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On the Power Graph of a Finite Group

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Abstract. The power graph $P(G)$ of a group G is the graph whose vertex set is the group elements and two elements are adjacent if one is a power of the other. In this paper, we consider some graph theoretical properties of a power graph $P(G)$ that can be related to its group theoretical properties. As consequences of our results, simple proofs for some earlier results are presented.

1. Introduction

All groups and graphs in this paper are finite. Throughout the paper, we follow the terminology and notation of [11, 12] for groups and [18] for graphs.

Groups are the main mathematical tools for studying symmetries of an object and symmetries are usually related to graph automorphisms, when a graph is related to our object. Groups linked with graphs have been arguably the most famous and productive area of algebraic graph theory, see [1, 11] for details. The power graphs is a new representation of groups by graphs. These graphs were first used by Chakrabarty et al. [4] by using semigroups. It must be mentioned that the authors of [4] were motivated by some papers of Kelarev and Quinn [8–10] regarding digraphs constructed from semigroups. We also encourage interested readers to consult papers by Cameron and his co-workers on power graphs constructed from finite groups [2, 3].

Suppose G is a finite group. The *power graph* $P(G)$ is a graph in which $V(P(G)) = G$ and two distinct elements x and y are adjacent if and only if one of them is a power of the other. If G is a finite group then it can be easily seen that the power graph $P(G)$ is a connected graph of diameter 2. In [4], it is proved that for a finite group G, $P(G)$ is complete if and only if G is a cyclic group of order 1 or p^m, for some prime number p and positive integer m.

Following [12, 13], two finite groups G and H are said to be conformal if and only if they have the same number of elements of each order. In [13], the following question was investigated:

Question: *For which natural numbers n are any two conformal groups of order n isomorphic?*

Let G be a group and $x \in G$. We denote by $o(x)$ the order of x and G is said to be EPO−group, if all non-trivial element orders of G are prime. An EPPO−group is that its element orders are prime power.

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The set of all elements order of G is called its *spectrum*, denoted by $\pi_e(G)$, A maximal subgroup H of G is denoted by $H < \cdot\, G$ and the set of all elements of G of order k is denoted by $\Omega_k(G)$.

Suppose Γ is a graph. A subset X of the vertices of Γ is called a *clique* if the induced subgraph on X is a complete graph. The maximum size of a clique in Γ is called the *clique number* of Γ and denoted by $\omega(\Gamma)$. A subset Y of $V(\Gamma)$ is an *independent set* if the induced subgraph on X has no edges. The maximum size of an independent set is called the *independence number* of G and denoted by $\alpha(G)$. The *chromatic number* of Γ is the smallest number of colors needed to color the vertices of Γ so that no two adjacent vertices share the same color. This number is denoted by $\chi(\Gamma)$.

Throughout this paper our notation is standard and they are taken from the standard books on graph theory and group theory such as [12, 18].

2. Main Results

Suppose G is a finite group of order n. Chakrabarty, Ghosh and Sen [4] proved that the number of edges of $P(G)$ can be computed by the following formula:

$$e = \frac{1}{2}\sum_{a\in G}\{2o(a) - \phi(o(a)) - 1\},$$

where ϕ is the Euler's totient function. In the case that G is cyclic, we have:

$$e = \frac{1}{2}\sum_{d|n}\{2d - \phi(d) - 1\}\phi(d).$$

Moreover, $P(Z_n)$ is nonplanar when $\phi(n) > 7$ or $n = 2^m$, $m \geq 3$. Finally, if $n \geq 3$ then $P(Z_n)$ is Hamiltonian.

Suppose $D(n)$ denotes the set of all positive divisors of n. It is well-known that $(D(n), |)$ is a distributive lattice. $D(n)$ is a Boolean algebra if and only if n is square-free. In the following theorem we apply the structure of this lattice to compute the clique and chromatic number of $P(Z_n)$.

Lemma 1 Suppose G is a group and $A \subseteq G$. The vertices of A constitute a complete subgraph in $P(G)$ if and only if $\{\langle x\rangle \mid x \in A\}$ is a chain.

