109. On the Power Semigroup of the Group of Integers

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If $G(\cdot)$ is a group, the power semigroup $\mathcal{P}(G)$ is the semigroup of all nonempty subsets of G with respect to the operation defined by $AB=\{ab : a \in A,\ b \in B\}$ for all $A, B \in \mathcal{P}(G)$. The author and Shafer [5] obtained the group of units of $\mathcal{P}(G)$, and Putcha [4] studied the greatest semilattice decomposition of $\mathcal{P}(G)$ of a finite group G, but we know little about archimedean components of $\mathcal{P}(G)$ of an infinite group G.

Let Z be the group of integers under addition and Z_+ the subsemigroup of positive integers. The operation in $\mathcal{P}(Z)$ is denoted by $X+Y=\{x+y : x \in X,\ y \in Y\}$. For $X \in \mathcal{P}(Z)$ and $m \in Z_+$, we let $mX = \underbrace{X+\cdots+X}_{m}$ and $[a, b]=\{z \in Z : a \leq z \leq b\}$ if $a, b \in Z$ with $a \leq b$. For undefined terminology and basic information on commutative semigroups, the reader should refer to [1], [3].

Let $\mathcal{P}^*(Z)$ denote the subsemigroup of $\mathcal{P}(Z)$ consisting of all finite nonempty subsets of Z. If $X \in \mathcal{P}^*(Z)$, the archimedean component of $\mathcal{P}(Z)$ containing X coincides with that of $\mathcal{P}^*(Z)$ containing X. Let $\mathcal{A}\{0,1\}$ denote the archimedean component of $\mathcal{P}(Z)$ containing the element $\{0,1\}$. The purpose of this paper is to investigate the structure of $\mathcal{A}\{0,1\}$.

Let $X=\{x_1, x_2, \cdots, x_k\} \in \mathcal{P}^*(Z)$ where $x_1<x_2<\cdots<x_k$. We define $\min X=x_1$, $\max X=x_k$, $\mathrm{id}(X)=x_2-x_1$, $\mathrm{fd}(X)=x_k-x_{k-1}$, and $\mathrm{md}(X)=\max\{x_2-x_1, \cdots, x_k-x_{k-1}\}$. Note $\mathrm{md}(X) \geqq 1$ unless X is a singleton. If $\mathrm{md}(X)=1$, i.e. $X=[x_1, x_k]$, then X is called *consecutive*. If $\mathrm{id}(X)=\mathrm{fd}(X)=1$, X is called *semi-consecutive*. The following is a main theorem in this paper.

Theorem 1. *Let $X \in \mathcal{P}(Z)$. The following are equivalent*:

(1.1) $X \in \mathcal{A}\{0,1\}$.

(1.2) $nX=\{0,1\}+Y$ *for some* $n \in Z_+$ *and some* $Y \in \mathcal{P}(Z)$.

(1.3) $nX=m\{0,1\}+b$ *for some* $n, m \in Z_+$ *and some* $b \in Z$.

(1.4) X *is semi-consecutive.*

(1.5) nX *is consecutive for some* $n \in Z_+$.

Proof. (1.1)$\to$(1.2) is obvious by archimedeaness.

(1.2)$\to$(1.4). If $X=\{x_1, x_2, \cdots, x_k\}$, $\min(nX)=nx_1$ and the second element of nX is $(n-1)x_1+x_2$. This implies $\mathrm{id}(nX)=\mathrm{id}(X)$. Similarly

$\mathrm{fd}(nX)=\mathrm{fd}(X)$. Since $\{0,1\}+Y$ is semi-consecutive, we have $\mathrm{id}(X)=\mathrm{fd}(X)=1$.

(1.4)→(1.5). First the following lemma is obvious:

Lemma 1.6. *Let $V, W \in \mathcal{P}^*(Z)$ and assume $[a,b]\cap V\neq\phi$. If $V\subseteq W$, then* $\mathrm{md}([a,b]\cap W)\leq \mathrm{md}([a,b]\cap V)$.

To prove "(1.4)→(1.5)" it suffices to prove the following by induction on l.

Lemma 1.7. *Let $n=\mathrm{md}(X)$. If X is semi-consecutive, then* $\mathrm{md}(lX)\leq n-l+1$ *for each l with $1\leq l\leq n$.*

If $l=1$, it is obvious. Assume $l>1$ and Lemma 1.7 holds for l. Let $X=\{0,1,x_2,\cdots,x_{k-1},x_k\}$, $0<1<x_2<\cdots<x_{k-1}<x_k$ and $x_k-x_{k-1}=1$, and let $(l+1)X=D_1\cup D_2$ where $D_1=[0, lx_k+1]\cap(l+1)X$, $D_2=[lx_k, (l+1)x_k]\cap(l+1)X$. Now $lX+\{0,1\}\subset D_1$ and $\mathrm{md}(lX+\{0,1\})\leq n-l$ since $\mathrm{md}(lX)\leq n-l+1$ by induction hypothesis. By Lemma 1.6 we have

$$\mathrm{md}(D_1)\leq \mathrm{md}(lX+\{0,1\})\leq n-l.$$

Next we want to show $\mathrm{md}(D_2)\leq n-l$. The subset lX contains a consecutive subset $C=[lx_{k-1}, lx_k]=\{c_0, c_1,\cdots,c_l\}$ where

