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On the potentials of galactic discs

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ABSTRACT
The standard Bessel function formula for the potential of a thin axisymmetric disc is extended to include arbitrary vertical structure, yielding a formula which reduces the potential to a single quadrature in the important cases of a disc with constant scale-height and either exponential or Gaussian vertical density profiles. The general solution of Poisson's equation for an axisymmetric body is also given as a double integral over a Legendre function. A new formulation for the potential of thin axisymmetric discs is also given. The potential is expressed as a double integral over elementary functions in the most general case, and can usually be reduced to a single quadrature. This yields a more convenient form (from a numerical point of view) for the potentials of discs which are not known completely analytically, including in particular that of the exponential disc. Finally, the potential corresponding to an arbitrary distribution of matter is given as an integral of a four-dimensional Fourier transform, which reduces to a four-dimensional Fourier transform in the case of triaxial discs. The link between the Green's function and Bessel function formulations is shown up explicitly by this formula, which should prove useful in practice for the computation of many triaxial disc potentials. This solution is then illustrated by the analytical calculation of the potential of a particular family of triaxial discs.

Key words: methods: analytical – celestial mechanics, stellar dynamics – galaxies: kinematics and dynamics.

1 INTRODUCTION

Most treatments of the dynamics of disc galaxies are restricted to infinitesimally thin discs. For many purposes this is a good approximation, since in reality the scaleheights of galactic discs are much smaller than the other length-scales involved, and an orbiting star will effectively see an infinitesimally thin disc provided it spends most of its time at a sufficient vertical distance from the plane of the disc. Any infinitesimal disc has, however, the fatal property that the vertical force is a discontinuous function of vertical displacement z on $z=0$, making any numerical study of the three-dimensional dynamics impossible. Moreover, for a detailed treatment of the orbital properties of many stellar populations it is desirable to include the vertical structure in the description of the disc potential. There do not appear to be any treatments of this in the literature.

Fortunately, the well-known observational results of van der Kruit & Searle (1982) suggest that galactic discs have scale-heights which are nearly independent of radius. This means that we can represent the three-dimensional density law $\rho(R, z)$ by the separable form $\rho=\rho_1(R)\rho_2(z)$. The radial dependence is almost universally taken to be a simple exponential, $\rho_1(R)\propto\exp(-R/h)$, but considerable controversy exists as to the best functional form for the vertical dependence (see e.g. van der Kruit 1988 for a review). For present purposes, it will suffice to note that various authors have claimed good fits to the observations with the following list of vertical density profiles: the exponential law $\rho_2(z)\propto\exp(-|z|/z_0)$; the Gaussian law $\rho_2(z)\propto\exp(-z^2/z_0^2)$; the hyperbolic secant law $\rho_2(z)\propto\operatorname{sech} z/z_0$; and the hyperbolic secant law squared $\rho_2(z)\propto\operatorname{sech}^2 z/z_0$. Kuijken & Gilmore (1989) gave a very simple way of determining the potential of such discs of constant scaleheight, which is generalized here to other radial and vertical profiles. This representation of the potential can be easily generalized to include density laws $\rho(R, z)$ that are not separable in cylindrical (or any other) coordinates.

To a reasonable first approximation, the disc populations of spiral and lenticular galaxies can be modelled as axisymmetric discs with infinitesimal thickness (zero scaleheight). The problem of finding the gravitational potential $\Phi(R, z)$ at any point in cylindrical coordinates (R, z) for such systems with a given surface density profile $\Sigma(R)$ has a number of well-known general solutions (see Binney & Tremaine 1987, chapter 2, for a review). The most direct approach is simply to use the integral form of Poisson's equation

$$\Phi(\boldsymbol{x}) = -G\iiint \frac{\rho(\boldsymbol{x}')}{|\boldsymbol{x}-\boldsymbol{x}'|}\,\mathrm{d}^3\boldsymbol{x}', \tag{1}$$

where the reciprocal distance $1/|\boldsymbol{x}-\boldsymbol{x}'|$ plays the role of a Green's function in the formal solution of Poisson's equation (e.g. Garabedian 1986). The power of equation (1) lies in the fact that it is valid for any configuration of matter, irrespective of any symmetry or scaleheight properties. In the special case of a thin axisymmetric disc, it is easy to show that, in cylindrical coordinates, two of the integrals in (1) can be performed analytically, leading to the general solution

$$\Phi(R, z) = -4G\int_0^\infty K\left[\sqrt{\frac{4RR'}{(R+R')^2+z^2}}\right]\frac{\Sigma(R')\,R'\,\mathrm{d}R'}{\sqrt{(R+R')^2+z^2}}, \tag{2}$$

where K is a complete elliptic integral of the first kind. Hence the solution of Poisson's equation for a thin axisymmetric disc has been reduced to the evaluation of a single quadrature. There are, however, two clear disadvantages of this formulation: first, the integrand in (2) is sufficiently complicated that it is very difficult indeed to recover analytic Σ, Φ pairs from (2); secondly, on $z=0$, the integrand in (2) goes through a singularity at $R'=R$ due to the divergence of $K(k)$ at $k=1$. This is numerically inconvenient.