Proof Suppose C is a clique in $P(G)$. To prove that $\{\langle x\rangle \mid x \in C\}$ is a chain, we proceed by induction on $|V(C)|$. If $|C| = 2$ the result is obvious. If $V(C) = \{x_1, x_2, \cdots, x_n\}$ then by induction hypothesis, $\{\langle x_i\rangle \mid 1 \leq i \leq n-1\}$ is a chain in $P(G)$. Without loss of generality we can assume that $1 \subseteq \langle x_1\rangle \subseteq \langle x_2\rangle \subseteq \cdots \subseteq \langle x_{n-1}\rangle$. Consider $t = max\{i \mid \langle x_i\rangle \subseteq \langle x_n\rangle\}$. If $t = n-1$ then the result is proved. Otherwise, $\langle x_t\rangle \subseteq \langle x_n\rangle \subseteq \langle x_{t+1}\rangle$, as desired. Conversely, by definition of power graph, every chain of cyclic subgroups is a clique. □

Theorem 2 Suppose $n = p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_r^{\alpha_r}$, where $p_1 < p_2 < ... < p_r$ are prime numbers. Then

$$\omega(P(Z_n)) = \chi(P(Z_n)) = p_r^{\alpha_r} + \sum_{j=0}^{r-2}(p_{r-j-1}^{\alpha_{r-j-1}} - 1)\left(\prod_{i=0}^{j}\phi(p_{r-i}^{\alpha_{r-i}})\right).$$

Proof Define the relation $\sim$ on Z_n by $a \sim b$ if and only if they have the same order. Then it can easily seen that $\sim$ is an equivalence relation on Z_n and $\frac{Z_n}{\sim}$ can be equipped with an order such that $\frac{Z_n}{\sim} \cong D(n)$. Here $\frac{a}{\sim} \leq \frac{b}{\sim}$ if and only if $o(a)|o(b)$. Choose an element $a \in Z_n$. By our definition, the elements of $\frac{a}{\sim}$ are adjacent in $P(Z_n)$. Moreover, for each chain $\frac{v_1}{\sim}, \frac{v_2}{\sim}, \cdots, \frac{v_t}{\sim}$ of elements in $\frac{Z_n}{\sim}$, $\bigcup_{i=1}^{t}\frac{v_i}{\sim}$ is a complete subgraph of $P(Z_n)$. For an arbitrary element $\frac{u}{\sim} \in \frac{Z_n}{\sim}$, define $d(\frac{o}{\sim}, \frac{u}{\sim})$ to be the same as distance between corresponding elements of $D(n)$.

To find a maximal complete subgraph of $P(Z_n)$, by Lemma 1 it is enough to obtain a maximal chain

$$Q : \frac{a_0}{\sim} = \frac{o}{\sim}, \frac{a_1}{\sim}, \frac{a_2}{\sim}, \cdots, \frac{a_l}{\sim}, \frac{n}{\sim} = \frac{a_{l+1}}{\sim} \tag{1}$$

such that Q has the maximum length, $\frac{a_1}{\sim} \cup \frac{a_2}{\sim} \cup \cdots \cup \frac{a_l}{\sim}$ has the maximum possible size and $l+1 = \alpha_1 + \cdots + \alpha_r$. To do this, it is enough to choose a_1 to be an element of order p_r, a_2 to be an element of order p_r^2, ..., a_{α_r} to be an element of order $p_r^{\alpha_r}$, $a_{\alpha_{r+1}}$ to be an element of order $p_r^{\alpha_r} p_{r-1}$ and so on. Therefore,

$$\begin{aligned}
\omega(P(Z_n)) &= |\frac{a_0}{\sim}| + |\frac{a_1}{\sim}| + \cdots + |\frac{a_{l+1}}{\sim}| \\
&= (\phi(p_r) + \phi(p_r^2) + \cdots + \phi(p_r^{\alpha_r})) \\
&+ \phi(p_r^{\alpha_r})(\phi(p_{r-1}) + \cdots + \phi(p_{r-1}^{\alpha_{r-1}})) \\
&+ \cdots \\
&+ \phi(p_r^{\alpha_r}) \cdots \phi(p_2^{\alpha_2})(\phi(p_1) + \cdots + \phi(p_1^{\alpha_1})) + 1 \\
&= p_r^{\alpha_r} + \sum_{j=0}^{r-2} (p_{r-j-1}^{\alpha_{r-j-1}} - 1) \left(\prod_{i=0}^{j} \phi(p_{r-i}^{\alpha_{r-i}}) \right).
\end{aligned}$$

