\ (y-x)^{\alpha-1}\,, & 1 \le x < x+1 < y < 2x\,, \\ x^{\alpha/2}y^{\alpha/2-1}\,, & 1 \le x < 2x < y\,. \end{cases}$$

Proof. The proof is straightforward, but long. It uses only the Green function formula (6.2) and estimates (6.1) for v. It consists of analyzing several cases and subcases. We will give the complete proof for $0 < \alpha < 1$. Cases 1–3 below work also for $1 \le \alpha < 2$.

Case 1: $0 < x < y < 2$.

Since $0 < t < x < 2$, $v(t) \asymp 1$. Also, $0 < y-x < y-x+t < y < 2$, hence $v(y-x+t) \asymp 1$. Therefore, $G(x,y) \asymp \int_0^x 1 \cdot 1\, dt = x$.

Case 2: $0 < x < 1 < 2 < y$.

Again, $v(t) \asymp 1$. Further, $y-x+t \geq y-x > 1$, hence $v(y-x+t) \asymp (y-x+t)^{\alpha/2-1}$. Therefore, by the mean value theorem,

$$G(x,y) \asymp \int_0^x 1 \cdot (y-x+t)^{\alpha/2-1}\, dt = \int_{y-x}^{y} s^{\alpha/2-1}\, ds \asymp \theta^{\alpha/2-1} x\,,$$

where $y-x < \theta < y$. Further, $x < 1 < y/2$, hence $y-x > y/2$. Thus, for $\theta \in (y-x, y)$ it holds that $\theta^{\alpha/2-1} \asymp y^{\alpha/2-1}$. Therefore, $G(x,y) \asymp xy^{\alpha/2-1}$.

Case 3: $1 \leq x < 2x < y$.

Note that $1 + x < 2x < y$ and thus $1 < y-x$. Hence, $y-x+t > 1$ and thus $v(y-x+t) \asymp (y-x+t)^{\alpha/2-1}$. Further,

$$G(x,y) = \int_0^1 v(t)v(y-x+t)\, dt + \int_1^x v(t)v(y-x+t)\, dt =: I_1 + I_2\,.$$

For the first integral we have

$$I_1 = \int_0^1 v(t)v(y-x+t)\, dt \asymp \int_0^1 1 \cdot (y-x+t)^{\alpha/2-1} dt = \int_{y-x}^{y-x+1} s^{\alpha/2-1}\, ds \asymp \theta^{\alpha/2-1}$$

for some $\theta \in (y-x, y-x+1)$. Therefore

$$I_1 = \int_0^1 v(t)v(y-x+t)\, dt \asymp (y-x)^{\alpha/2-1} \asymp y^{\alpha/2-1} \tag{6.3}$$

since $y/2 < y-x < y$. For the second integral, we use that $y/2 < y-x < y-x+1 < y-x+t < y$, and hence

$$I_2 = \int_1^x v(t)v(y-x+t)\, dt \asymp \int_1^x t^{\alpha/2-1}(y-x+t)^{\alpha/2-1}\, dt \asymp \int_1^x t^{\alpha/2-1} y^{\alpha/2-1}\, dt \asymp (x^{\alpha/2}-1)y^{\alpha/2-1}\,. \tag{6.4}$$

Putting (6.3) and (6.4) together, we obtain

$$G(x,y) \asymp y^{\alpha/2-1} + (x^{\alpha/2}-1)y^{\alpha/2-1} \asymp x^{\alpha/2} y^{\alpha/2-1}\,.$$

From now on we assume that $0 < \alpha < 1$.

Case 4: $1 \leq x < y < x+1$.

For $0 < t < 1$ we have $y-x < y-x+t < y-x+1 < 2$, hence

$$I_1 = \int_0^1 v(t)v(y-x+t)\, dt \asymp \int_0^1 1 \cdot 1\, dt = 1\,.$$

For I_2 we have

$$I_2 \asymp \int_1^x t^{\alpha/2-1}(y-x+t)^{\alpha/2-1}\, dt \leq \int_1^x t^{\alpha/2-1} t^{\alpha/2-1}\, dt \asymp 1 - x^{\alpha-1} \asymp 1\,,$$

and

$$\begin{aligned} I_2 &\asymp t^{\alpha/2-1}(y-x+t)^{\alpha/2-1}\,dt \ge \int_1^x t^{\alpha/2-1}(1+t)^{\alpha/2-1}\,dt \\ &\ge \int_1^x (1+t)^{\alpha/2-1}(1+t)^{\alpha/2-1}\,dt \asymp 2^{\alpha-1}-(x+1)^{\alpha-1} \asymp 1\,. \end{aligned}$$

Hence $G(x,y) \asymp 1$.