An alternative formulation for thin axisymmetric discs, first developed for the study of the potentials of electrified discs (Weber 1873; Beltrami 1881; see also Jackson 1975), was first applied to galactic discs by Toomre (1963). The idea of this method is to separate Laplace's equation in cylindrical coordinates and use Gauss's theorem to obtain the particular Σ, Φ pair

$$\Sigma_k(R) = \frac{k}{2\pi}J_0(kR), \tag{3a}$$

$$\Phi_k(R, z) = -GJ_0(kR)\exp(-k|z|), \tag{3b}$$

where J_0 is a cylindrical Bessel function and k is a free parameter. The potential $\Phi(R, z)$ corresponding to a given surface density profile $\Sigma(R)$ is then expressed as a weighted sum of the basic components in (3). The linearity of the Poisson equation is exploited (see Binney & Tremaine 1987, chapter 2) and an integral equation for a weight function is solved to give the general solution for the potential as

$$\Phi(R, z) = -2\pi G\int_0^\infty\int_0^\infty \Sigma(R')J_0(kR')R'J_0(Rk)\exp(-|z|k)\,\mathrm{d}R'\,\mathrm{d}k. \tag{4}$$

Even though this solution requires two quadratures rather than one, the relative simplicity of the integrand (and the existence of extensive tables of integrals over Bessel functions) means that in practice one can sometimes use (4) to obtain analytic Σ, Φ pairs. Furthermore, the integral over R' can be performed analytically for a rather wide range of simple $\Sigma(R')$ forms. One is, however, still forced to evaluate integrals over special functions, and the oscillatory behaviour of $J_0(Rk)$ at large k makes numerical evaluation a non-trivial task.

In an attempt to derive a convenient solution to Poisson's equation for thin axisymmetric discs that does not (necessarily) involve special functions, Evans & de Zeeuw (1992) have recently replaced the separable building blocks (3) by the simple and elementary Σ, Φ pair of Kuzmin's disc (Kuzmin 1956). This gives a generalized Stieltjes integral equation for the weight function $S(k)$ (cf. equation 6.11 of Evans & de Zeeuw), in terms of which the potential is given by

$$\Phi(R, z) = -2\pi G\int_0^\infty \frac{S(k)\,\mathrm{d}k}{(a^2+k)^{1/2}(R^2+[\sqrt{a^2+k}+|z|]^2)^{1/2}}, \tag{5}$$

from which it can be seen that special functions are only involved if $S(k)$ turns out to contain them. The price to pay for this situation is that the integral equation involved is not easily solved in practice. Evans & de Zeeuw do succeed in finding the general solution for $S(k)$, but this involves performing the combination of an Euler transform, an (inverse) Abel transform and an inverse Stieltjes transform. Remarkably, all this can be done analytically for a reasonable range of simple $\Sigma(R)$ forms. However, an inverse Stieltjes transform involves the evaluation of a discontinuity along a cut in the complex plane on the negative real axis, and therefore does not lend itself to numerical work.

In Section 2, I derive an integral expression for the potential of a thick axisymmetric disc that has the special property of a constant scaleheight, which, as argued by van der Kruit & Searle (1982), is a good approximation for the discs of spiral galaxies. For the cases of exponential or Gaussian density dependence in the vertical direction, as is relevant for spiral galaxies, the expression for the potential reduces to a single quadrature for most radial density profiles.

In Section 3, I derive a different formulation of the solution of Poisson's equation for thin axisymmetric discs which shares the advantage of the Evans & de Zeeuw formulation of not necessarily involving special functions, while involving a much more simple and convenient integral equation for the weight function.

In Section 4 the solution of Poisson's equation for a general elliptic disc is given in terms of Fourier transforms, which shows the connection between the elliptic integral and Bessel function formulations and provides a numerically convenient route to the potentials of triaxial systems. Section 4 also contains the calculation of a particular family of triaxial discs that illustrates this method; Section 5 then concludes.

2 DISCS WITH CONSTANT SCALEHEIGHT

First note that for discs with constant scaleheight, such that $\rho=\rho_1(x, y)\rho_2(z)$, a simple identification can be made between the planar dependence $\rho_1(x, y)$ and the column density $\Sigma(x, y)$, the latter being the surface density of the equivalent thin disc. We have

$$\Sigma(x, y)=\int_{-\infty}^{\infty} \rho_1(x, y)\rho_2(z)\,\mathrm{d}z, \tag{6}$$

and therefore $\Sigma(x, y)=N\rho_1(x, y)$, where the normalization constant N is given by $N=\int_{-\infty}^{\infty}\rho_2(z)\,\mathrm{d}z$.

Suppose we add various thin discs, with surface density profiles proportional to $\rho_1(x, y)$, at different z-levels with a weight function $w(z)$ to give the potential

$$\Phi_{\mathrm{thick}}(x, y, z)\equiv\int_{-\infty}^{\infty} w(z')\Phi_{\mathrm{thin}}(x, y, z-z')\,\mathrm{d}z', \tag{7}$$

and try to solve for the weight function $w(z)$.

Using the Green's function form (1) of the general solution of Poisson's equation in Cartesian coordinates, we have

$$\Phi_{\mathrm{thin}}(x, y, z)=-G\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\Sigma(x', y')\,\mathrm{d}x'\,\mathrm{d}y'}{\sqrt{(x-x')^2+(y-y')^2+z^2}}. \tag{8}$$

Therefore,

$$\begin{aligned}\Phi_{\mathrm{thick}}(x, y, z)&=-G\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\Sigma(x', y')w(z')\,\mathrm{d}x'\,\mathrm{d}y'\,\mathrm{d}z'}{\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}}\\&=-G\iiint\frac{\Sigma(x', y')w(z')\,\mathrm{d}^3\boldsymbol{x}'}{|\boldsymbol{x}-\boldsymbol{x}'|}.\end{aligned} \tag{9}$$

It is easy to show (e.g. Binney & Tremaine 1987, equation 2-12) that

$$\nabla^2\left(\frac{1}{|\boldsymbol{x}-\boldsymbol{x}'|}\right)=-4\pi\delta(\boldsymbol{x}-\boldsymbol{x}'), \tag{10}$$

so that

$$\nabla^2\Phi_{\mathrm{thick}}=4\pi G\Sigma(x, y)\,w(z). \tag{11}$$

The weight function $w(z)$ is therefore simply given by

$$w(z)=\frac{\rho_1(x, y)}{\Sigma(x, y)}\rho_2(z)=\frac{\rho_2(z)}{N}, \tag{12}$$

so that the potential of a disc with constant scaleheight is related to that of the equivalent thin disc with the same surface density profile through

$$\Phi_{\rm thick}(x, y, z)=\frac{1}{N}\int_{-\infty}^{\infty} \rho_2(z')\Phi_{\rm thin}(x, y, z-z')\, dz'. \tag{13}$$

The physical interpretation of this result is obvious: a disc with constant scaleheight is simply a superposition of thin discs situated at various z-levels, with a weight function equal to the (normalized) vertical density profile.