To complete the proof we have to prove that $\omega(P(Z_n)) = \chi(P(Z_n))$ and this is a direct consequence of the strong perfect graph theorem [5]. □

The *exponent* of a finite group G is defined as the least common multiple of all elements of G, denoted by $Exp(G)$. It is easy to see that if G is nilpotent then there exists an element $a \in G$ such that $o(a) = Exp(G)$. Such groups are said to be *full exponent*.

Theorem 3 Suppose that G is a full exponent group and $n = Exp(G) = p_1^{\beta_1} p_2^{\beta_2} \cdots p_r^{\beta_r}$, where $p_1 < p_2 < ... < p_r$ are prime numbers. If x is an element of order n then

$$\omega(P(G)) = \chi(P(G)) = p_r^{\beta_r} + \sum_{j=0}^{r-2} (p_{r-j-1}^{\beta_{r-j-1}} - 1) \left(\prod_{i=0}^{j} \phi(p_{r-i}^{\beta_{r-i}}) \right).$$

Proof By Lemma 1, a subset A of G constitutes a clique in $P(G)$ if and only if $\{\langle x \rangle \mid x \in A\}$ is a chain. To obtain a maximal clique in $P(G)$, we have to choose a chain $1 \subseteq \langle x_1 \rangle \subseteq \langle x_2 \rangle \subseteq \cdots \subseteq \langle x_t \rangle$ such that $o(x_t) = o(x)$ and $1 + \sum_{i=1}^{t} \varphi(o(x_i))$ has maximum value among all possible chains of subgroups of $\langle x \rangle$. Now a similar argument as given in the proof of Theorem 2, completes the proof. □

Our calculations on the small group library of GAP [15] suggest the following conjecture:

Conjecture 1: The Theorem 3 is correct in general.

Corollary 4 Let G be a finite group. Then the power graph $P(G)$ is planar if and only if $\pi_e(G) \subseteq \{1, 2, 3, 4\}$.

Proof Suppose $P(G)$ is planar. Then $P(G)$ does not have the complete graph K_5 as its induced subgraph and the Theorem 3 concludes the result. Conversely, if $\pi_e(G) \subseteq \{1, 2, 3, 4\}$ then it can easily seen that $P(G)$ can be embedded into sphere, as desired. □

In [4, Lemma 4.7], the authors proved that if G is a cyclic group of order n, $n \geq 3$ and $\phi(n) > n$ then $P(G)$ is not planar. Also, in [4, Lemma 4.8] it is proved that a cyclic group of order 2^n, $n \geq 3$, is not planar. In the following corollary we apply Corollary 4 to find a simple classification for planarity of the power graph of cyclic groups.

Corollary 5 The power graph of a cyclic group of order n is planar if and only if $n = 2, 3, 4$.

In what follows, U_n denotes the groups of units in the ring Z_n. In the following corollary a new simple proof for [4, Lemma 4.10] is presented.

Corollary 6 The power graph of U_n is planar if and only if $n|240$.

Proof Suppose $n = p_1^{e_1}p_2^{e_2}\cdots p_k^{e_k}$, where $p_1, p_2, \cdots, p_k$ are distinct primes. Then by [7, Theorems 6.11, 6.13 and Corollary 6.14], U_{p^e} is cyclic for odd p, $U_2 \cong 1$, $U_4 \cong Z_2$, $U_{2^n} \cong Z_2 \times Z_{2^{n-2}}$ and $U_n \cong U_{p_1^{e_1}} \times \cdots \times U_{p_k^{e_k}}$. Therefore, by Corollary 4, $n|240$. □

Consider the dihedral group D_{2n} presented by

$$D_{2n} = \langle x, y \mid x^n = y^2 = e \ \&\ y^{-1}xy = x^{-1}\rangle.$$

From [4, Corollary 4.3], we can deduce that the number of edges of $P(D_{2n})$ is given by $e = \frac{1}{2}\sum_{d|n}\{2d\phi(d) - \phi(d)^2\} + n$. This graph is neither Eulerian nor hamitonian, since the group has elements of order 2.