Case 5: $1 \le x < x+1 < y < 2x$.

For $0<t<1$, we have $1 < 1+t < y-x+t$, and hence

$$I_1 \asymp \int_0^1 1\cdot(y-x+t)^{\alpha/2-1}dt \asymp (y-x)^{\alpha/2-1}\,,$$

where we used the fact that $y-x \le y-x+t < y-x+1 \le 2(y-x)$.

To get the upper bound for I_2 we use the change of variable:

$$\begin{aligned} I_2 &\asymp \int_1^x t^{\alpha/2-1}(y-x+t)^{\alpha/2-1}\,dt = (y-x)^{\alpha-1}\int_{\frac{1}{y-x}}^{\frac{x}{y-x}} s^{\alpha/2-1}(1+s)^{\alpha/2-1}\,ds \\ &\le (y-x)^{\alpha-1}\int_0^\infty s^{\alpha/2-1}(1+s)^{\alpha/2-1}\,ds = c(y-x)^{\alpha-1}\,. \end{aligned}$$

For the lower bound, we consider separately three cases: (i) and $y-x \ge x/2$ and $x \ge 2$, (ii) $y-x \ge x/2$ and $1 \le x \le 2$, (iii) $y-x \le x/2$.

In case (i), by use of $y-x+t < x+x = 2x$, it follows that

$$I_2 \asymp \int_1^x t^{\alpha/2-1}(y-x+t)^{\alpha/2-1}\,dt \ge 2^{\alpha/2-1}x^{\alpha/2-1}\int_1^x t^{\alpha/2-1}\,dt \ge 2^{\alpha/2-1}x^{\alpha/2-1}\int_{x/2}^x t^{\alpha/2-1}\,dt = cx^{\alpha-1}\,.$$

Since $y-x \ge x/2$, we have $2x > y > y-x > x/2$, thus $(y-x)^{\alpha-1} \asymp x^{\alpha-1}$. Therefore in this case we have that $I_2 \ge c(y-x)^{\alpha-1}$.

In case (ii), $I_2 \ge 0$. Note that for this case $1 < y-x < x \le 2$, hence $I_1 + I_2 \ge (y-x)^{\alpha/2-1} \asymp (y-x)^{\alpha-1}$.

In case (iii), we have that $x \ge 2(y-x)$, hence again by a change of variable

$$\begin{aligned} I_2 &\asymp \int_1^x t^{\alpha/2-1}(y-x+t)^{\alpha/2-1}\,dt \ge \int_1^{2(y-x)} t^{\alpha/2-1}(y-x+t)^{\alpha/2-1}\,dt \\ &= (y-x)^{\alpha-1}\int_{\frac{1}{y-x}}^2 s^{\alpha/2-1}(1+s)^{\alpha/2-1}\,ds \ge (y-x)^{\alpha-1}\int_1^2 s^{\alpha/2-1}(1+s)^{\alpha/2-1}\,ds \\ &= c(y-x)^{\alpha-1}\,. \end{aligned}$$

Note, further, that for $y-x \ge 1$ it holds that $(y-x)^{\alpha/2-1} \le (y-x)^{\alpha-1}$. Hence, by combining the expression for I_1, the upper and the lower bound for I_2, we obtain that $G(x,y) \asymp (y-x)^{\alpha-1}$. $\square$

Remark 6.2 In case of a symmetric α-stable process, the sharp bounds for the Green function $G_\alpha^{(0,\infty)}$ o f the process killed upon exiting $(0,\infty)$ can be easily deduced from [4]. They read

$$\begin{aligned} G_\alpha^{(0,\infty)}(x,y) &\asymp x^{\alpha/2}y^{\alpha/2-1}\,, \quad 1<\alpha<2, \\ G_\alpha^{(0,\infty)}(x,y) &\asymp \begin{cases} x^{1/2}y^{-1/2}\,, & 0<x<y/2\,, \\ \log\frac{x}{y-x}\,, & 0<y/2<x<y\,, \end{cases} \qquad \alpha=1, \\ G_\alpha^{(0,\infty)}(x,y) &\asymp \begin{cases} x^{\alpha/2}y^{\alpha/2-1}\,, & 0<x<y/2\,, \\ (y-x)^{\alpha-1}\,, & 0<y/2<x<y\,, \end{cases} \qquad 0<\alpha<1. \end{aligned}$$