In the special case of an axisymmetric disc we can use the Bessel function expression (4) for $\Phi_{\rm thin}$ to obtain the two alternative general solutions of Poisson's equation for constant-scaleheight axisymmetric discs:

$$\Phi(R, z)=\frac{1}{N}\int_0^{\infty} S(k)J_0(Rk)\int_{-\infty}^{\infty} \rho_2(z')\exp(-k|z-z'|)\, dz'\, dk \tag{14a}$$

$$=\frac{1}{N}\int_{-\infty}^{\infty} \rho_2(z')\int_0^{\infty} S(k)J_0(Rk)\exp(-k|z-z'|)\, dk\, dz'. \tag{14b}$$

For the double-exponential density dependence, this reduces to the expressions of Kuijken & Gilmore, who also give results for $\rho(z)\propto\cosh^{-\xi}z$.

The solution (14a) can be cast in a form analogous to the solution in the thin disc case by defining an integral weight function $f(z;k)$ as

$$f(z; k)=\frac{1}{N}\int_{-\infty}^{\infty} \exp(-k|z-z'|)\rho_2(z')\, dz', \tag{15}$$

so that

$$\Phi(R, z)=\int_0^{\infty} S(k)J_0(Rk)f(z; k)\, dk. \tag{16}$$

2.1 Exponential vertical density profile

If the vertical density profile is exponential, we have $\rho_2(z')=\exp(-|z'|/z_0)$, where z_0 is the exponential scaleheight. The normalization constant in this case is $N=2z_0$. By (15), the solution for $f(z;k)$ can be shown to be

$$f_{\rm exp}(z; k)=\frac{\exp(-k|z|)+\exp(-|z|/z_0)}{2(kz_0+1)}+\frac{\exp(-|z|/z_0)-\exp(-k|z|)}{2(kz_0-1)}, \tag{17}$$

which tends to $\exp(-|z|/z_0)(1+|z|/z_0)/2$ as $k\rightarrow 1/z_0$. Thus the potential of a constant-scaleheight disc with an exponential vertical density profile is given by the single quadrature $\Phi_{\rm exp}(R, z)=\int_0^\infty S(k)J_0(Rk)f_{\rm exp}(z; k)\, dk$. For many radial density profiles the Hankel transform $S(k)$ is known analytically. In particular, the potential of the double exponential $\rho(R, z)=\rho(0, 0)\exp(-R/h)\exp(-|z|/z_0)$ can be written

$$\Phi_{2\,\rm exp}(R, z)=-4\pi Gz_0\rho(0, 0)h^2\int_0^{\infty}\frac{J_0(Rk)f_{\rm exp}(z; k)}{(1+h^2k^2)^{3/2}}\, dk, \tag{18}$$

which can be evaluated swiftly and accurately on a computer.

The radial force $F_R(R, z)$ is given by the simple expression

$$F_R(R, z)=\frac{\partial\Phi_{2\,\rm exp}}{\partial R}(R, z)=4\pi Gz_0\rho(0, 0)h^2\int_0^{\infty}\frac{kJ_1(Rk)f_{\rm exp}(z; k)}{(1+h^2k^2)^{3/2}}\, dk, \tag{19}$$

while the vertical force $F_z(R, z)$ is obtained from the quadrature

$$F_z(R, z)=\frac{\partial\Phi_{2\,\rm exp}}{\partial z}(R, z)=-4\pi Gz_0\rho(0, 0)h^2\int_0^{\infty}\frac{J_0(R, k)[(\partial/\partial z)f_{\rm exp}(z; k)]}{(1+h^2k^2)^{3/2}}\, dk, \tag{20}$$

where the derivative of $f_{\rm exp}(z; k)$ is equal to

$$\frac{\partial f_{\rm exp}}{\partial z}(z; k)=\frac{-\mathrm{sgn}(z)}{2z_0(kz_0+1)}[kz_0\exp(-k|z|)+\exp(-|z|/z_0)]+\frac{\mathrm{sgn}(z)}{2z_0(kz_0-1)}[kz_0\exp(-k|z|)-\exp(-|z|/z_0)]. \tag{21}$$

The vertical force tends to zero as $z \to 0$ both from above and below, and is therefore a continuous function of z at $z=0$. It also converges to $-z\exp(-|z|/z_0)/(2z_0^2)$ as $k \to 1/z_0$.

I am indebted to the referee for pointing out that the double-exponential disc was also calculated by Kuijken & Gilmore (1989, p. 597, appendix A).

2.2 Gaussian vertical density profile

If the vertical density profile is Gaussian, we can write $\rho_2(z') = \exp(-z'^2/z_0^2)$, where z_0 is a Gaussian scaleheight. The normalization constant in this case is $N = \sqrt{\pi}\, z_0$. A simple application of (15) then shows that $f(z;\, k)$ is given by

$$f_G(z;\, k) = \frac{1}{2}\exp\left(\frac{k^2 z_0^2}{4} - kz\right)\operatorname{erfc}\left(\frac{kz_0}{2} - \frac{z}{z_0}\right) + \frac{1}{2}\exp\left(\frac{k^2 z_0^2}{4} + kz\right)\operatorname{erfc}\left(\frac{kz_0}{2} + \frac{z}{z_0}\right), \tag{22}$$

which converges to $2\exp(-z^2/z_0^2)/(\sqrt{\pi}\, kz_0)$ as $k \to \infty$. Equation (22) can be used to express the potential as a single quadrature: again, in the special case of an exponential radial profile we have

$$\Phi_{G\,\exp}(R, z) = -2\pi^{3/2} G z_0 \rho(0, 0) h^2 \int_0^\infty \frac{J_0(Rk)\, f_G(z;\, k)}{(1 + h^2 k^2)^{3/2}}\, dk. \tag{23}$$