By corollary 5, it is easy to prove the power graph of a dihedral group of order $2n$ is planar if and only if $n = 2, 3, 4$.

Corollary 7 $\chi(P(D_{2n})) = \omega(P(D_{2n})) = \chi(P(Z_n))$.

Proof Notice that the power graph $P(D_{2n})$ is a union of $P(Z_n)$ and n copy of K_2 that share in the identity element of D_{2n}. □

The semi−dihedral group SD_{2^n} is presented by

$$SD_{2^n} = \langle x, y \mid x^{2^{n-1}} = y^2 = 1, yxy = r^{2^{n-2}-1}\rangle.$$

Corollary 8 The power graph $P(SD_{2^n})$ is a union of a complete graph of order 2^n and 2^n copies of K_2 that share in the identity vertex. This graph is non-Eulerian, non-hamiltonian and nonplanar, for $n \geq 3$. Moreover, $\chi(P(SD_{2^n})) = \omega(P(SD_{2^n})) = \alpha(P(SD_{2^n})) = 2^n$.

Following [6] we assume that P is a finite partially ordered set (poset for short) which possesses a *rank function* $r : P \longrightarrow \mathbb{N}$ with the property that $r(p) = 0$, for some minimal element p of P and $r(q) = r(p) + 1$ whenever q covers p. Let $N_k := \{p \in P \ :\ r(p) = k\}$ be its k^{th} level and let $r(P) := max\{r(p) \ :\ p \in P\}$ be the rank of P. An *antichain* or *Sperner family* in P is a subset of pairwise incomparable elements of P. It is clear that each level is an antichain. The *width* (*Dilworth* or *Sperner number*) of P is the maximum size $d(P)$ of an antichain of P. The poset P is said to have the *Sperner property* if $d(P) = max_k|N_k|$. A *k−family* in P, $1 \leq k \leq r(P)$, is a subset of P containing no $(k+1)$−chain in P, and P has the *strong Sperner property* if for each k the largest size of a k−family in P equals the largest size of a union of k levels.

Theorem 9 Suppose that $n = p_1^{\beta_1}\cdots p_r^{\beta_r}$ is the prime decomposition of n and $m = \beta_1 + \cdots + \beta_r$. Then $\alpha(P(Z_n))$ is the coefficient of the middle or the two middle term of $\Pi_{j=1}^m(1 + x + \cdots + x^{\beta_j})$.

Proof It is well-known that the lattice of divisors of a natural number, ordered by divisibility, has strong Sperner property and so its largest antichain is its largest rank level. □

Let Γ be a graph. The minimum number of vertices of Γ which need to be removed to disconnect the remaining vertices of Γ from each other is called the *connectivity* of Γ, denoted by $\kappa(\Gamma)$. If G is finite group then we define:

$$M(G) = \{x \in G \ ;\ \langle x\rangle < \cdot\, G\}.$$

Theorem 10 Suppose G is a non-cyclic group and $x \in G$ such that $\langle x\rangle < \cdot\, G$. Define $r(x) = \cup_{y \in M(G)-\langle x\rangle}(\langle x\rangle \cap \langle y\rangle)$. Then,

$$\kappa(P(G)) \leq Min\{|r(x)| \ ;\ \langle x\rangle < \cdot\, G\}.$$

Proof Suppose $\langle x\rangle$ is a maximal cyclic subgroup of G. We claim that $r(x)$ is a cut set of $P(G)$. Since G is noncyclic, there exists another maximal cyclic subgroup $\langle y\rangle$ different from $\langle x\rangle$. If $r(x)$ is not a cut set of $P(G)$ then there exists a shortest path $Q : x = x_0, x_1, x_2, ..., x_{n-1}, x_n = y$ in $P(G)$ connecting x and y. Without loss of generality we can assume that $x_{2k}, 0 \le k \le \lceil \frac{n}{2} \rceil$, are generators of maximal cyclic subgroups of G. Thus, $x_1 \in \langle x\rangle \cap \langle x_2\rangle \subseteq r(x)$ contradict by our assumption. This completes the proof. □

For a finite group G, the set of all maximal cyclic subgroups of G is denoted by $MaxCyc(G)$.