$$c_0=lx_{k-1},\cdots,c_i=(l-i)x_{k-1}+ix_k,\cdots,c_l=lx_k.$$

Let $K=[lx_k,(l+1)x_k]$ and $C_0=\{c_l\}$, $C_i=[c_{l-i},c_l]$, $i=1,\cdots,l-1$, $C_l=C$. By induction on i, $\mathrm{md}(K\cap(C_i+X))=\mathrm{md}(K\cap(C_{i-1}+X))-1$, $i=1,\cdots,l$. Since $\mathrm{md}(C_0+X)=n$, $\mathrm{md}(K\cap(C+X))=n-l$. By Lemma 1.6

$$\mathrm{md}(D_2)\leq \mathrm{md}(K\cap(C+X))=n-l$$

because $C+X\subset(l+1)X$. Combining $\mathrm{md}(D_1)\leq n-l$ with $\mathrm{md}(D_2)\leq n-l$, we have $\mathrm{md}((l+1)X)\leq n-l$. Hence Lemma 1.7 holds for all l with $1\leq l\leq n$. In particular, let $l=n$ in Lemma 1.7, then $\mathrm{md}(nX)\leqq 1$. Since nX is not a singleton, $\mathrm{md}(nX)=1$.

(1.5)→(1.3). Since nX is consecutive, there is $b\in Z$ such that $nX-b=[0,m]=m\{0,1\}$ for some $m\in Z_+$.

(1.3)→(1.1). Straightforward.

By Theorem 1, $\{0,1\}+Y\in\mathcal{A}\{0,1\}$ for all $Y\in\mathcal{P}^*(Z)$, so that $\mathcal{A}\{0,1\}$ is an ideal of $\mathcal{P}^*(\mathrm{Z})$. By using the results of [2] and [6] we can describe the structure of $\mathcal{A}\{0,1\}$.

(2) *$\mathcal{A}\{0,1\}$ is homomorphic onto the group Z under $h: X\mapsto$* $\min(X)$.

(3) *Let $X, Y\in\mathcal{A}\{0,1\}$. Then $m\{0,1\}+X=n\{0,1\}+Y$ for some $m,n\in \mathrm{Z}_+$ if and only if* $\min(X)=\min(Y)$.

Let $\mathcal{A}_z=\{X\in\mathcal{A}\{0,1\}: h(X)=z\}$. Then $\mathcal{A}\{0,1\}=\bigcup_{z\in Z}\mathcal{A}_z$, in particular $\mathcal{A}_0$ is a subsemigroup. Define a partial order $\prec$ on each $\mathcal{A}_z$ as follows:

$X, Y\in\mathcal{A}_z$, $X\prec Y$ iff $X=m\{0,1\}+Y$ for some $m\in\mathrm{Z}_+^0=\mathrm{Z}_+\cup\{0\}$.

(4) *$\mathcal{A}_z(\prec)$ forms a tree for each $z\in\mathrm{Z}$, and $\mathcal{A}_0(\prec)$ is order-isomorphic onto $\mathcal{A}_z(\prec)$ for every $z\in\mathrm{Z}$ under $X\mapsto X+z$.*

Theorem 5. *$\mathcal{A}\{0,1\}$ is isomorphic to the direct product of the idempotent-free power joined semigroup $\mathcal{A}_0$ and the group Z.*

Every element X of $\mathcal{A}\{0,1\}$ has the form $X=\bigcup_{i=1}^{l} X_i$, $l\geqq 1$, where each X_i is consecutive, $|X_1|^{*)}\geq 2$, $|X_l|\geq 2$ and if $l>1$, $X_i\cap X_j=\phi$ $(i\neq j)$; $x<y$ for all $x\in X_i$, $y\in X_j$ with $i<j$. Let $X\in\mathcal{A}\{0,1\}$. Then $\{0,1\}|X$ in $\mathcal{A}\{0,1\}$ if and only if (i) $|X_1|\geq 3$ and $|X_l|\geq 3$, and (ii) if $l>2$, $|X_i|\geq 2$ for all i with $i\neq 1$, $i\neq l$.

Theorem 6. *$\mathcal{A}_0$ consists of $\{0,1\}$, $\{0,1,2\}$, $\{0,1,2,3\}$ and $\{0,1\}\cup Y\cup\{i-1,i\}$ where $i\geq 4$ and Y is any subset of $[2,i-2\}$, Y may be empty. If X is not consecutive, $n\{0,1\}+X$ is consecutive for some $n\in Z_+$ where the least n is $(\mathrm{md}(X))-1$. The homomorphism $h_c:\mathcal{A}_0\to Z_+$ defined by $h_c(X)=\max(X)$ is the greatest cancellative homomorphism of $\mathcal{A}_0$.*

Theorem 7. *Let C be the set of all consecutive elements of $\mathcal{A}\{0,1\}$. Then C is a subsemigroup of $\mathcal{A}\{0,1\}$, and C is also the greatest cancellative homomorphic image of $\mathcal{A}\{0,1\}$, that is, $C\cong\mathcal{A}\{0,1\}/\rho_1$ where ρ_1 is defined by $X\rho_1 Y$ iff $\max(X)=\max(Y)$. C is a cancellative idempotent-free archimedean semigroup and C is isomorphic to the direct product of Z_+ and Z.*

References

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*) $|X_1|$ denotes the number of elements of X_1.