The radial force $F_R(R, z)$ is given by the simple expression

$$F_R(R, z) = \frac{\partial \Phi_{G\,\exp}}{\partial R}(R, z) = 2\pi^{3/2} G z_0 \rho(0, 0) h^2 \int_0^\infty \frac{kJ_1(Rk)\, f_G(z;\, k)}{(1 + h^2 k^2)^{3/2}}\, dk, \tag{24}$$

while the vertical force $F_z(R, z)$ is obtained from the quadrature

$$F_z(R;\, z) = \frac{\partial \Phi_{G\,\exp}}{\partial z}(R, z) = -2\pi^{3/2} G z_0 \rho(0, 0) h^2 \int_0^\infty \frac{J_0(Rk)\,[(\partial/\partial z)\, f_G(z;\, k)]}{(1 + h^2 k^2)^{3/2}}\, dk, \tag{25}$$

where the derivative of $f_G(z;\, k)$ is equal to

$$\frac{\partial}{\partial z} f_G(z;\, k) = \frac{k}{2}\exp\left(\frac{k^2 z_0^2}{4}\right)\left[\exp(kz)\operatorname{erfc}\left(\frac{kz_0}{2} + \frac{z}{z_0}\right) - \exp(-kz)\operatorname{erfc}\left(\frac{kz_0}{2} - \frac{z}{z_0}\right)\right], \tag{26}$$

which converge to $-4z\exp(-z^2/z_0^2)/(\sqrt{\pi}\, kz_0^3)$ as $k \to \infty$.

2.3 General solution for discs with varying scaleheight

The procedure given at the beginning of this section for constant-scaleheight discs can be applied to general density laws $\rho = \rho(R, z)$, replacing $S(k)\, f(z;\, k)$ by $f(z;\, k)$. For the potential this yields the triple integral

$$\Phi(R, z) = -2\pi G \int_0^\infty J_0(Rk) \int_{-\infty}^\infty \exp(-k|z - z'|) \int_0^\infty \rho(R', z') J_0(kR') R'\, dR'\, dz'\, dk, \tag{27}$$

which can be expressed in five alternative ways by various rearrangements of the order of integration. By writing the k integral first, this equation can always be reduced to a double integral over a Legendre function,

$$\Phi(R, z) = -\frac{2G}{\sqrt{R}} \int_{-\infty}^\infty \int_0^\infty \rho(R', z') R'^{1/2} Q_{-1/2}\left[\frac{R^2 + R'^2 + (z - z')^2}{2RR'}\right] dR'\, dz'. \tag{28}$$

This has the same problem as the elliptic integral formulation in that $Q_{-1/2}(x) \to \infty$ as $x \to 1$.

3 FORMULATION FOR THIN AXISYMMETRIC DISCS

In this section I will build a thin axisymmetric disc of arbitrary surface density profile as a weighted superposition of infinitely flattened homoeoids. It can be shown (e.g. Binney & Tremaine 1987, equation 2-137) that the surface density $\Sigma_H(R)$ of an infinitely flattened homoeoid is

$$\Sigma_{\mathrm{H}}(R; a)=\begin{cases}\dfrac{1}{\sqrt{a^2-R^2}}, & R<a,\\ 0, & R\geq a,\end{cases} \tag{29}$$

where a is the semimajor axis length. I now use the Bessel function formulation (equation 4) to calculate the potential $\Phi_{\mathrm{H}}(R, z; a)$ corresponding to this surface density profile. With the help of equations (6.554.2) and (6.752.1) of Gradshteyn & Ryzhik (1965), the potential of an infinitely flattened homoeoid is given as the elementary function

$$\Phi_{\mathrm{H}}(R, z; a)=-2\pi G\sin^{-1}\left[\frac{2a}{\sqrt{z^2+(a+R)^2}+\sqrt{z^2+(a-R)^2}}\right]. \tag{30}$$

Equations (29) and (30) now constitute an elementary Σ, Φ pair to use as building blocks to find the potential of an arbitrary disc. The integral equation for the weight function $S(a)$ becomes, in this case,

$$\Sigma(R)=\int_R^\infty \frac{S(a)}{\sqrt{a^2-R^2}}\,\mathrm{d}a. \tag{31}$$

The motivation for choosing the flattened homoeoid as the building block now becomes apparent: equation (31) has the convenient form of an Abel integral equation (the squares a^2 and R^2 can be transformed to the more familiar linear forms by the obvious substitution). Direct inversion then yields, for the weight function $S(a)$,

$$S(a)=\frac{-2}{\pi}\frac{\mathrm{d}}{\mathrm{d}a}\int_a^\infty \frac{\Sigma(R')R'}{\sqrt{R'^2-a^2}}\,\mathrm{d}R'. \tag{32}$$

The potential $\Phi(R, z)$ of a thin axisymmetric disc with surface density $\Sigma(R)$ can therefore be written

$$\Phi(R, z)=4G\int_0^\infty \sin^{-1}\left[\frac{2a}{\sqrt{z^2+(a+R)^2}+\sqrt{z^2+(a-R)^2}}\right]\left[\frac{\mathrm{d}}{\mathrm{d}a}\int_a^\infty \frac{\Sigma(R')R'}{\sqrt{R'^2-a^2}}\,\mathrm{d}R'\right]\mathrm{d}a. \tag{33}$$