Lemma 11 Suppose G is a non-cyclic finite group, $S \subseteq G - M(G)$, $MaxCyc(G) = \{\langle x_1\rangle, ..., \langle x_r\rangle\}$ and $A = \{x_1, ..., x_r\}$. S is a minimal cut set with this property that each component of $P(G) - S$ has exactly one element of A if and only if $S = \cup_{x \in M(G)} r(x)$.

Proof If $S = \cup_{x\in M(G)} r(x)$ then by an argument similar to the proof of Theorem 10, one can see that if $x, y \in M(G)$ and $\langle x\rangle \neq \langle y\rangle$ then $\{x_1, x_3, \cdots\} \subseteq S$, where $x = x_0, x_1, x_2, ..., x_{n-1}, x_n = y$ is a shortest path in $P(G)$ connecting x and y. Therefore, if $x, y \in M(G)$, $\langle x\rangle \neq \langle y\rangle$ then x and y are not in the same component of $P(G) - S$.

Conversely, we assume that S is a cut set with this property that each component of $P(G) - S$ has exactly one element of A and $x, y \in A$. Suppose $ł \in \langle x\rangle \cap \langle y\rangle$ and $ł \notin S$. Then $ł$ is adjacent to x and y and so there exists a component of $P(G) - S$ containing both of x and y, a contradiction. Therefore, $\cup_{x\in M(G)} r(x) \subseteq S$. On the other hand, we assume that $z \in S$ and $\langle t\rangle$ is a maximal cyclic subgroup of G containing z. By minimality of S, there are at least two components X_1 and X_2 of $P(G) - S$ such that z is adjacent to a vertex $v_1 \in X_1$ and a vertex $v_2 \in X_2$. Without loss of generality, we can assume that X_1 is the component containing t and $v_1 = t$. Obviously, $\langle v_2\rangle \nsubseteq \langle t\rangle$ and so there exists a vertex $t' \in A \cap X_2$ such that $\langle v_2\rangle \subseteq \langle t'\rangle$. Since z is adjacent to v_2, $\langle z\rangle \subseteq \langle v_2\rangle$ or $\langle v_2\rangle \subseteq \langle z\rangle$. If $\langle z\rangle \subseteq \langle v_2\rangle$ then $z \in \langle t\rangle \cap \langle t'\rangle$, as desired. If $\langle v_2\rangle \subseteq \langle z\rangle$ then v_2 is adjacent to t which is impossible. This completes our argument. □

It is easily seen that the power graph of a $p-$group Q is a union of some complete graphs of order p which share in identity vertex if and only if Q has exponent p. In the following theorem we investigate the same problem for an arbitrary group.

Theorem 12 $P(G)$ is a union of complete graphs which share the identity element of G if and only if G is an EPPO-group and for every maximal cyclic subgroup A and B with $A \neq B$, $A \cap B = \{e\}$.

Proof Suppose there exist $x \in G$ and prime numbers p_1 and p_2 such that $p_1, p_2 | o(x)$. Then the cyclic subgroup $\langle x\rangle$ is containing non-adjacent elements x_1 of order p_1 and x_2 of order p_2. Since x_1 and x_2 are adjacent to x, they are in the same block of $P(G)$, a contradiction. If $A = \langle a\rangle$ and $B = \langle b\rangle$ are maximal cyclic subgroup of G such that $e \neq x \in A \cap B$ then x, a and b are mutually adjacent and so $A \subseteq B$ or $B \subseteq A$, which is impossible. Conversely, we assume that maximal cyclic subgroups of G have prime power order and for every maximal cyclic subgroup A and B with $A \neq B$, $A \cap B = \{e\}$. By Lemma 11, $S = \cup_{x\in M(G)} r(x) = \{e\}$. On the other hand, if $MaxCyc(G) = \{\langle x_1\rangle, ..., \langle x_r\rangle\}$ and $A = \{x_1, ..., x_r\}$ then by Lemma 11, each component of $P(G) - \{e\}$ is of form $\langle x_i\rangle - \{e\}$, for some i, $1 \le i \le r$, which is a complete subgraph of $P(G)$. This completes the proof. □

Corollary 13 If G is an EPO$-$group then $P(G)$ is a union of some complete graphs which share in the identity element of G.