Now we recall the formula (4.1) for the Poisson kernel of X in $(0,r)$

$$P^{(0,r)}(x,z) = \int_0^r G^{(0,r)}(x,y)j(y-z)\,dy, \quad z\in[0,r]^c\,.$$

In the case of $\alpha/2$-stable subordinator, it turns out that $j(z-y)=c(\alpha)|z-y|^{-1-\alpha}$. For $0<x<r<z$ let

$$\widetilde{P}^{(0,r)}(x,z) := \frac{1}{r}\int_0^x (r-x)y(z-y)^{-1-\alpha}\,dy + \frac{1}{r}\int_x^r x(r-y)(z-y)^{-1-\alpha}\,dy\,.$$

It follows from the Green function estimates in Theorem 3.5 that $\widetilde{P}^{(0,r)}(x,z)\asymp P^{(0,r)}(x,z)$.

Lemma 6.3 *For $x\in(0,r)$ and $z>r$ we have*

$$\widetilde{P}^{(0,r)}(x,z) = \begin{cases} \frac{\kappa(\alpha)}{r}\left(r(z-x)^{1-\alpha} - x(z-r)^{1-\alpha} - (r-x)z^{1-\alpha}\right)\,, & \alpha\in(0,1)\cup(1,2), \\ \frac{1}{r}\left(x\log\frac{z-x}{z-r} + (r-x)\log\frac{z-x}{z}\right)\,, & \alpha=1\,, \end{cases}$$

where $\kappa(\alpha)=1/(\alpha(\alpha-1))$.

Proof. This follows by straightforward integration. □

For $z\in[0,r]^c$, let $\delta(z)=\mathrm{dist}(z,(0,r))$. By combining the above lemma with $\widetilde{P}^{(0,r)}(x,z)\asymp P^{(0,r)}(x,z)$, one can show the following sharp bounds for the Poisson kernel. We will omit the proof.

Theorem 6.4 *Let $R>0$. There exists a constant $C_{10}=C_{10}(R)>1$ such that for all $r\in(0,R]$, for all $x\in(0,r)$, and for all $z\notin[0,r]$ it holds that*

$$\begin{aligned} C_{10}^{-1}\frac{\delta(x)}{1+\delta(z)}|z-x|^{-\alpha} &\le P^{(0,r)}(x,z) \le C_{10}\frac{\delta(x)}{1+\delta(z)}|z-x|^{-\alpha}\,, \quad 0<\alpha<1\,, \\ C_{10}^{-1}\frac{\delta(x)|\log(\delta(z)|}{(1+\delta(z))\log(2+\delta(z))} &\le P^{(0,r)}(x,z) \le C_{10}\frac{\delta(x)|\log(\delta(z))|}{(1+\delta(z))\log(2+\delta(z))}|z-x|^{-1}\,, \quad \alpha=1\,, \\ C_{10}^{-1}\frac{\delta(x)}{(1+\delta(z))\delta(z)^{\alpha-1}} &\le P^{(0,r)}(x,z) \le C_{10}\frac{\delta(x)}{(1+\delta(z))\delta(z)^{\alpha-1}}|z-x|^{-1}\,, \quad 1<\alpha<2\,. \end{aligned}$$

The Poisson kernel $P_\alpha^{(0,r)}$ for the symmetric α-stable process was computed in [3], and it turns out that

$$P_\alpha^{(0,r)}(x,z) \asymp \frac{\delta(x)^{\alpha/2}}{(1+\delta(z))\delta(z)^{\alpha/2}}\,|z-x|^{-1}\,, \quad x\in(0,r), z\in[0,r]^c, \quad 0<\alpha<2\,.$$

An interesting new feature of $P^{(0,r)}$ is that in case $0<\alpha<1$ there is no singularity in $\delta(z)$. This is not surprising in view of Lemma 2.1(b) and Remark 2.2.

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