An alternative form can be obtained by an integration by parts, which yields

$$\Phi(R, z)=\frac{-4G}{\sqrt{2}}\int_0^\infty \frac{[(a+R)/\sqrt{+}]-[(a-R)/\sqrt{-}]}{\sqrt{R^2-z^2-a^2+\sqrt{+}\sqrt{-}}}\int_a^\infty \frac{\Sigma(R')R'\,\mathrm{d}R'}{\sqrt{R'^2-a^2}}\,\mathrm{d}a, \tag{34}$$

where I have used the obvious shorthand notations

$$\sqrt{+}=\sqrt{z^2+(a+R)^2}, \tag{35a}$$

$$\sqrt{-}=\sqrt{z^2+(a-R)^2}. \tag{35b}$$

The advantages of the formulation given by (33) or (34) are that the solution is given in terms of elementary functions: while these are not as simple as those of the Kuzmin disc formulation given by Evans & de Zeeuw (1992), the integral equation for the weight function is directly invertible by a single inverse Abel transform to yield an expression which can be treated very easily by standard numerical procedures. This is because the kernel involved in the quadrature formula is both elementary and everywhere non-singular.

Note that the homoeoid method (traditionally used to find the potentials of ellipsoidal bodies) has also been used by Mestel (1963), Lynden-Bell & Pineault (1978) and Lynden-Bell (1989) for thin axisymmetric discs. However, the approach taken by these authors, which follows that of Chandrasekhar (1969), is different from that taken here, and results in a final expression for the potential which is less convenient than (33) and (34) (see equations 31 and 41 of Lynden-Bell 1989). Specifically, even though their formulation is equivalent to the present one on the plane of the disc $z=0$ (see below), away from this plane the former obtains an 'impossible integral for the potential at a general point' (Lynden-Bell & Pineault 1978) – so impossible that the expression is not explicitly written in any of these papers. On the other hand, the *derivation* in this paper of the $\Sigma(R)\rightarrow\Phi(R, z)$ inversion is based on the building-block/linearity of the Poisson equation method, which is both clearer and more elegant (in my opinion). Moreover, the present method yields (for the first time in the flattened homoeoid method) a convenient expression for $\Phi(R, z)$ at *all* R and z that is both elegant *and* convenient numerically. The expression for the three-dimensional potential of the exponential disc, given below (equations 51 and 52), as an infinite integral over the modified Bessel function K_1 (which decays exponentially without changing sign, as opposed to the standard Bessel functions J_0 and J_1 which decay very slowly and in an oscillatory fashion) is just one (important) illustrative example of this fact.

Note finally that the total mass M_T and central surface density Σ_0 are given in terms of the weight function $S(a)$ by

$$M_T = 2\pi \int_0^\infty S(a)\, a\, da, \tag{36}$$

$$\Sigma_0 = \int_0^\infty \frac{S(a)}{a}\, da. \tag{37}$$

The mass $M(R)$ contained within radius R can be written

$$M(R) = M_T - 2\pi \int_R^\infty S(a)\sqrt{a^2 - R^2}\, da. \tag{38}$$

3.1 Potential and density on plane and rotation curve

Putting $z=0$ in equation (34) and remembering that $\lim_{z\to 0}\sqrt{-} = |a-R|$, we obtain for the potential $\Phi(R, 0)$ on the equatorial plane the double Abel transform

$$\Phi(R,0) = -4G \int_0^R \frac{1}{\sqrt{R^2-a^2}} \int_a^\infty \frac{\Sigma(R')R'\, dR'}{\sqrt{R'^2-a^2}}\, da. \tag{39}$$

A check on the validity of this formula can be performed by a careful reversal of the order of integration in (39), which yields

$$\Phi(R,0) = -4G \int_0^R \Sigma(R')R' \int_0^{R'} \frac{da}{\sqrt{R^2-a^2}\sqrt{R'^2-a^2}}\, dR' - 4G \int_R^\infty \Sigma(R')R' \int_0^R \frac{da}{\sqrt{R^2-a^2}\sqrt{R'^2-a^2}}\, dR'. \tag{40}$$

The inner integrals are complete elliptic integrals, so that

$$\Phi(R,0) = \frac{-4G}{R} \int_0^R \Sigma(R')R' K\left(\frac{R'}{R}\right) dR' - 4G \int_R^\infty \Sigma(R') K\left(\frac{R}{R'}\right) dR' = -4G \int_0^\infty R'\Sigma(R') \frac{1}{R_>} K\left(\frac{R_<}{R_>}\right) dR', \tag{41}$$

where $R_> = \max(R, R')$ and $R_< = \min(R, R')$. Equation (41) is exactly the expression that is obtained from the Bessel function formulation on performing the integral over k on $z=0$ in equation (4).

It is easy to see from (39) that the circular speed $v_c(R)$ is given by

$$v_c^2(R) = R\frac{\partial\Phi}{\partial R}(R,0) = -4G \int_0^R \frac{a}{\sqrt{R^2-a^2}} \left[\frac{d}{da} \int_a^\infty \frac{\Sigma(R')R'}{\sqrt{R'^2-a^2}}\, dR'\right] da. \tag{42}$$

Due to the Abel and inverse-Abel nature of (42), this double integral equation can be inverted to give an expression for the surface density as a function of the rotation curve:

$$\Sigma(R) = \frac{1}{\pi^2 G} \int_R^\infty \frac{1}{a\sqrt{a^2-R^2}} \left[\frac{d}{da} \int_0^a \frac{v_c^2(R')R'\, dR'}{\sqrt{a^2-R'^2}}\right] da. \tag{43}$$

This result was effectively already known to Kuzmin (1952) and, later, to Brandt (1960).