Lemma 14 A finite group G is *EPPO* if and only if the vertices of every maximal clique of $P(G)$ is a maximal cyclic subgroup of G.

Proof ($\Longleftarrow$) Suppose H is a maximal clique in $P(G)$ and $x \in H$. If $o(x)$ has at least two prime divisors p and q then there are elements of these orders in H which is impossible.

($\Longrightarrow$) By Lemma 1, we map the maximal clique H in $P(G)$ to the chain $1 \subseteq \langle x_1\rangle \subseteq \langle x_2\rangle \subseteq \cdots \subseteq \langle x_t\rangle$. Then x_t has prime power order p^α and since G is *EPPO* group, $p^\alpha = 1 + \varphi(p) + \cdots + \varphi(p^\alpha)$. This implies that $H = \langle x_t\rangle$. □

A Chinese group theorist Wujie Shi [14] conjectured that a finite group and a finite simple group are isomorphic if they have the same orders and sets of element orders, see also [16, Question 12.39]. Vasiliev, Grechkoseeva and Mazurov gave an affirmative answer to this question in [17]. In the following theorem this result is applied to obtain a new characterization of finite simple groups by their power graphs.

Theorem 15 If G_1 is one of the following finite groups

a) A simple group,
b) A cyclic group,
c) A symmetric group,
d) A diheral group,
e) A generalised quaternion group,

and G_2 is a finite group such that $P(G_1) \cong P(G_2)$ then $G_1 \cong G_2$.

Proof Since $P(G_1) \cong P(G_2)$, by [3, Corollary 3] G_1 and G_2 have the same numbers of elements of each order. To prove (a) it is enough to use this corollary and the main result of [17] mentioned in Introduction.

b) If $P(G_2) \cong P(Z_n)$ then by the mentioned result of Cameron, G_2 have to exists an element of order n.

c) By [14], $G_2 \cong S_n$ if and only if $\pi_e(G_2) = \pi_e(S_n)$ and $|G_2| = |S_n|$, proving the part (c).

d) Suppose $P(G_2) \cong P(D_{2n})$ then $|G_2| = 2n$ and G_2 has an element a of order n. Since G has the same number of elements of order 2 as the dihedral group D_{2n}, we can choose an element b of order 2 in G_2 such that $\langle a \rangle \cap \langle b \rangle = 1$. This implies that G_2 is a semi-direct product of the cyclic group Z_n by Z_2. Therefore, $G_2 \cong D_{2n}$.

e) Suppose Q_{4n} denotes the generalized quaternion group of order $4n$ and $P(G_2) \cong P(Q_{4n})$. Then $|S| > 1$, where S is the set of vertices of the power graph $P(G_2)$ which are joined to all other vertices. We now apply [3, Proposition 4] to deduce that G_2 is isomorphic to Q_{4n}. $\square$

Let p be an odd prime number. Two groups of order $2p^2$ have isomorphic power graph if and only if they are isomorphic. This is a direct consequence of [13, Lemma 1]. In [2, Theorem 1], Peter Cameron characterized abelian groups by their power graphs. In the following theorem a simple proof for this result is presented.

Theorem 16 If G_1 and G_2 are finite abelian groups such that $P(G_1) \cong P(G_2)$ then $G_1 \cong G_2$.