As a simple illustration of (43), consider the disc which generates a constant circular velocity everywhere, $v_c^2(R') = 2\pi G\Sigma_0 R_0$, where Σ_0 and R_0 are surface density and radius scales respectively. For this rotation curve, the inner integral is simply $2\pi G\Sigma_0 R_0 a$, so that

$$\begin{aligned}\Sigma(R) &= \frac{2\Sigma_0 R_0}{\pi} \int_R^\infty \frac{da}{a\sqrt{a^2-R^2}} \\ &= \frac{2\Sigma_0 R_0}{\pi R} \int_0^\infty dx/\cosh x \\ &= \frac{\Sigma_0 R_0}{R},\end{aligned} \tag{44}$$

where I have used the fact that $(2\tan^{-1} e^x)' = 1/\cosh x$. The surface density profile given by (44) is that of Mestel's disc (Mestel 1963): this well-known result has been obtained here by the evaluation of two elementary integrals – special functions were not needed.

Note also that equation (39) can be inverted for $\Sigma(R)$ in the same way as (42). This gives the following expression for the surface density as a function of the potential on the plane:

$$\Sigma(R) = \frac{1}{\pi^2 GR}\frac{d}{dR}\int_R^\infty \frac{a}{\sqrt{a^2-R^2}}\left(\frac{d}{da}\int_0^a \frac{\Phi(R',0)R'\,dR'}{\sqrt{a^2-R'^2}}\right)da. \tag{45}$$

This can be expressed as a double inverse Abel transform in terms of the more familiar linear factors by

$$\Sigma(R) = \frac{1}{\pi^2 G}\frac{d}{dR^2}\int_{R^2}^\infty \frac{t^{1/2}}{\sqrt{t-R^2}}\left[\frac{d}{dt}\int_0^t \frac{\psi(R')\,dR'}{\sqrt{t-R'}}\right]dt, \tag{46}$$

where ψ is defined by $\psi(R') \equiv [\Phi(R, 0); R \to \sqrt{R'}]$. It may be worth noting that (46) now provides a way of writing $\Sigma(R)$ formally in terms of $\Phi(R, 0)$ in such a way that only derivatives appear, so that an analogue of Poisson's equation is obtained for two-dimensional systems:

$$\Sigma(R) = \frac{1}{\pi G}\mathrm{D}_\infty^{1/2}\{t^{1/2}\mathrm{D}_0^{1/2}[\psi(R'); t]; R^2\}. \tag{47}$$

The inverse Abel operators $\mathrm{D}_{0,\infty}^{1/2}$ are known as fractional derivatives (e.g. Erdèlyi et al. 1954; Oldham & Spanier 1974), and are defined by

$$\mathrm{D}_0^{1/2}[f(y); x] = \frac{1}{\sqrt{\pi}}\frac{d}{dx}\int_0^x \frac{f(y)}{\sqrt{x-y}}\,dy, \tag{48}$$

$$\mathrm{D}_\infty^{1/2}[f(y); x] = \frac{1}{\sqrt{\pi}}\frac{d}{dx}\int_x^\infty \frac{f(y)}{\sqrt{y-x}}\,dy. \tag{49}$$

The resemblance of the fractional derivative form (47) to the planar part of Poisson's equation can be seen by (heuristically) ignoring the vertical derivatives in Poisson's equation to obtain

$$\rho = \frac{1}{4\pi GR}\frac{d}{dR}\left(R\frac{d\Phi}{dR}\right) = \frac{1}{4\pi G}\frac{d}{dR^2}\left(R^{1/2}\frac{d\Phi}{dR^{1/2}}\right).$$

However, it is unfortunately not immediately apparent how to generalize the differintegral expression (47) to the elliptic disc case.

3.2 The exponential disc

Freeman (1970) showed that the light profiles in the radial direction of spiral galaxies can be modelled extremely well by a simple exponential law. The assumption of a constant mass-to-light ratio for the disc gives the well-known surface density profile $\Sigma(R) = \Sigma_0 \exp(-R/h)$. In the Bessel function formulation, the potential $\Phi_{\exp}(R, z)$ corresponding to this profile is given as a single infinite integral of a function involving $J_0(kR)$ (e.g. Binney & Tremaine 1987, equation 2-167). Numerically this is quite awkward, because this Bessel function behaves as a slowly decaying cosine at infinity, thus giving an oscillatory integrand there. A similar problem arises in the Evans & de Zeeuw (1992) formulation (with the Bessel function J_1). The present formulation provides a more convenient representation of the three-dimensional potential of the exponential disc (and indeed for the potentials of all discs that cannot be found completely analytically). Inserting $\Sigma = \Sigma_0 \exp(-R'/h)$ gives, for the inner integral in (33) and (34),

$$\int_a^\infty \frac{\Sigma(R')R'\,dR'}{\sqrt{R'^2-a^2}} = \Sigma_0\int_a^\infty \frac{\exp(-R'/h)R'\,dR'}{\sqrt{R'^2-a^2}} = \Sigma_0 a K_1\left(\frac{a}{h}\right), \tag{50}$$

where I have used equation (3.365.2) of Gradshteyn & Rhyzhik (1965). K_1 is a modified Bessel function, which is positive definite and which decays exponentially at infinity. Equations (33) and (34) now yield the two alternative integral formulae for the potential of the exponential disc,

$$\Phi_{\exp}(R, z) = \frac{-4G\Sigma_0}{h} \int_0^\infty \sin^{-1}\left(\frac{2a}{\sqrt{+} + \sqrt{-}}\right) aK_0\left(\frac{a}{h}\right) \mathrm{d}a \tag{51}$$

$$= \frac{-4G\Sigma_0}{\sqrt{2}} \int_0^\infty \frac{[(a+R)/\sqrt{+}] - [(a-R)/\sqrt{-}]}{\sqrt{R^2 - z^2 - a^2 + \sqrt{+}\sqrt{-}}} aK_1\left(\frac{a}{h}\right) \mathrm{d}a, \tag{52}$$

which should prove significantly more convenient for numerical evaluation than previous formulae.

4 TRIAXIAL POTENTIALS

The problem of solving Poisson's equation for triaxial potentials is significantly more difficult than for the axisymmetric case. A number of algorithms exist for this purpose, including Kalnajs' (1971) method of decomposition into logarithmic spirals and the triaxial extension of the Bessel function formulation (Binney & Tremaine 1987, problem 2-9). Both these methods require, for a triaxial disc potential, an infinite summation over three integrals. Recently, Evans & de Zeeuw (1992) have devised a powerful new method for generating triaxial disc potentials, which is very useful in cases where the Stieltjes integral equation involved can be solved analytically.

In this section I present the general solution of Poisson's equation for an arbitrary distribution of matter in a form involving Fourier transforms, which I do not believe to have previously appeared in the literature. This formulation shows that the link between the elliptic integral and Bessel function formulations is a manifestation of the properties of the various integral transforms involved, and should prove useful in practice for the computation of a number of triaxial disc potentials.

By separating Poisson's equation in Cartesian coordinates, and using the same method as in Section 2, it is easy to show that the potential generated by an arbitrary distribution of matter $\rho(x, y, z)$ can be written as the integral of a four-dimensional Fourier transform:

$$\Phi(x, y, z) = \frac{-G}{2\pi} \int_{-\infty}^{\infty} \exp(\mathrm{i}x\lambda) \int_{-\infty}^{\infty} \frac{\exp(\mathrm{i}yk)}{\sqrt{k^2+\lambda^2}} \int_{-\infty}^{\infty} \exp(-\sqrt{k^2+\lambda^2}\,|z-z'|)$$
$$\times \int_{-\infty}^{\infty} \exp(\mathrm{i}ky') \int_{-\infty}^{\infty} \exp(\mathrm{i}\lambda x')\rho(x', y', z')\, \mathrm{d}x'\, \mathrm{d}y'\, \mathrm{d}z'\, \mathrm{d}k\, \mathrm{d}\lambda. \tag{53}$$

It is also easy to show, by rearranging the order of integration and performing the integrals over k and λ, that (53) is equivalent to the Green's function solution (1).

If the triaxial mass distribution is that of an infinitesimal disc, we can write $\rho(x', y', z') = \Sigma(x', y')\,\delta(z')$, so that (53) reduces to the four-dimensional Fourier transform

$$\Phi(x, y, z) = \frac{-G}{2\pi} \int_{-\infty}^{\infty} \exp(\mathrm{i}x\lambda) \int_{-\infty}^{\infty} \frac{\exp(\mathrm{i}yk)}{\sqrt{k^2+\lambda^2}} \exp(-|z|\sqrt{k^2+\lambda^2}) \int_{-\infty}^{\infty} \exp(\mathrm{i}ky') \int_{-\infty}^{\infty} \exp(\mathrm{i}\lambda x')\Sigma(x', y')\, \mathrm{d}x'\, \mathrm{d}y'\, \mathrm{d}k\, \mathrm{d}\lambda. \tag{54}$$

When expressed in cylindrical coordinates, this becomes

$$\Phi(R, \phi, z) = \frac{-G}{2\pi} \int_0^{2\pi} \int_0^\infty \exp[\mathrm{i}\alpha R\cos(\phi-\beta)] \exp(-|z|\alpha) \int_0^{2\pi} \int_0^\infty \exp[\mathrm{i}\alpha R'\cos(\phi'-\beta)]\Sigma(R', \phi')R'\, \mathrm{d}R'\, \mathrm{d}\phi'\, \mathrm{d}\alpha\, \mathrm{d}\beta. \tag{55}$$

If the surface density assumes the special, purely multipole form

$$\Sigma(R', \phi') = \Sigma_1(R') \exp(\mathrm{i}n\phi'), \tag{56}$$

then it follows from the well-known integral

$$\int_0^{2\pi} \exp\{\mathrm{i}[r\rho\cos(\theta-\alpha) + n\theta]\}\, \mathrm{d}\theta = 2\pi \exp[\mathrm{i}n(\alpha+\pi/2)]J_n(r\rho) \tag{57}$$

(e.g. Hochstadt 1989) that the potential can be written

$$\Phi(R, \phi, z) = -2\pi G \exp[\mathrm{i}n(\phi+\pi)] \int_0^\infty \int_0^\infty \exp(-|z|\alpha) J_n(\alpha R') J_n(\alpha R) \Sigma_1(R') R'\, \mathrm{d}R'\, \mathrm{d}\alpha, \tag{58}$$

which is the correct Bessel function expression for Φ, reducing to the well-known integral over zeroth-order Bessel functions in the axisymmetric case. This result is a consequence of the fact that the two-dimensional Fourier transform of an axially symmetric function is equal to the Hankel transform of that function. Equation (58) provides the best way of calculating, for example, the potential of a bar.

Note finally that writing (55) such that the inner integral is over α gives a simple Laplace transform to yield

$$\Phi(R, \phi, z) = \frac{-G|z|}{2\pi} \int_0^{2\pi} \int_0^\infty R' \Sigma(R', \phi') \int_0^{2\pi} \frac{1}{z^2 + [R\cos(\phi-\beta) + R'\cos(\phi'-\beta)]^2} \, d\beta \, dR' \, d\phi'. \tag{59}$$

Equation (59) resembles the expression obtained by Kalnajs' method (Binney & Tremaine 1987, equation 2-181), but the former is valid for all z-values.