Proof Suppose G_1 and G_2 are finite abelian groups such that $P(G_1) \cong P(G_2)$. Then by [3, Corollary 3], G_1 and G_2 are conformal. On the other hand, by [12, pp 107-109], finite abelain conformal groups are isomorphic. Therefore, $G_1 \cong G_2$. $\square$

Suppose p is prime. Then there are five groups of order p^3 up to isomorphism. From the cyclic decomposition of finite abelian groups, there are three abelian groups isomorphic to $G_1 \cong Z_p \times Z_p \times Z_p$, $G_2 \cong Z_p \times Z_{p^2}$, $G_3 \cong Z_{p^3}$. There are also two non-abelian groups, G_4 and G_5, of order p^3. If $p = 2$ then these groups are isomorphic to D_8 and Q_8, respectively. If p is odd then

$$G_4 \cong \langle a, b | a^{p^2} = b^p = bab^{p-1}a^{p^2-p-1} = e \rangle,$$

is a non-abelian group of order p^3. It has $p^2 - 1$ elements of order p, which fall into two conjugacy classes, of sizes $p - 1$ and $p^2 - p$; and $p^3 - p^2$ elements of order p^2, forming a single conjugacy class. There is also another group isomorphic to semi-direct product $Z_{p^2} \propto Z_p$. It has $p^3 - 1$ elements of order p falling into three conjugacy classes of sizes $p - 1$, $p^2 - p$ and $p^3 - p^2$. Suppose $G = G_1$ and $H = G_4$. An easy calculation shows that $P(G) \cong P(H)$. Therefore, non-cyclic abelian groups cannot be characterized by their power graphs.

Theorem 17 Let G be a finite group. The power graph $P(G)$ is bipartite if and only if G is an elementary abelian group of even order.

Proof Suppose $P(G)$ is bipartite. If an odd prime p divides $|G|$ then the complete graph K_p can be embedded into $P(G)$, a contradiction. On the other hand, if G has an element of order 4 then $P(G)$ is containing a copy of K_4 which is impossible. Therefore, G is an elementary abelian group of even order. The converse is trivial. □

A matching on a graph G is a set of edges of G such that no two of them share a vertex in common. A maximum matching of G is a matching with the largest size among all matchings in G. A vertex cover of G is a subset $Q \subseteq V(G)$ that contains at least one end point of each edge. The König-Egerváry theorem [18, Theorem 3.1.16], states that in any bipartite graph, the number of edges in a maximum matching equals the number of vertices in a minimum vertex cover.

Theorem 18 The power graph $P(Z_{p^n})$ has the maximum number of edges among all power graphs of $p-$groups of order p^n.

Proof Suppose G is a non-cyclic $p-$groups of order p^n. We construct a bipartite graph $\Gamma = (X, Y)$ as follows:

$$X = G, Y = Z_{p^n} \text{ and } E(\Gamma) = \{ab \mid a \in X,\ b \in Y \textit{ and } o(a) \leq o(b)\}.$$

We first assume that Γ has a perfect matching M and $f : G \longrightarrow Z_{p^n}$ is a bijective mapping such that for each $a \in G$, a and $f(a)$ are saturated by M. Thus, $o(a) - \varphi(o(a)) \leq o(f(a)) - \varphi(o(f(a))$ and since G is not cyclic,

$$\sum_{a\in G}[2o(a) - \varphi(o(a))] < \sum_{a\in G}[2o(f(a)) - \varphi(o(f(a))].$$

But $\frac{1}{2}\left[\sum_{a\in G}[2o(a) - \varphi(o(a))] - 1\right]$ and $\frac{1}{2}\left[\sum_{a\in G}[2o(f(a)) - \varphi(o(f(a))] - 1\right]$ are the number of edges in $P(G)$ and $P(Z_{p^n})$, respectively. So, it is enough to prove that Γ has a perfect matching. By König-Egerváry theorem we have to show that a minimum vertex cover of Γ has exactly p^n elements. Suppose that A is a minimum vertex cover of Γ and $p^\gamma = max\{o(x) \mid x \in G\}$. If $A = X$ then there is nothing to prove that $|A| = p^n$. Otherwise, elements of orders $p^\gamma + 1, p^{\gamma+2}, \cdots, p^n$ of Y are adjacent to all elements of G and so these elements are in A. We claim that A contains all elements of Y of order $p^k, k \leq \gamma$. Define,