I now give an example of the use of equation (54) by calculating the potential–density pair for a particular family of triaxial discs. Consider the surface density law

$$\Sigma(x, y) = \frac{\Sigma_0 x_0^a y_0^b}{|x|^a |y|^b}, \tag{60}$$

where $0 < a, b < 1$. This surface density law gives rise to equidensity contours which are somewhat diamond-shaped. The singularity at the origin is meant to mimic the behaviour of Mestel's disc. By (54), two simple Fourier transforms can be performed to give the three-dimensional potential corresponding to this surface density as

$$\Phi(x, y, z) = C \int_{-\infty}^{\infty} \frac{\exp(ix\lambda)}{|\lambda|^{1-a}} \left[\int_{-\infty}^{\infty} \frac{\exp(iyk)\exp(-|z|\sqrt{k^2+\lambda^2})}{\sqrt{k^2+\lambda^2}\,|k|^{1-b}} \, dk \right] d\lambda, \tag{61}$$

where the constant C is given by $C = -2G\Sigma_0 x_0^a y_0^b \Gamma(1-a)\Gamma(1-b)\sin(a\pi/2)\sin(b\pi/2)/\pi$. By equation (3.773.4) of Gradshteyn & Ryzhik (1965), the integral over k can be performed when $z=0$ to yield the potential on the plane as the single Fourier transform

$$\Phi(x, y, 0) = C \int_{-\infty}^{\infty} \exp(ix\lambda) \frac{|\lambda|^{1-b}}{|\lambda|^{1-a}} G_{13}^{21} \left[\frac{y^2\lambda^2}{4} \middle| \begin{matrix} b/2 \\ (b-1)/2, 0, 1/2 \end{matrix} \right] d\lambda, \tag{62}$$

where G is Meijer's G-function. Using the fact that Fourier transforms can sometimes be found from tables of Laplace transforms with a suitable change of variable, I finally obtain, with the help of equation (4.23.18) of Erdèlyi et al. (1954), the triaxial potential on the plane of symmetry according to the hypergeometric expression

$$\Phi(x, y, 0) = 2C \left[D|x|^{b-a-1} {}_3F_2 \left(\frac{2-b}{2}, \frac{a-b+1}{4}, \frac{a-b+3}{4}; \frac{3-b}{2} \frac{1}{2}; \frac{-y^2}{x^2} \right) + E \frac{|x|^{-a}}{|y|^{1-b}} {}_3F_2 \left(\frac{1}{2}, \frac{a}{4}, \frac{a+2}{4}; \frac{b}{2}; \frac{b+1}{2}; \frac{-y^2}{x^2} \right) \right], \tag{63}$$

where the constants D and E are given in terms of a and b by

$$D = \Gamma\left(\frac{2-b}{2}\right) \Gamma\left(\frac{b-1}{2}\right) \Gamma(a-b+1) \cos\left[\frac{\pi(b-a-1)}{2}\right] \Big/ \sqrt{\pi},$$

$$E = 2^{1-b} \sqrt{\pi}\, \Gamma\left(\frac{1-b}{2}\right) \Gamma(a) \cos\left(\frac{\pi a}{2}\right) \Big/ \Gamma(b/2).$$

This hypergeometric expression converges for $|y/x| < 1$; the potential for $|y/x| > 1$ can be obtained from this by the simultaneous transpositions $x \leftrightarrow y$, $a \leftrightarrow b$.

5 CONCLUSION

In this paper the standard Bessel function formula for the potential of a thin axisymmetric disc is generalized to include arbitrary vertical structure. In particular, when the disc has a constant scaleheight with either an exponential or a Gaussian vertical density profile (both of which provide good fitting laws to external spiral galaxies), the potential and its derivatives can be expressed as single quadratures. [The case of a disc with exponential profiles in both directions was already known to Kuijken & Gilmore (1989, appendix A), although that result does not seem to have been noticed by many workers.]

An alternative formulation for the potential of axisymmetric thin discs is also presented. This formulation is obtained physically by using the potential–density pair of an infinitely flattened homoeoid as a building block and exploiting the linearity of the Poisson equation. This yields a double integral over elementary functions, with the inner integral being a simple Abel transform of the surface density (so that in practice the potential will often be expressible as a single quadrature, often over elementary functions). The advantage of this formulation over the standard Bessel function formula lies in the fact that, in the latter, one is always forced to perform integrals over Bessel functions (or worse). The Abel transform formulation also has the advantage with respect to the new Kuzmin disc formulation of Evans & de Zeeuw (1992) that the integral equation for the weight function is always directly invertible by an inverse Abel transform, as opposed to the case of a generalized Stieltjes equation whose formal inversion is not amenable to numerical treatment. Furthermore, in the important case of the exponential disc, the potential in the Abel transform formulation is a single quadrature over the product of an elementary function with a *modified* Bessel function K_0 or K_1 (as opposed to a normal Bessel function J_0 or J_1 in the other two formulations). This gives rise to a much more convenient numerical evaluation, since K_0 and K_1 decay exponentially and are positive definite at infinity, as opposed to the slow oscillatory decay of J_0 and J_1. I stress, however, that the present formulation is complementary to the methods of Toomre (1962) and Evans & de Zeeuw (1992), since the latter two are more useful in obtaining completely analytical potential–density pairs (albeit that the number of such pairs is very small). The formulation presented here expresses the potential on the plane as a double Abel transform of the surface density. This can be inverted to give the surface density as a double inverse Abel transform of the potential on the plane, which is also a combination of half-derivatives of the planar potential. This yields an expression formally similar to the radial part of Poisson's equation. No analogous result has yet been found for elliptic discs (but see Dejonghe's equation 2-13 in Binney & Tremaine 1987).

Finally, the solution of Poisson's equation for an arbitrary distribution of matter has been given in Cartesian coordinates as an integral of a four-dimensional Fourier transform, which reduces to a four-dimensional Fourier transform for a thin triaxial disc. This formulation shows that the link between the Green's function and Bessel function solutions of Poisson's equation is a consequence of the close relation between Fourier and Hankel transforms. For density laws which are naturally expressible in Cartesian coordinates, this provides a new and convenient way of calculating the potentials of triaxial discs, since fast and efficient algorithms exist for the numerical evaluation of Fourier transforms. A particular family of potential–density pairs for triaxial discs is also calculated analytically by this method.

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