$$L_k = \{(x, y) \in X \times Y \mid o(x) = o(y) = p^k\},$$

where $k \leq \gamma$. By our definition, if $(x, y) \in L_k$ then x is adjacent to y and so if $(x, y) \in L_k$ then $x \in A$ or $y \in A$. One can easily seen that L_k induces a complete bipartite induced subgraph of Γ and hence $\Omega_{p^k}(G) \subseteq A$ or $\Omega_{p^k}(Z_{p^n}) \subseteq A$. Since $|\Omega_{p^k}(G)| \leq |\Omega_{p^k}(Z_{p^n})|$, by minimality we can assume that $\Omega_{p^k}(Z_{p^n}) \subseteq A$, where $1 \leq k \leq \gamma$. Therefore, $A = Y$ and Γ has a perfect matching. This completes the proof. □

Corollary 19 If G is a non-cyclic $p-$group of order p^n then $\sum_{x\in G} o(x) < \sum_{x\in Z_{p^n}} o(x)$.

Our calculations with groups of small order suggest the following conjecture:

Conjecture 2: The power graph $P(Z_n)$ has the maximum number of edges among all power graphs of groups of order n.

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References

[1] N. Biggs, Algebraic Graph Theory, Second ed., Cambridge Univ. Press, Cambridge, 1993.
[2] P. J. Cameron, S. Ghosh, The power graph of a finite group, Discrete Mathematics, 311 (2011) 1220–1222.
[3] P. J. Cameron, The power graph of a finite group, II, Journal of Group Theory 13 (2010) 779–783.
[4] I. Chakrabarty, S. Ghosh, M. K. Sen, Undirected power graphs of semigroups, Semigroup Forum 78 (2009) 410–426.
[5] M. Chudnovsky, N. Robertson, P. Seymour, R. Thomas, The strong perfect graph theorem, Annals of Mathematics 164 (1) (2006) 51–229.
[6] Encyclopaedia of Mathematics, Supplement Vol. III, Edited by M. Hazewinkel, Kluwer Academic Publishers, Dordrecht, 2001.
[7] G. A. Jones, J. M. Jones, Elementary Number Theory, Springer, New Delhi, 2005.
[8] A. V. Kelarev, S. J. Quinn, A combinatorial property and power graphs of groups, Contributions to general algebra, 12 (Vienna, 1999), 229–235, Heyn, Klagenfurt, 2000.

[9] A. V. Kelarev, S. J. Quinn, Directed graph and combinatorial properties of semigroups, Journal of Algebra 251 (2002) 16–26.
[10] A. V. Kelarev, S. J. Quinn, A combinatorial property and power graphs of semigroups, Commentationed Mathematicae Universitatis Carolinae 45 (2004) 1–7.
[11] J. Lauri, R. Scapellato, Topics in Graphs Automorphisms and Reconstruction, London Mathematical Society Student Texts 54, Cambridge University Press, Cambridge, 2003.
[12] G. A. Miller, H. F. Blichfeldt, L. E. Dickson, Theory and applications of finite groups, Dover Publications, Inc., New York, 1961.
[13] R. Scapellato, Finite groups with the same number of elements of each order, Rendiconti di Matematica, Serie VII, 8 (1988) 339–344.
[14] W. J. Shi, A new characterization of the sporadic simple groups, in Group Theory, Proc., 1987, Singapore Group Theory Conference, Walter de Gruyter, New York , 531–540, 1989.
[15] The GAP Team, GAP, Groups, Algorithms and Programming, Lehrstuhl De fr Mathematik, RWTH, Aachen, 1995.
[16] Unsolved problems in group theory, The Kourovka notebook, Thirteenth augmented edition, Edited by V.D. Mazurov, E. I. Khukhro, Russian Academy of Sciences Siberian Division, Institute of Mathematics, Novosibirsk, 1995.
[17] A. V. Vasiliev, M. A. Grechkoseeva, V. D. Mazurov, Characterization of the finite simple groups by spectrum and order, Algebra and Logic 48 (2009) 385–409.
[18] D. B. West, Introduction to Graph Theory, Prentice Hall. Inc. Upper Saddle River, NJ, 1996.